COMPARISON THEOREM OF SOLUTIONS TO BSDE WITH JUMPS, AND VISCOSITY SOLUTION TO A GENERALIZED HAMILTON-JACOBI-BELLMAN EQUATION

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Abstract: Consider the following controlled FSDE and BSDE:

$$dx_s^{t,x,v} = b(s, x_s^{t,x,v}, v_s)ds + \sigma(s, x_s^{t,x,v}, v_s)dw_s + \int_{\mathbb{Z}} c(s, x_s^{t,x,v}, v_s, z)\tilde{N}_k(ds, dz),$$

$$x_t^{t,x,v} = x \in R^d, s \in [t, T];$$

$$-dy_s^{t,x,v} = f(s, x_s^{t,x,v}, y_s^{t,x,v}, q_s^{t,x,v}, p_s^{t,x,v}, v_s)ds - q_s^{t,x,v}dw_s - \int_{\mathbb{Z}} p_s^{t,x,v}(z)\tilde{N}_k(ds, dz),$$

$$y_T^{t,x,v} = \Phi(x_T^{t,x,v}), s \in [t, T],$$

where $v_s \in \mathbb{E}$, $\mathbb{E}$ is the admissible control set of all square integrable $\mathfrak{F}_s$ -predictable processes valued in a compact set $U \subset R^m$. Let $u(t, x) = \sup_{v(\cdot)} y_s^{t,x,v} |_{s=t}$. Introduce the Hamiltonian function $H(t, x, r, \beta, A, \eta) : [0, T] \times R^d \times R^1 \times R^d \times R^{d \otimes d} \times C_b^1(R^d) \mapsto R^1$, as

$$H(t, x, r, \beta, A, \eta) = \sup_{v \in U} \left\{ \begin{aligned} &f(t, x, r, \sigma^*(t, x, v)\beta, \eta(x + c(t, x, v, \cdot)) - \eta(x), v) \\ &+ \langle \beta, b(t, x, v) \rangle + \frac{1}{2}Tr.(\sigma(t, x, v)\sigma^*(t, x, v)A) \\ &+ \int_{\mathbb{Z}} [\eta(x + c(t, x, v, z)) - \eta(x) - \eta'(x) \cdot c(t, x, v, z)]\pi(dz) \end{aligned} \right\}$$

Then by means of our comparison theorem of solutions to the BSDE with jumps under appropriate condition we show that $u(t, x)$ is a viscosity solution of the following H-J-B equation:

$$\partial_t u(t, x) + H(t, x, u(t, x), Du(t, x), D^2 u(t, x), u(t, \cdot)) = 0.$$

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1 COMPARISON THEOREMS OF SOLUTIONS TO BSDE WITH JUMPS

Backward stochastic differential equations (BSDE) have important applications in the financial market and optimal stochastic control.^{[2],[6],[10],[11]} The comparison theorem of solutions to BSDE is a powerful tool to deal with such kind of problems.^{[2],[6],[11]} In this paper we shall investigate the case with jumps. Consider the following two BSDE: $i = 1, 2, t \in [0, T]$

$$y_t^i = Y^i + \int_t^T b^i(s, y_s^i, q_s^i, p_s^i, \omega) ds - \int_t^T q_s^i dw_s - \int_t^T \int_Z p_s^i(z) \tilde{N}_k(ds, dz), \quad (1)$$

where w_t is a r -dimensional standard Brownian Motion (BM), $k(\cdot)$ is a Poisson point process, $\tilde{N}_k(ds, dz)$ is the Poisson martingale measure generated by $k(\cdot)$ satisfying

$$\tilde{N}_k(ds, dz) = N_k(ds, dz) - \pi(dz)ds,$$

where $\pi(\cdot)$ is a σ -finite measure on a measurable space $(Z, \mathfrak{R}(Z))$, $N_k(ds, dz)$ is the Poisson counting measure generated by $k(\cdot)$, Y^i is $\mathfrak{F}_T$ -measurable, and $\mathfrak{F}_t = \sigma(w_s, k_s, s \leq t)$ (means the completion of the σ -algebra).

For the precise definition of solution (1) we need the following denotation.

$$L_{\mathfrak{F}_t}^2(R^1) = \left\{ \begin{array}{l} f(t, \omega) : f(t, \omega) \text{ is } \mathfrak{F}_t\text{-adapted, } R^1\text{-valued such that} \\ E \int_0^T |f(t, \omega)|^2 dt < \infty \end{array} \right\};$$

$L_{\mathfrak{F}_t}^2(R^{1 \otimes r})$ is defined similarly.

$$F_{\mathfrak{F}_t}^2(R^1) = \left\{ \begin{array}{l} f(t, z, \omega) : f(t, z, \omega) \text{ is } R^1\text{-valued, } \mathfrak{F}_t\text{-predictable} \\ \text{such that } E \int_0^T \int_Z |f(t, z, \omega)|^2 \pi(dz) dt < \infty \end{array} \right\}.$$

Definition 1. (y_t, q_t, p_t) is said to be a solution of (1), say, for $i = 1$, iff $(x_t, q_t, p_t) \in L_{\mathfrak{F}_t}^2(R^1) \times L_{\mathfrak{F}_t}^2(R^{1 \otimes r}) \times F_{\mathfrak{F}_t}^2(R^1)$, and it satisfies (1) for $i = 1$.

From definition 1 it is seen that for discussing the solution of (1) we need to assume that b satisfies the following assumption

(A) $b : [0, T] \times R^1 \times R^{1 \otimes r} \times L_{\pi(\cdot)}^2(R^1) \times \Omega \rightarrow R^1$ is jointly measurable, $\mathfrak{F}_t$ -adapted, where

$$L_{\pi(\cdot)}^2(R^1) = \left\{ \begin{array}{l} f(z) : f(z) \text{ is } R^1\text{-valued, and} \\ \|f\|^2 = \int_Z |f(z)|^2 \pi(dz) < \infty \end{array} \right\}.$$

We have the following comparison theorem for solutions to (1).

Theorem 1. Assume that

$$1^\circ b^1(t, y, q, p, \omega) \geq b^2(t, y, q, p, \omega)$$

$$2^\circ b^1(t, y, q, p, \omega) = \beta(t, y, q, \omega) + \int_Z C_t(z, \omega) p(z) \pi(dz),$$

$$|\beta(t, y_1, q_1, \omega) - \beta(t, y_2, q_2, \omega)| \leq k_0 (|y_1 - y_2| + |q_1 - q_2|),$$

where $k_0 \geq 0$ is a constant, and $C_t(z, \omega)$ satisfies the condition

$$(B) C_t(z) \geq -1, C_t(z) \in F_{\mathfrak{F}_t}^2(R^1),$$

$$3^\circ Y^i \in \mathfrak{F}_T, E|Y^i|^2 < \infty, i = 1, 2, \text{ and } Y^1 \geq Y^2.$$

Then $P - a.s.$

$$y_t^1 \geq y_t^2, \text{ for all } t \in [0, T].$$

Theorem 1 can be shown by using the following lemma, which is given in ^[8].

Lemma 1. Under assumption in Theorem 1 and additional assumption that

$$\int_Z \left| 1 - \sqrt{1 + C_t(z)} \right|^2 \pi(dz) dt \leq k_0;$$

the conclusion of Theorem 1 is true.

Proof.

Take $Z_m \uparrow Z$ such that $\pi(Z_m) < \infty$. Denote

$$\tilde{N}_k^{(m)}(dt, dz) = \tilde{N}_k(dt, dz \cap Z_m), \text{ etc.}$$

Then for each $m = 1, 2, \dots$ there exists a unique solution (x_t^m, q_t^m, p_t^m) satisfying the following BSDE [9], [10]

$$\begin{aligned} y_t^m &= E(Y | \mathfrak{S}_T^{w, \tilde{N}_k^{(m)}}) + \int_t^T (a_s(\omega)y_s^m + b_s(\omega)q_s^{m*} + \int_Z C_s(z)p_s^m(z)\pi^{(m)}(dz) \\ &\quad + f_0(s, \omega))ds - \int_t^T q_s^m dw_s - \int_t^T \int_Z p_s^m \tilde{N}_k^{(m)}(ds, dz) \\ &= E(Y | \mathfrak{S}_T^{w, \tilde{N}_k^{(m)}}) + \int_t^T (a_s(\omega)y_s^m + b_s(\omega)q_s^{m*} + \int_{Z_m} C_s(z)p_s^m(z)\pi(dz) \\ &\quad + f_0(s, \omega))ds - \int_t^T q_s^m dw_s - \int_t^T \int_{Z_m} p_s^m \tilde{N}_k^{(m)}(ds, dz), \end{aligned}$$

where $Y = Y_1 - Y_2$, q_s^{m*} means the transpose of q_s^m ,

$$a_t(\omega) = I_{(y_t^1 \neq y_t^2)}(\tilde{\beta}(t, y_t^1, q_t^1, \omega) - \tilde{\beta}(t, y_t^2, q_t^1, \omega))(y_t^1 - y_t^2)^{-1},$$

$$b_{it}(\omega) = I_{(q_{it}^1 \neq q_{it}^2)}(\tilde{\beta}(t, y_t^2, \tilde{q}_{i-1,t}^2, \omega) - \tilde{\beta}(t, y_t^2, \tilde{q}_{i,t}^2, \omega))(q_{it}^1 - q_{it}^2)^{-1}, i = 1, \dots, r,$$

$$f_0(t, \omega) = b^1(t, y_t^2, q_t^2, p_t^2, \omega) - b^2(t, y_t^2, q_t^2, p_t^2, \omega) \geq 0,$$

where $\tilde{q}_t^2 = (q_{1t}^2, \dots, q_{it}^2, q_{(i+1)t}^1, \dots, q_{rt}^1)$, $q_t^2 = (q_{1t}^2, \dots, q_{rt}^2)$, etc.

Since $(\hat{y}_t, \hat{q}_t, \hat{p}_t) = (y_t^1 - y_t^2, q_t^1 - q_t^2, p_t^1 - p_t^2)$ satisfies a similar BSDE as follows:

$$\hat{y}_t = Y + \int_t^T (a_s(\omega)\hat{y}_s + b_s(\omega)\hat{q}_s + \int_Z C_s(z)\hat{p}_s\pi(dz) + f_0(s, \omega))ds - \int_t^T \hat{q}_s dw_s - \int_t^T \int_Z \hat{p}_s \tilde{N}_k^{(m)}(ds, dz).$$

Hence by Ito's formula to $|y_t^m - \hat{y}_t|^2$ and Gronwall's inequality one has that

$$\begin{aligned} E(|y_t^m - \hat{y}_t|^2 + \frac{1}{2} \int_t^T (|q_s^m - q_s|^2 + \int_Z |p_s^m I_{Z_m}(z) - p_s|^2 \pi(dz))ds \\ \leq \tilde{k}_0 E \left| E(Y | \mathfrak{S}_T^{w, \tilde{N}_k^{(m)}}) - Y \right|^2. \end{aligned}$$

Since

$$\mathfrak{S}_T^{w, \tilde{N}_k^{(m)}} \subset \mathfrak{S}_T^{w, \tilde{N}_k^{(m+1)}},$$

and

$$\mathfrak{S}_T^{w, \tilde{N}_k} = \vee_m \mathfrak{S}_T^{w, \tilde{N}_k^{(m)}}.$$

Hence by Levy's theorem (Ikeda^[4], Theorem 1.6.6, pp31) as $m \rightarrow \infty$,

$$E(Y | \mathfrak{S}_T^{w, \tilde{N}_k^{(m)}}) \rightarrow Y, \text{ a.s.,}$$

and $\left\{ E(|Y|^2 | \mathfrak{S}_T^{w, \tilde{N}_k^{(m)}}) \right\}_{m=1}^\infty$ is uniformly integrable. Hence as $m \rightarrow \infty$

$$E \left| E(Y | \mathfrak{S}_T^{w, \tilde{N}_k^{(m)}}) - Y \right|^2 \rightarrow 0.$$

Therefore as $m \rightarrow \infty$

$$E|y_t^m - \hat{y}_t|^2 \rightarrow 0.$$

However, by Lemma 1 it is true that as $t \in [0, T]$

$$y_t^m \geq 0, \forall m.$$

Hence $P - \text{a.s.}$

$$\hat{y}_t \geq 0, \text{ for all } t \in [0, T].$$

□

Sometimes we need to consider the following BSDE with jumps, for example, in the application to the financial market^[1], which is a little bit different from (1): $i = 1, 2$,

$y_t^i = Y^i + \int_t^T b^i(s, y_s^i, q_s^i, p_s^i, \omega) ds - \int_t^T q_s^i dw_s - \int_t^T p_s^i d\tilde{N}_k(s)$, $0 \leq t \leq T$, (1)', where $b^i : [0, T] \times R^1 \times R^{1 \otimes r} \times R^1 \times \Omega \rightarrow R^1$ is jointly measurable, $\mathfrak{F}_t$ -adapted, $\tilde{N}_k(t)$ is a centralized Poisson process, such that

$$d\tilde{N}_k(t) = \tilde{N}_k(dt, Z),$$

and we assume that $\pi(Z) = 1$, i.e.

$$\tilde{N}_k(t) = N_k(t) - t,$$

$N_k(t)$ is a Poisson process with $EN_k(t) = t$. In this case we can even remove the condition (B) in 2° of Theorem 1 as follows:

Theorem 1'. Assume that

$$1^\circ b^1(t, y, q, p, \omega) \geq b^2(t, y, q, p, \omega),$$

$$2^\circ |b^1(t, y_1, q_1, p_1, \omega) - b^1(t, y_2, q_2, p_2, \omega)| \leq k_0 (|y_1 - y_2| + |q_1 - q_2| + |p_1 - p_2|).$$

$$3^\circ Y^i \in \mathfrak{F}_T, E|Y^i|^2 < \infty, i = 1, 2, \text{ and } Y^1 \geq Y^2.$$

Then $P - a.s.$

$$y_t^1 \geq y_t^2, \text{ for all } t \in [0, T].$$

To show Theorem 1' we need the following preparation. Consider the following simpler RSDE:

$$y_t = Y + \int_t^T (a_s(\omega)y_s + b_s(\omega)q_s^T + c_s(\omega)p_s + f_0(\omega)) ds - \int_t^T q_s dw_s - \int_t^T p_s d\tilde{N}_k(s),$$

$$0 \leq t \leq T,$$

where we assume that a_t, b_t are $\mathfrak{F}_t$ -adapted, c_t is $\mathfrak{F}_t$ -predictable, and they are all bounded, i.e. there exists a constant $k_0 \geq 0$ such that

$$|a_t| + |b_t| + |c_t| \leq k_0.$$

Proposition 1. If

$$f_0(\omega), Y \geq 0,$$

then $P - a.s.$

$$y_t \geq 0, \text{ for all } t \in [0, T].$$

To show Proposition 1 let us use the Girsanov measure transformation and to make an additional assumption that

$$(B_1) : c_t > -1.$$

The general case can be reduced to this simple case by using finite steps of induction. Let

$$z_t = \exp\left(\int_0^t b_s \cdot dw_s - \frac{1}{2} \int_0^t |b_s|^2 ds\right) \cdot \prod_{0 < s \leq t} (1 + c_s \Delta N_s) e^{\int_0^t c_s d(\tilde{N}_s - N_s)}.$$

Then under addition condition (B_1) the following formula defines a new probability measure:

$$d\tilde{P} = z_T dP.$$

By Ito's formula it is not difficult to show that

$$y_t = E_{\tilde{P}} \left[Y e^{\int_t^T a_s ds} + \int_t^T e^{\int_t^u a_s ds} f_0(u) du \mid \mathfrak{F}_t \right].$$

Hence the conclusion of Proposition 1 is derived. For the general case $c_t > -1 - k_0$ we can write

$$c_t = \sum_{i=1}^{n_0} c_i(t)$$

such that all $c_i(t) > -1$. Then through finite steps of transformation we can arrive at the conclusion. Now Theorem 1' is not difficult to derive from Proposition 1 as the proof of Theorem 1.

2 OPTIMAL CONTROL AND BELLMAN PRINCIPLE

Now we will apply the comparison theorem of solutions to BSDE (Theorem 1) to discuss the optimal control problem for the BSDE systems with jumps, which is corresponding to some optimal recursive utility function problem in the financial market.^{[2],[11]} Consider the following controlled Forward SDE (FSDE):

$$\begin{aligned} dx_s^{t,x,v} &= b(s, x_s^{t,x,v}, v_s)ds + \sigma(s, x_s^{t,x,v}, v_s)dw_s \\ &+ \int_Z c(s, x_s^{t,x,v}, v_s, z)\tilde{N}_k(ds, dz), \quad x_t^{t,x,v} = x \in R^d, \quad s \in [t, T]; \quad (2) \end{aligned}$$

where $b(s, x, v)$ is non-random, etc., and $v(\cdot) \in \mathbb{E}$, $\mathbb{E}$ is the admissible control set of all square integrable $\mathfrak{F}_s$ -predictable processes valued in a compact set $U \subset R^m$; i.e.

$$\mathbb{E} = \left\{ v = v(t, \omega) : \text{it is } \mathfrak{F}_t - \text{predictable, } v(t, \omega) \in U \text{ such that } \int_0^T |v(t, \omega)|^2 dt < \infty \right\}.$$

It is well known that FSDE (2) has a unique solution $x_s^{t,x,v} \in R^d, s \in [t, T]$, provided that the following conditions are satisfied:^[7]

$$(H1) \quad \begin{cases} |b(t, x, v)|^2 + |\sigma(t, x, v)|^2 + \int_Z |c(t, x, v, z)|^2 \pi(dz) \\ \leq k_0(1 + |x|^2 + |v|^2), \\ |b(t, x, v) - b(t, x', v')|^2 + |\sigma(t, x, v) - \sigma(t, x', v')|^2 \\ + \int_Z |c(t, x, v, z) - c(t, x', v', z)|^2 \pi(dz) \\ \leq k_0(|x - x'|^2 + |v - v'|^{2\alpha}), \quad \alpha \in (0, 1], \end{cases}$$

where $b \in R^d, \sigma \in R^{d \otimes r}, c \in R^d$. Now consider the following BSDE:

$$\begin{aligned} -dy_s^{t,x,v} &= f(s, x_s^{t,x,v}, y_s^{t,x,v}, q_s^{t,x,v}, p_s^{t,x,v}, v_s)ds - q_s^{t,x,v}dw_s \\ &- \int_Z p_s^{t,x,v}(z)\tilde{N}_k(ds, dz), \quad y_T^{t,x,v} = \Phi(x_T^{t,x,v}), \quad s \in [t, T]. \end{aligned} \quad (3)$$

It is also known that (3) has a unique solution $(y_s^{t,x,v}, q_s^{t,x,v}, p_s^{t,x,v})$, provided that the following conditions are satisfied:^{[9],[10]}

$$(H2) \quad \begin{cases} |f(t, x, 0, 0, 0, v)|^2 + |\Phi(x)|^2 \leq k_0(1 + |x|^2 + |v|^2), \\ |f(t, x, y, q, p, v) - f(t, x', y', q', p', v')|^2 + |\Phi(x) - \Phi(x')|^2 \\ \leq k_0(|x - x'|^{2\alpha} + |y - y'|^2 + |q - q'|^2 + \|p - p'\|^2 + |v - v'|^{2\alpha}), \\ \alpha \in (0, 1], \end{cases}$$

where $f, \Phi \in R^1$. Let

$$\begin{aligned} J(t, x, v(\cdot)) &= y_s^{t,x,v} |_{s=t}, \\ u(t, x) &= \sup_{v(\cdot) \in \mathbb{E}} J(t, x, v(\cdot)). \end{aligned} \quad (4)$$

In the financial market (2), (3) and (4) can be interpreted as the price processes of stocks, the wealth process of an investor, and the optimal recursive utility function, respectively. Our object here is to discuss (4). Let us denote the solution of (3) by

$$y_s^{t,x,v} = G_{s,T}^{t,x,v}(\Phi(x_T^{t,x,v})),$$

and introduce the following assumption:

(H3) $f(t, x, y, q, p, v)$ satisfies the condition 2° in Theorem 1 for each $v \in U, x \in R^d$.

We have

Theorem 2. (Bellman Principle). Under assumption (H1) - (H3) $\forall 0 \leq \delta < T - t$

$$u(t, x) = \sup_{v(\cdot) \in \mathbb{E}} G_{t, t+\delta}^{t, x, v}(u(t + \delta, x_{t+\delta}^{t, x, v})).$$

Theorem 2 tells us that to find the optimal value functional on $[t, T]$, we can divide it into two steps: First, find out the optimal value functional $u(t + \delta, x_{t+\delta}^{t, x, v})$ on $[t + \delta, T]$. Second, set $u(t + \delta, x_{t+\delta}^{t, x, v})$ as the terminal value then find out the optimal value functional on $[t, t + \delta]$. Theorem 2 can be proved by using our Comparison theorem of solutions to BSDE with jumps (Theorem 1), the following lemmas and similar approaches as that in [6].

Lemma 2. Under assumption (H1) $\forall x, x' \in R^d, \forall v(\cdot), v'(\cdot) \in \mathbb{E}$,

$$E^{\mathfrak{S}_t} \sup_{t \leq s \leq T} |x_s|^2 \leq k_T < \infty,$$

$$E^{\mathfrak{S}_t} \sup_{t \leq s \leq T} \left| x_s^{t, x, v} - x_s^{t, x', v'} \right|^2 \leq k_T (|x - x'|^2 + E^{\mathfrak{S}_t} \int_t^T |v(s) - v'(s)|^{2\alpha} ds).$$

Lemma 3. Under assumption (H1)-(H2) $\forall x, x' \in R^d, \forall v(\cdot), v'(\cdot) \in \mathbb{E}$,

$$\begin{aligned} & \left| y_t^{t, x, v} \right|^2 + E^{\mathfrak{S}_t} \int_t^T (|q_s^{t, x, v}|^2 + \|p_s^{t, x, v}\|^2) ds \leq k_T (1 + |x|^2), \\ & \left| y_t^{t, x, v} - y_t^{t, x', v'} \right|^2 + E^{\mathfrak{S}_t} \int_t^T (|q_s^{t, x, v} - q_s^{t, x', v'}|^2 + \|p_s^{t, x, v} - p_s^{t, x', v'}\|^2) ds \\ & \leq k_T (|x - x'|^{2\alpha} + (E^{\mathfrak{S}_t} \int_t^T |v_s - v'_s|^{2\alpha} ds)^\alpha + E^{\mathfrak{S}_t} \int_t^T |v_s - v'_s|^{2\alpha} ds). \end{aligned}$$

3 VISCOSITY SOLUTION OF HJB EQUATION

Now introduce the Hamiltonian function $H(t, x, r, \beta, A, \eta) : [0, T] \times R^d \times R^1 \times R^d \times R^{d \otimes d} \times C_b^1(R^d) \mapsto R^1$, as

$$\begin{aligned} & H(t, x, r, \beta, A, \eta) \\ & = \sup_{v \in U} \left\{ \begin{aligned} & f(t, x, r, \sigma^*(t, x, v)\beta, \eta(x + c(t, x, v, \cdot)) - \eta(x), v) \\ & + \langle \beta, b(t, x, v) \rangle + \frac{1}{2} Tr.(\sigma(t, x, v)\sigma^*(t, x, v)A) \\ & + \int_Z [\eta(x + c(t, x, v, z)) - \eta(x) - \eta'(x) \cdot c(t, x, v, z)] \pi(dz) \end{aligned} \right\} \quad (5) \end{aligned}$$

By means of our comparison theorem of solutions to the BSDE with jumps under appropriate condition one can show that $u(t, x)$ is a viscosity solution of the following H-J-B equation:

$$\partial_t u(t, x) + H(t, x, u(t, x), Du(t, x), D^2 u(t, x), u(t, \cdot)) = 0. \quad (6)$$

Let us introduce the definition of viscosity solution as follows:

Definition 2. A continuous function $u(t, x)$ is called a viscosity sub-solution (super-solution) of (6), iff $\forall \varphi \in C_b^{1,3}([0, T] \times R^d)$ (the totality of all real functions, which has a bounded continuous first derivative with respect to t , and up to the third derivatives with respect to x) at all minimum point (maximum point) (t, x) of function $\varphi - u$ one has

$$\partial_t \varphi(t, x) + H(t, x, u(t, x), D\varphi(t, x), D^2 \varphi(t, x), \varphi(t, \cdot)) \geq 0 \ (\leq 0).$$

If $u(t, x)$ is both a viscosity super-solution and sub-solution, then it is called a viscosity solution.

Since now actually H is concerned with the jump coefficient c , and we need some α -Hölder continuous of the function in supremum in (5) for x . Hence we put one more assumption

(H4) one of the following conditions is satisfied:

$$(i) |b(s, x, v)| + |\sigma(s, x, v)| + \|c(s, x, \cdot, v)\| + \|c_x(s, x, z, v)\| \leq k_0, \int_Z |c| \pi(dz) \leq k_0;$$

$$(ii) \|c_x(s, x, z, v)\| \leq k_0(1 + |x|), \int_Z |c| \pi(dz) \leq k_0(1 + |x|), \\ E(\int_0^T \int_Z |c(s, x_{s-}(\omega), z, v)|^2 N_k(ds, dz))^p \leq k_0(1 + E \int_0^T |x_s(\omega)|^{2p} ds), p = 1, \dots, 4;$$

for any R^d -valued $\mathfrak{F}_t$ -adapted cadlag process $x_s(\omega)$ holds, where c_x means the gradient of c with respect to x , which is assumed to be existed in both cases.

We have

Theorem 3. Under assumption (H1)-(H4) $u(t, x)$ defined by (4) is a viscosity solution of (6).

To show Theorem 3 we need several lemmas. For a given φ let us denote

$$F(s, x, y, q, p, v) = \partial_s \varphi(s, x) + \mathcal{L}(s, x, v) \varphi(s, x) \\ + f(s, x, y + \varphi(s, x), q + D\varphi(s, x) \sigma(s, x, v), p + \varphi(s, x + c(s, x, \cdot, v)) - \varphi(s, x), v),$$

where

$$\mathcal{L}(s, x, v) = \langle D\varphi, b(t, x, v) \rangle + \frac{1}{2} \text{tr}[\sigma(t, x, v) \sigma^*(t, x, v) D^2 \varphi] \\ + \int_Z (\varphi(s, x + c(s, x, z, v)) - \varphi(s, x) - \langle D\varphi, c(t, x, z, v) \rangle) \pi(dz).$$

For the property of F we have the following

Lemma 4. $F(s, x, y, q, p, v)$ still satisfies the Lipschitzian condition for (y, q, p) , i.e. the second condition in (H2) for (y, q, p) holds. Moreover, in case (i) of (H4)

$$|F(s, x_1, y, q, p, v) - F(s, x_2, y, q, p, v)| \\ \leq k_0(|x_1 - x_2|^\alpha + |x_1 - x_2|), \quad (7)$$

in case (ii) of (H4)

$$|F(s, x_1, y, q, p, v) - F(s, x_2, y, q, p, v)| \\ \leq k_0(1 + \max(|x_1|, |x_2|))(|x_1 - x_2|^\alpha + |x_1 - x_2|), \quad (8)$$

where $\alpha \in (0, 1]$ comes from assumption (H2). Furthermore, in case (ii) of (H4) holds, one has that

$$E \sup_{t \leq T} |x_t|^{2p} \leq k_0, \forall p = 1, \dots, 4. \quad (9)$$

By using Lemma 4, Comparison theorem (Theorem 1), and Theorem 2 one can show theorem 3 similar to [6].

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