DOMAIN DECOMPOSITION IN OPTIMAL CONTROL OF ELLIPTIC SYSTEMS ON 2-D NETWORKS

John E. Lagnese

Department of Mathematics

Georgetown University

Washington, DC 20057-1233 USA

`lagnese@math.georgetown.edu`

Abstract: A method of decomposing the optimality system arising in optimal boundary control of elliptic systems of partial differential equations on 2-d networks into subproblems on individual elements of the network is presented.

1 ELLIPTIC SYSTEMS ON 2-D NETWORKS

Let $\mathcal{P}$ be a two-dimensional polygonal topological network in $\mathbb{R}^N$, $N \geq 3$. Thus $\mathcal{P}$ is a finite union of nonempty subsets $\mathcal{P}_I$, $I \in \mathcal{I}$, such that

- (i) each $\mathcal{P}_I$ is a simply connected open polygonal subset of a plane Π_I in $\mathbb{R}^N$;
- (ii) $\bigcup_{I \in \mathcal{I}} \overline{\mathcal{P}_I}$ is connected;
- (iii) for all $I, J \in \mathcal{I}$, $\overline{\mathcal{P}_I} \cap \overline{\mathcal{P}_J}$ is either empty, a common vertex, or a whole common side.

For each $I \in \mathcal{I}$ we fix a system of Cartesian coordinates (x_1, x_2) in Π_I . Thus

$$\Pi_I = \{\mathbf{p}_{I0} + x_1 \boldsymbol{\eta}_{I1} + x_2 \boldsymbol{\eta}_{I2}\}, \quad x_1, x_2 \in \mathbb{R},$$

where $\boldsymbol{\eta}_{I1}$, $\boldsymbol{\eta}_{I2}$ are orthonormal vectors that span Π_I and $\mathbf{p}_{I0}$ denotes the origin of coordinates in Π_I . The linear segments that form the boundary $\partial \mathcal{P}_I$ of $\mathcal{P}_I$ are written $\overline{\Gamma}_{IJ}$, $J = 1, \dots, N_I$. It is convenient to assume that Γ_{IJ} is open in $\partial \mathcal{P}_I$. The collection of all Γ_{IJ} are the *edges* of $\mathcal{P}$ and is denoted by $\mathcal{A}$. An edge Γ_{IJ} corresponding to an $A \in \mathcal{A}$ is denoted by Γ_{IA} or Γ_A and the *index set* $\mathcal{I}_A$ of A is $\mathcal{I}_A = \{I \mid A = \Gamma_{IA}\}$. The *degree* of an edge is the cardinality of $\mathcal{I}_A$ and is denoted by d_A . For each $I \in \mathcal{I}_A$ we denote by ν_{IA} the unit outer normal to $\mathcal{P}_I$ along Γ_{IA} . The coordinates of ν_{IA} in the given coordinate system of Π_I are written (ν_{IA}^1, ν_{IA}^2) .

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Let $m \geq 1$ be a given integer. For a function $\mathbf{W} : \mathcal{P} \mapsto \mathbb{R}^m$, $\mathbf{W}_I$ will denote the restriction of $\mathbf{W}$ to $\mathcal{P}_I$, that is $\mathbf{W}_I : \mathcal{P}_I \mapsto \mathbb{R}^m : x \mapsto \mathbf{W}(x)$. We introduce $m \times m$ matrices $A_I^{\alpha\beta}$, B_I^β , C_I , $I \in \mathcal{I}$, $\alpha, \beta = 1, 2$, $A_I^{\alpha\beta} = (A_I^{\beta\alpha})^T$ and $C_I = C_I^T$, and where the superscript T indicates transpose, and we define the symmetric bilinear form

$$\mathcal{B}(\mathbf{W}, \mathbf{V}) = \sum_{I \in \mathcal{I}} \int_{\mathcal{P}_I} [A_I^{\alpha\beta}(\mathbf{W}_{I,\beta} + B_I^\beta \mathbf{W}_I) \cdot (\mathbf{V}_{I,\alpha} + B_I^\alpha \mathbf{V}_I) + C_I \mathbf{W}_I \cdot \mathbf{V}_I], \quad (1.1)$$

where repeated lower case Greek indices are summed over 1,2. A subscript following a comma indicated differentiation with respect to the corresponding variable, e.g., $\mathbf{W}_{I,\beta} = \partial \mathbf{W}_I / \partial x_\beta$. The matrices $A_I^{\alpha\beta}$, B_I^β , C_I may depend on $(x_1, x_2) \in \mathcal{P}_I$ and $\mathcal{B}$ is required to be $\mathcal{V}$ -elliptic for an appropriate function space $\mathcal{V}$.

We shall consider the variational equation

$$\mathbf{W} \in \mathcal{V}, \quad \mathcal{B}(\mathbf{W}, \mathbf{V}) = \sum_{A \in \mathcal{N}^{\text{ext}}} \int_{\Gamma_{IA}} \mathbf{F}_A \cdot \mathbf{V}_I, \quad \forall \mathbf{V} \in \mathcal{V}, \quad (1.2)$$

where the space $\mathcal{V}$ is defined in terms of certain *geometric edge conditions*, $\mathcal{N}^{\text{ext}}$ is the set of *exterior Neumann edges* (see below), I is the unique index associated with an edge $A \in \mathcal{N}^{\text{ext}}$, and $\mathbf{F}_A \in [L^2(\Gamma_A)]^m$. To describe the space $\mathcal{V}$ we partition the edges of $\mathcal{A}$ into two disjoint subsets $\mathcal{D}$, the *Dirichlet edges*, and $\mathcal{N}$, the *Neumann edges*. Each edge $A \in \mathcal{D}$ is assumed to be an *exterior edge*, that is, $d_A = 1$. We denote by $\mathcal{N}^{\text{ext}}$ the exterior edges in $\mathcal{N}$ and set $\mathcal{N}^{\text{int}} = \mathcal{N} \setminus \mathcal{N}^{\text{ext}}$, the set of *interior Neumann edges*. We define

$$\mathcal{H} = \{\mathbf{V} : \mathcal{P} \mapsto \mathbb{R}^m \mid \mathbf{V}_I \in [L^2(\mathcal{P}_I)]^m\}$$

with the standard norm,

$$\mathcal{V} = \{\mathbf{V} \in \mathcal{H} \mid \mathbf{V}_I \in [H^1(\mathcal{P}_I)]^m, \mathbf{V}_I = 0 \text{ on } \Gamma_{IA} \text{ if } \Gamma_{IA} \in \mathcal{D}, \mathbf{V}_I = \mathbf{V}_J \text{ on } \Gamma_A \text{ if } \Gamma_{IA} = \Gamma_{JA}\}.$$

We shall assume that the bilinear form $\mathcal{B}$ satisfies

$$\mathcal{B}(\mathbf{V}, \mathbf{V}) \geq c \|\mathbf{V}\|_{\mathcal{H}^1(\mathcal{P})}^2, \quad \forall \mathbf{V} \in \mathcal{V},$$

for some constant $c > 0$. This ellipticity condition will hold in examples 2.2 and 2.3 below if $\mathcal{D} \neq \emptyset$, and also in example 2.1 if, for instance, for each $I \in \mathcal{I}$ the 2×2 matrix $(a_I^{\alpha\beta})$ is positive definite and $b_I^\alpha = c_I = 0$.

Let

$$\mathcal{U} = \prod_{A \in \mathcal{N}^{\text{ext}}} \mathcal{L}^2(\Gamma_A), \quad \mathcal{L}^2(\Gamma_A) := [L^2(\Gamma_A)]^m.$$

For $\mathbf{F} \in \mathcal{U}$ and $A \in \mathcal{N}^{\text{ext}}$, $\mathbf{F}_A$ will denote the restriction of $\mathbf{F}$ to Γ_A . Consider the optimal control problem

$$\inf_{\mathbf{F} \in \mathcal{U}} \left\{ \sum_{I \in \mathcal{I}} \int_{\mathcal{P}_I} |\mathbf{W}_I - \mathbf{Z}_I|^2 + k \sum_{A \in \mathcal{N}^{\text{ext}}} \int_{\Gamma_A} |\mathbf{F}_A|^2 \right\}$$

where $\mathbf{Z} = \{\mathbf{Z}_I\}_{I \in \mathcal{I}} \in \mathcal{H}$ is given, $k > 0$ is the penalization parameter and $\mathbf{W}_I$, $I \in \mathcal{I}$ is the solution of (1.2). By standard theory, the optimality system is

$$\begin{aligned} \mathcal{B}(\mathbf{W}, \mathbf{V}) + \sum_{A \in \mathcal{N}^{\text{ext}}} \frac{1}{k} \int_{\Gamma_{IA}} \Phi_I \cdot \mathbf{V}_I &= 0, \\ \mathcal{B}(\Phi, \mathbf{V}) &= \sum_{I \in \mathcal{I}} \int_{\mathcal{P}_I} (\mathbf{W}_I - \mathbf{Z}_I) \cdot \mathbf{V}_I, \quad \forall \mathbf{V} \in \mathcal{V}. \end{aligned}$$

This is the variational formulation of the boundary value problem

$$\begin{aligned} \mathcal{L}_I \mathbf{W}_I &= 0, \quad \mathcal{L}_I \Phi_I = \mathbf{W}_I - \mathbf{Z}_I \text{ in } \mathcal{P}_I, \\ \mathbf{W}_I &= \Phi_I = 0 \text{ on } \Gamma_{IA} \text{ if } \Gamma_{IA} \in \mathcal{D}, \\ \mathbf{W}_I &= \mathbf{W}_J, \quad \Phi_I = \Phi_J \text{ on } A \text{ if } \Gamma_{IA} = \Gamma_{JA}, \end{aligned} \quad (1.3)$$

$$\begin{aligned} D_{\nu_{IA}} \Phi_I &= 0, \quad D_{\nu_{IA}} \mathbf{W}_I + \frac{1}{k} \Phi_I = 0 \text{ on } \Gamma_{IA} \text{ if } \Gamma_{IA} \in \mathcal{N}^{\text{ext}}, \\ \sum_{I \in \mathcal{I}_A} D_{\nu_{IA}} \mathbf{W}_I &= \sum_{I \in \mathcal{I}_A} D_{\nu_{IA}} \Phi_I = 0 \text{ on } A \text{ if } A \in \mathcal{N}^{\text{int}}, \end{aligned} \quad (1.4)$$

where we have set

$$\mathcal{L} \mathbf{W}_I := -\frac{\partial}{\partial x_\alpha} [A_I^{\alpha\beta} (\mathbf{W}_{I,\beta} + B_I^\beta \mathbf{W}_I)] + (B_I^\alpha)^T A_I^{\alpha\beta} (\mathbf{W}_{I,\beta} + B_I^\beta \mathbf{W}_I) + C_I \mathbf{W}_I,$$

$$D_{\nu_{IA}} \mathbf{W}_I := \nu_{IA}^\alpha A_I^{\alpha\beta} (\mathbf{W}_{I,\beta} + B_I^\beta \mathbf{W}_I).$$

Conditions (1.3) and (1.4) are referred to as *geometric junction conditions* and *mechanical junction conditions*, respectively.

2 EXAMPLES

Example 2.1 (Scalar elliptic equations on 2-d networks) Suppose that $m = 1$. In this case the matrices $A_I^{\alpha\beta}$, B_I^α , C_I reduce to scalars $a_I^{\alpha\beta}$, b_I^α , c_I , where $a_I^{\alpha\beta} = a_I^{\beta\alpha}$. The geometric and mechanical junction conditions are respectively

$$\begin{aligned} W_I &= W_J \text{ on } A \text{ if } \Gamma_{IA} = \Gamma_{JA}, \quad A \in \mathcal{N}^{\text{int}}, \\ \sum_{I \in \mathcal{I}_A} \nu_{IA}^\alpha &= a_I^{\alpha\beta} (W_{I,\beta} + b_I^\beta W_I) = 0 \text{ on } A, \quad A \in \mathcal{N}^{\text{int}}. \end{aligned}$$

The reader is referred to the book of Nicaise (1993) for a detailed existence-regularity theory for scalar elliptic equations on 2-d networks.

Example 2.2 (Networks of homogeneous isotropic membranes in $\mathbb{R}^3$) In this case $m = N = 3$. For each $I \in \mathcal{I}$ we choose a basis $\{\xi_{Ik}\}_{k=1}^3$ so that $\xi_{I1} = \eta_{I1}$, $\xi_{I2} = \eta_{I2}$, $\xi_{I3} = \xi_{I1} \times \xi_{I2}$, where η_{I1} , η_{I2} are the unit coordinate vectors in Π_I . We set $B_I^\alpha = C_I = 0$. With respect to the $\{\xi_{Ik}\}_{k=1}^3$ basis the matrices

$A_I^{\alpha\beta}$ are given by

$$\begin{aligned} A_I^{11} &= \text{diag}(2\mu_I + \lambda_I, \mu_I, \mu_I), \\ A_I^{22} &= \text{diag}(\mu_I, 2\mu_I + \lambda_I, \mu_I), \\ A_I^{12} &= \begin{pmatrix} 0 & \lambda_I & 0 \\ \mu_I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_I^{21} = \begin{pmatrix} 0 & \mu_I & 0 \\ \lambda_I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Write $\mathbf{W}_I = W_{Ik}\boldsymbol{\xi}_{Ik}$ (summation convention over k from 1 to m), $\mathbf{w}_I = W_{I\alpha}\boldsymbol{\xi}_{I\alpha}$,

$$\varepsilon_{\alpha\beta}(\mathbf{w}_I) = \frac{1}{2}(W_{I\alpha,\beta} + W_{I\beta,\alpha}),$$

$$S_I^{\alpha\beta}(\mathbf{w}_I) = 2\mu_I\varepsilon_{\alpha\beta}(\mathbf{w}_I) + \lambda_I\varepsilon_{\gamma\gamma}(\mathbf{w}_I)\delta^{\alpha\beta}.$$

The bilinear form (1.1) may be written

$$\mathcal{B}(\mathbf{W}, \mathbf{V}) = \sum_{I \in \mathcal{I}} \int_{\mathcal{P}_I} [S_I^{\alpha\beta}(\mathbf{w}_I)\varepsilon_{\alpha\beta}(\mathbf{v}_I) + \mu_I W_{I3,\alpha} V_{I3,\alpha}].$$

The geometric and mechanical junction conditions for $\mathbf{W}$ are respectively

$$\mathbf{W}_I = \mathbf{W}_J \text{ on } A \text{ if } \Gamma_{IA} = \Gamma_{JA},$$

$$\sum_{I \in \mathcal{I}_A} [S_I^{\alpha\beta}(\mathbf{w}_I)\boldsymbol{\xi}_{I\beta} + \mu_I W_{I3,\alpha}\boldsymbol{\xi}_{I3}] \nu_{IA}^\alpha = 0 \text{ on } A \text{ when } A \in \mathcal{N}^{\text{int}},$$

which have the interpretations of continuity of displacements and balance of forces, respectively, at the membrane junctions (see Lagnese, Leugering and Schmidt (1994)).

Example 2.3 (Interface problems for serial Reissner-Mindlin plates) We take $N = m = 3$ and assume that the regions $\mathcal{P}_I$ are coplanar. The region $\cup_{I \in \mathcal{I}} \overline{\mathcal{P}_I}$ represents the middle surface of a thin plate of uniform thickness h . Within the region $\mathcal{P}_I \times (-h/2, h/2)$ the plate is assumed homogeneous and elastically isotropic with Lamé parameters λ_I, μ_I and shear modulus K_I . Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2$ be unit coordinate vectors in the plane Π containing the regions $\mathcal{P}_I$, and set $\boldsymbol{\xi}_1 = \boldsymbol{\eta}_2, \boldsymbol{\xi}_2 = -\boldsymbol{\eta}_1$ and $\boldsymbol{\xi}_3 = \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2$. The matrix $C_I = 0$ and, with respect to the $\{\boldsymbol{\xi}_k\}_{k=1}^3$ basis,

$$\begin{aligned} A_I^{11} &= h \text{diag}\left(\frac{h^2}{12}(2\mu_I + \sigma_I), \frac{h^2}{12}\mu_I, K_I\right), \\ A_I^{22} &= h \text{diag}\left(\frac{h^2}{12}\mu_I, \frac{h^2}{12}(2\mu_I + \sigma_I), K_I\right) \end{aligned}$$

where $\sigma_I = 2\mu_I\lambda_I/(2\mu_I + \lambda_I)$,

$$A_I^{12} = \frac{h^3}{12} \begin{pmatrix} 0 & \sigma_I & 0 \\ \mu_I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_I^{21} = \frac{h^3}{12} \begin{pmatrix} 0 & \mu_I & 0 \\ \sigma_I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_I^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_I^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Write $\mathbf{W}_I = W_{Ik}\boldsymbol{\xi}_k$, $\mathbf{w}_I = W_{I\alpha}\boldsymbol{\xi}_\alpha$ and

$$S_I^{\alpha\beta}(\mathbf{w}_I) = \frac{h^3}{12}(2\mu_I\varepsilon_{\alpha\beta}(\mathbf{w}_I) + \sigma_I\varepsilon_{\gamma\gamma}(\mathbf{w}_I)\delta^{\alpha\beta}) \quad (1.5)$$

With this notation the bilinear form (1.1) is

$$\mathcal{B}(\mathbf{W}, \mathbf{V}) = \sum_{I \in \mathcal{I}} \int_{\mathcal{P}_I} [S_I^{\alpha\beta}(\mathbf{w}_I)\varepsilon_{\alpha\beta}(\mathbf{v}_I) + K_I h(W_{I3,\alpha} + W_{I\alpha})(V_{I3,\alpha} + V_{I\alpha})].$$

The function $W_{I3}(x_1, x_2)$ represents the transverse displacement of the material particle situated at $(x_1, x_2) \in \mathcal{P}_I$ in the reference configuration, and W_{I1} , W_{I2} measure transverse shearing. The geometric junction conditions for $\mathbf{W}$ are

$$W_{Ik} = W_{Jk} \quad \text{on } A \text{ when } \Gamma_{IA} = \Gamma_{JA}, \quad k = 1, 2, 3,$$

(the degree $d_A = 2$ for each interior edge) while the mechanical junction conditions are

$$\begin{aligned} \sum_{I \in \mathcal{I}_A} \nu_{IA}^\alpha &= S_I^{\alpha\beta}(\mathbf{w}_I) = 0, \quad \beta = 1, 2, \\ \sum_{I \in \mathcal{I}_A} K_I \nu_{IA}^\alpha (W_{I3,\alpha} + W_{I\alpha}) &= 0 \text{ on } A \text{ when } A \in \mathcal{N}^{\text{int}}. \end{aligned}$$

These have the interpretations of balance of angular and linear momenta, respectively (see Lagnese, Leugering and Schmidt (1994)).

3 DOMAIN DECOMPOSITION

In the present context the object of domain decomposition is to replace the above global optimality system on the network $\mathcal{P}$ by a sequence of local problems on the elements $\mathcal{P}_I$ whose solutions converge to a solution of the global optimality system in an appropriate sense. Each local problem is itself the optimality system for some local optimal control problem. In the decomposition, the geometric and mechanical junction conditions that couple adjacent regions in the global network are replaced by local Robin-type boundary conditions in which the effect of adjacent regions is modeled by certain inhomogeneities in the boundary conditions. This strategy was first employed in the context of optimal control problems for single scalar equations by Benamou (see, e.g., Benamou (1995), (1998) and references therein) and later extended by Leugering (1998) to construct domain decompositions for the optimal control of certain mechanical networks of 1-dimensional elements, where multiple nodes naturally appear.

For each index $I \in \mathcal{I}$ let $\mathcal{D}_I$ denote the Dirichlet edges in $\partial\mathcal{P}_I$ and set $\mathcal{N}_I = \partial\mathcal{P}_I \setminus \mathcal{D}_I$, which is partitioned into $\mathcal{N}_I^{\text{ext}} \cup \mathcal{N}_I^{\text{int}}$, where $\mathcal{N}_I^{\text{ext}} = \mathcal{N}_I \cap \mathcal{N}^{\text{ext}}$. We write $\mathcal{H}_I^s$ for $[H^s(\mathcal{P}_I)]^m$, $\mathcal{H}_I = \mathcal{H}_I^0$, and we define

$$\mathcal{V}_I = \{\mathbf{V}_I \mid \mathbf{V}_I \in \mathcal{H}_I^1, \mathbf{V}_I = 0 \text{ on } \Gamma_{IA} \text{ if } \Gamma_{IA} \in \mathcal{D}_I\}$$

A continuous, symmetric bilinear form $\mathcal{B}_I$ is defined on $\mathcal{V}_I$ by

$$\mathcal{B}_I(\mathbf{W}_I, \mathbf{V}_I) = \int_{\mathcal{P}_I} [A_I^{\alpha\beta}(\mathbf{W}_{I,\beta} + B_I^\beta \mathbf{W}_I) \cdot (\mathbf{V}_{I,\alpha} + B_I^\alpha \mathbf{V}_I) + C_I \mathbf{W}_I \cdot \mathbf{V}_I]$$

and we assume that

$$\mathcal{B}_I(\mathbf{V}_I, \mathbf{V}_I) \geq c_I \|\mathbf{V}_I\|_{\mathcal{H}_I^1}^2, \quad \forall \mathbf{V}_I \in \mathcal{V}_I,$$

for some constant $c_I > 0$.

For each $I \in \mathcal{I}$ and $n = 0, 1, \dots$, we consider the following sequence of local problems for functions $\mathbf{W}_I^{n+1}, \Phi_I^{n+1}$:

$$\begin{aligned} B_I(\mathbf{W}_I^{n+1}, \mathbf{V}_I) + \frac{1}{k} \sum_{A \in \mathcal{N}_I^{\text{ext}}} \int_{\Gamma_{IA}} \Phi_I^{n+1} \cdot \mathbf{V}_I + \sum_{A \in \mathcal{N}_I^{\text{int}}} k_A \int_{\Gamma_{IA}} \Phi_I^{n+1} \cdot \mathbf{V}_I \\ = \sum_{A \in \mathcal{N}_I^{\text{int}}} \int_{\Gamma_{IA}} \mathbf{V}_I \cdot \Lambda_{IA}(\mathbf{W}^n, \Phi^n), \quad \forall \mathbf{V}_I \in \mathcal{V}_I, \end{aligned} \quad (1.6)$$

$$\begin{aligned} B_I(\Phi_I^{n+1}, \mathbf{V}_I) - \sum_{A \in \mathcal{N}_I^{\text{int}}} k_A \int_{\Gamma_{IA}} \mathbf{W}_I^{n+1} \cdot \mathbf{V}_I - \int_{\mathcal{P}_I} (\mathbf{W}_I^{n+1} - \mathbf{Z}_I) \cdot \mathbf{V}_I \\ = \sum_{A \in \mathcal{N}_I^{\text{int}}} \int_{\Gamma_{IA}} \mathbf{V}_I \cdot \Pi_{IA}(\mathbf{W}^n, \Phi^n), \quad \forall \mathbf{V}_I \in \mathcal{V}_I, \end{aligned} \quad (1.7)$$

where $k_A > 0$ is constant,

$$\begin{aligned} \Lambda_{IA}(\mathbf{W}^n, \Phi^n) &= \frac{2k_A}{d_A} \sum_{J \in \mathcal{I}_A} \Phi_J^n - k_A \Phi_I^n - \frac{2}{d_A} \sum_{J \in \mathcal{I}_A} D_{\nu_{JA}} \mathbf{W}_J^n + D_{\nu_{IA}} \mathbf{W}_I^n, \\ \Pi_{IA}(\mathbf{W}^n, \Phi^n) &= \Lambda_{IA}(\Phi^n, -\mathbf{W}^n). \end{aligned}$$

The boundary value problem corresponding to (1.6), (1.7) is

$$\begin{aligned} \mathcal{L}_I \mathbf{W}_I^{n+1} &= 0, \quad \mathcal{L}_I \Phi_I^{n+1} = \mathbf{W}_I^{n+1} - \mathbf{Z}_I \text{ in } \mathcal{P}_I, \\ \mathbf{W}_I^{n+1} &= \Phi_I^{n+1} = 0 \text{ on } \Gamma_{IA} \text{ if } \Gamma_{IA} \in \mathcal{D}_I, \\ D_{\nu_{IA}} \mathbf{W}_I^{n+1} + \frac{1}{k} \gamma_{IA} \Phi_I^{n+1} &= 0, \quad D_{\nu_{IA}} \Phi_I^{n+1} = 0 \text{ on } \Gamma_{IA} \text{ if } \Gamma_{IA} \in \mathcal{N}_I^{\text{ext}}, \\ D_{\nu_{IA}} \mathbf{W}_I^{n+1} + k_A \gamma_{IA} \Phi_I^{n+1} &= \Lambda_{IA}(\mathbf{W}^n, \Phi^n), \\ D_{\nu_{IA}} \Phi_I^{n+1} - k_A \gamma_{IA} \mathbf{W}_I^{n+1} &= \Pi_{IA}(\mathbf{W}^n, \Phi^n) \end{aligned} \quad \text{on } \Gamma_{IA} \text{ if } \Gamma_{IA} \in \mathcal{N}_I^{\text{int}}$$

The inhomogeneities $\Lambda_{IA}(\mathbf{W}^n, \Phi^n)$, $\Pi_{IA}(\mathbf{W}^n, \Phi^n)$ reflects the influence of the regions adjacent to Γ_{IA} , $A \in \mathcal{N}_I^{\text{int}}$, on the solution in $\mathcal{P}_I$ at level $n+1$. The choice of $\{\mathbf{W}_I^0, \Phi_I^0\}_{I \in \mathcal{I}}$ is arbitrary save for a regularity requirement on $\Lambda_{IA}(\mathbf{W}^0, \Phi^0)$ and $\Pi_{IA}(\mathbf{W}^0, \Phi^0)$. One may verify that the above system for $\mathbf{W}_I^{n+1}, \Phi_I^{n+1}$ is the optimality system for the local optimal control problem

$$\inf_{\mathbf{F} \in \mathcal{L}^2(\mathcal{N}_I)} \left\{ \int_{\mathcal{P}_I} |\mathbf{W}_I^{n+1} - \mathbf{Z}_I|^2 + k \sum_{A \in \mathcal{N}_I^{\text{ext}}} \int_{\Gamma_{IA}} |\mathbf{F}_{IA}|^2 \right. \\ \left. + \sum_{A \in \mathcal{N}_I^{\text{int}}} \int_{\Gamma_{IA}} (|\mathbf{F}_{IA}|^2 + |\mathbf{W}_I^{n+1} + \Pi_{IA}(\mathbf{W}^n, \Phi^n)|^2) \right\}$$

subject to

$$\mathbf{W}_I^{n+1} \in \mathcal{V}_I, \quad \mathcal{B}_I(\mathbf{W}_I^{n+1}, \mathbf{V}_I) = \sum_{A \in \mathcal{N}_I^{\text{ext}}} \int_{\Gamma_{IA}} \mathbf{F}_{IA} \cdot \mathbf{V}_I \\ + \sum_{A \in \mathcal{N}_I^{\text{int}}} \frac{1}{k_A} \int_{\Gamma_{IA}} (\mathbf{F}_{IA} + \Lambda_{IA}(\mathbf{W}^n, \Phi^n)) \cdot \mathbf{V}_I, \quad \forall \mathbf{V}_I \in \mathcal{V}_I$$

4 CONVERGENCE

Suppose that $\Lambda_{IA}(\mathbf{W}^0, \Phi^0), \Pi_{IA}(\mathbf{W}^0, \Phi^0) \in \mathcal{L}^2(\Gamma_{IA}), \forall A \in \mathcal{N}_I^{\text{int}}$. Then, for each $n \geq 0$ and $I \in \mathcal{I}$, $\Lambda_{IA}(\mathbf{W}^n, \Phi^n), \Pi_{IA}(\mathbf{W}^n, \Phi^n) \in \mathcal{L}^2(\Gamma_{IA})$ and the local optimality system has a unique solution $\mathbf{W}_I^{n+1} \in \mathcal{V}_I, \Phi_I^{n+1} \in \mathcal{V}_I$.

Theorem 1 *Suppose that the solution of the global optimality system has the regularity*

$$D_{\nu_{IA}} \mathbf{W}_I, D_{\nu_{IA}} \Phi_I \in \mathcal{L}^2(\Gamma_{IA}), \quad I \in \mathcal{I}, A \in \mathcal{N}_I^{\text{int}}.$$

Then for each $I \in \mathcal{I}$,

$$\mathbf{W}_I^{n+1} \rightarrow \mathbf{W}_I \text{ strongly in } \mathcal{H}_I, \\ \Phi_I^{n+1}|_{\Gamma_{IA}} \rightarrow \Phi_I|_{\Gamma_{IA}} \text{ strongly in } \mathcal{L}^2(\Gamma_{IA}), A \in \mathcal{N}_I^{\text{ext}}.$$

In words, the solution $\mathbf{W} = 3D\{\mathbf{W}_I\}_{I \in \mathcal{I}}$ of the global optimal control problem is the strong limit in $\mathcal{H}$ of the solutions $\{\mathbf{W}_I^{n+1}\}_{I \in \mathcal{I}}$ of the local optimal control problems, and the global optimal control $\{-\frac{1}{k}\Phi_I|_{\Gamma_{IA}}\}_{A \in \mathcal{N}^{\text{ext}}}$ is the strong limit in $\mathcal{U}$ of the local optimal controls $\{-\frac{1}{k}\Phi_I^{n+1}|_{\Gamma_{IA}}\}_{A \in \mathcal{N}^{\text{ext}}}$.

Space limitations do not permit a detailed proof of this theorem. The idea is to set

$$\mathbf{U}_I^{n+1} = \mathbf{W}_I^{n+1} - \mathbf{W}_I, \quad \Psi_I^{n+1} = \Phi_I^{n+1} - \Phi_I,$$

and to introduce the quantity

$$E^{n+1} := \sum_{A \in \mathcal{N}^{\text{int}}} \sum_{I \in \mathcal{I}_A} \int_{\Gamma_{IA}} \{k_A(|\mathbf{U}_I^{n+1}|^2 + |\Psi_I^{n+1}|^2) \\ + \frac{1}{k_A}(|D_{\nu_{IA}} \mathbf{U}_I^{n+1}|^2 + |D_{\nu_{IA}} \Psi_I^{n+1}|^2)\}.$$

It is then possible to prove the recursion

$$E^{n+1} = E^n - 2 \sum_{I \in \mathcal{I}} \int_{\mathcal{P}_I} (|\mathbf{U}_I^{n+1}|^2 + |\mathbf{U}_I^n|^2) - \frac{2}{k} \sum_{A \in \mathcal{N}^{\text{ext}}} \int_{\Gamma_{IA}} (|\Psi_I^{n+1}|^2 + |\Psi_I^n|^2).$$

By iteration we obtain

$$E^{n+1} = E^1 - 4 \sum_{\ell=1}^{n+1} \left(\frac{2}{k} \sum_{A \in \mathcal{N}^{\text{ext}}} \int_{\Gamma_{IA}} |\Psi_I^\ell|^2 + \sum_{I \in \mathcal{I}} \int_{\mathcal{P}_I} |\mathbf{U}_I^\ell|^2 \right),$$

where $\sum_{\ell=1}^{n+1} a_\ell = 1/2(a_1 + a_{n+1}) + \sum_{\ell=2}^n a_\ell$. It follows in particular that

$$\sum_{\ell=1}^{\infty} \sum_{I \in \mathcal{I}} \int_{\mathcal{P}_I} |\mathbf{U}_I^\ell|^2 < \infty, \quad \sum_{\ell=1}^{\infty} \sum_{A \in \mathcal{N}^{\text{ext}}} \int_{\Gamma_{IA}} |\Psi_I^\ell|^2 < \infty.$$

Therefore

$$\{\mathbf{U}_I^n\}_{I \in \mathcal{I}} \rightarrow 0 \text{ strongly in } \mathcal{H}, \quad \Psi_I^n|_{\Gamma_{IA}} \rightarrow 0 \text{ strongly in } L^2(\Gamma_{IA}), \quad \forall A \in \mathcal{N}^{\text{ext}}.$$

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