OPTIMAL CONTROL OF A CLASS OF STRONGLY NONLINEAR EVOLUTION SYSTEMS *

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Abstract: In this paper, we investigate the existence of optimal controls of the first-order nonlinear evolution systems whose principle operator is pseudomonotone operator with nonmonotone nonlinearities perturbation.

Keywords. Nonmonotone nonlinearities, Pseudomonotone operator, Optimality.

1. Introduction

Recently, optimal control problems of nonlinear distributed parameter systems have been extensively studied. Many authors pay great attention to several class of semilinear and nonlinear evolution equations, such as, Papageorgiou ([1]-[2]), Ahmed ([4]) and Xiang ([5]). Meanwhile, to our knowledge, the class of nonlinear control systems with pseudomonotone operator has rarely been studied.

As well-known, in studying existence of solutions of nonlinear evolution equations, we usually consider the evolution triples ([3]), $X \hookrightarrow H \hookrightarrow X^*$, or more generally, consider the chains of spaces ([1]),

$$X \hookrightarrow Y \hookrightarrow H \hookrightarrow Y^* \hookrightarrow X^*.$$

The nonlinear perturbation is assumed by some authors as following:

$f : I \times Y \rightarrow Y^*([1])$, here I denotes the time horizon, or $f : I \times H \rightarrow H([2])$, or $f : I \times H \rightarrow X^*([5])$. In this paper, by developing the concept of pseudomonotone operator and using some technique of convergence, we deal with the existence of solutions of nonlinear evolution equation whose principle operator is pseudomonotone operator with nonmonotone nonlinearities perturbation being from $I \times Y$ into X^* , which includes the results of [1] and [2]. Further,

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we solve the existence of optimal controls of nonlinear control system.

2. Preliminaries

Let T be a fixed number, and $I = [0, T]$ the time horizon, H a separable Hilbert space, and X, Y subspaces of H carrying the structure of a separable reflexive Banach space. We will assume that $X \hookrightarrow Y \hookrightarrow H$, with all embeddings being continuous, dense, and X into Y is also compact. Identifying H with its dual, we have

$$X \hookrightarrow Y \hookrightarrow H \hookrightarrow Y^* \hookrightarrow X^*,$$

with all embeddings being continuous, dense, and the first and the last are also compact. In the literature, triples like (X, H, X^*) and (Y, H, Y^*) are usually called evolution triples ([3]). For simplicity, we denote by $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ the duality brackets of the pair (X, X^*) and the inner product of H , respectively. They are compatible in the sense that $\langle \cdot, \cdot \rangle|_{X \times H} = (\cdot, \cdot)$ (proposition 23.13, [3]). Also, by $\|\cdot\|$ (resp., $\|\cdot\|_Y, |\cdot|, \|\cdot\|_{Y^*}, \|\cdot\|_{X^*}$), we denote the norm of X (resp., Y, H, Y^*, X^*), and by $\rightarrow^s$ ($\rightarrow^w$), we denote the strongly (weakly) convergence. We set

$$W_{pq} = \{x \in L^p(I, X), \dot{x} \in L^q(I, X^*)\} (1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1),$$

where the derivative in the definition should be understood in the sense of vector-valued distributions, equipped with the norm

$$\|x\|_{W_{pq}} = (\|x\|_{L^p(I, X)}^2 + \|\dot{x}\|_{L^q(I, X^*)}^2)^{\frac{1}{2}}.$$

The space W_{pq} becomes a separable reflexive Banach space. Moreover, $W_{pq} \hookrightarrow L^p(I, X)$ and $W_{pq} \hookrightarrow C(I, H)$ continuously, and $W_{pq} \hookrightarrow L^p(I, Y)$ compactly (pp. 650, [1]). For convenience, we write the spaces

$$V = L^p(I, X), V^* = L^q(I, X^*), \mathcal{H} = L^2(I, H), Z = L^p(I, Y), Z^* = L^q(I, Y^*).$$

By $\|\cdot\|_V$ (resp., $\|\cdot\|_{V^*}, \|\cdot\|_{\mathcal{H}}, \|\cdot\|_Z, \|\cdot\|_{Z^*}$), we denote the norm of the space V (resp., $V^*, \mathcal{H}, Z, Z^*$). Let $\langle \cdot, \cdot \rangle$ and $((\cdot, \cdot))$ denote the duality brackets for the pair (V, V^*) and the inner product of $\mathcal{H}$, respectively.

Next we model the control space by a reflexive separable B-space E with the norm $\|\cdot\|_E$. By $P_{f(c)}(E)$, we denote a class of nonempty, closed (convex) subsets of E . Recall that a multi-function $U : I \rightarrow P_f(E)$ is called measurable if

$$GrU = \{(t, v) \in I \times E : v \in U(t)\} \in B(I) \times B(E),$$

with $B(I)(B(E))$ being the Borel σ -field of $I(E)$. By $\mathcal{U}_{ad}$, we denote the admissible control set of all selectors of $U(\cdot)$ that belong to Lebesgue-Bochner space $L^q(I, E)$, $1 < q < \infty$; that is,

$$\mathcal{U}_{ad} = \{u \in L^q(I, E) : u(t) \in U(t), \text{ a.e. on } I\}.$$

Note that

$$\mathcal{U}_{ad} \neq \emptyset, \text{ if } t \rightarrow |U(t)| = \sup\{\|u\|_E, u \in U(t)\} \in L^q_+,$$

in which case the multi-function is said to be L^q -bounded.

Finally recall that an operator $A : X \rightarrow X^*$ is said to be pseudomonotone if, for each $x \in X$ and each sequence (x_n) in X ,

$$x_n \xrightarrow{w} x \text{ in } X \text{ as } n \rightarrow \infty \text{ and } \overline{\lim}_{n \rightarrow \infty} \langle Ax_n, x_n - x \rangle \leq 0$$

imply

$$\langle Ax, x - w \rangle \leq \lim_{n \rightarrow \infty} \langle Ax_n, x_n - w \rangle, \text{ for all } w \in X.$$

In order to verify the existence of solutions of System (3.1) (See next section), we develop the concept of pseudomonotone.

Definition An operator $A : M \subseteq X \rightarrow X^*$ on the complete subspace M of the real Banach space X is called quasi-pseudomonotone if, for each $x \in M$ and each sequence (x_n) in M ,

$$x_n \xrightarrow{w} x \text{ in } M \text{ as } n \rightarrow \infty \text{ and } \overline{\lim}_{n \rightarrow \infty} \langle Ax_n, x_n - x \rangle \leq 0,$$

there exists a subsequence (x_{n_k}) of (x_n) , s.t.

$$\langle Ax, x - w \rangle \leq \lim_{k \rightarrow \infty} \langle Ax_{n_k}, x_{n_k} - w \rangle, \text{ for all } w \in X.$$

3. Existence of Solutions of Nonlinear Evolution Equation

We consider the following initial value problem:

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = f(t, x(t)), & \text{a.e. on } I \\ x(0) = x_0 \end{cases} \quad (3.1)$$

Recall that a function $x \in W_{pq}$ is a solution of (3.1) if, for all $v \in X$, x satisfies

$$\begin{cases} \frac{d}{dt}(x(t), v) + \langle A(t, x(t)), v \rangle = \langle f(t, x(t)), v \rangle, & \text{a.e. on } I \\ x(0) = x_0 \end{cases}$$

where $\dot{x}$ means the generalized distributed derivatives.

In order to investigate the existence of solutions of (3.1), we need the following hypotheses.

Hypothesis (A). $A : I \times X \rightarrow X^*$ is an operator s.t.

- (a) $t \rightarrow A(t, x)$ is measurable;
- (b) $x \rightarrow A(t, x)$ is pseudomonotone;
- (c) $\|A(t, x)\|_* \leq a_1(t) + b_1\|x\|^{p-1}$,
for all $x \in X$, with $a_1 \in L_+^q(0, T)$, $b_1 > 0$, $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$;
- (d) $\langle A(t, x) - A(t, y), x - y \rangle \geq c\|x - y\|^p - d\|x - y\|^2$,
for all $x, y \in X$, with $c > 0$ and $d \geq 0$.

Hypothesis (F). $f : I \times Y \rightarrow X^*$ is also an operator s.t.

- (a) $t \rightarrow f(t, x)$ is measurable;
- (b) $x \rightarrow f(t, x)$ is continuous from Y into X^* ;
- (c) $\|f(t, x)\|_* \leq a_2(t) + b_2\|x\|_Y^{p-1}$,
for all $x \in Y$, with $a_2 \in L_+^q(0, T)$ and $b_2 > 0$;
- (d) $\langle f(t, x), x \rangle \leq \phi(t)$, for all $x \in X$, with $\phi \in L^1(0, T)$.

Hypothesis (H₀). $x_0 \in H$.

Next we define the Nemyckii operators $\hat{A}$ and $\hat{f}$, given by

$$(\hat{A}x)(t) = A(t, x(t)), \text{ for } x \in V; (\hat{f}x)(t) = f(t, x(t)), \text{ for } x \in Z.$$

Let $\hat{A}|_{W_{pq}}$ denote the largest restriction of $\hat{A}$ on W_{pq} . We have some important properties on the operators $\hat{A}$, $\hat{f}$ and $\hat{A}|_{W_{pq}}$.

Lemma 1. Under Hypothesis (A), $\hat{A} : V \rightarrow V^*$ is bounded and satisfies

- (a) $\|\hat{A}x\|_{V^*} \leq \text{const}(\|a_1\|_q + b_1\|x\|_V^{p-1})$, for all $x \in V$;
 (b) $\langle\langle \hat{A}x - \hat{A}y, x - y \rangle\rangle \geq c\|x - y\|_V^p - d\|x - y\|_{\mathcal{H}}^2$,
 for all $x, y \in V$;
 (c) (i) $\hat{A}|_{W_{pq}}$ is quasi-pseudomonotone from $W_{pq} \subseteq V$ into V^* .
 (ii) $\hat{A}|_{W_{pq}}$ satisfies the condition(M), i.e.
 $x_n \xrightarrow{w} x$ in W_{pq} , $\hat{A}x_n \xrightarrow{w} b$ in V^* , and $\overline{\lim}_{n \rightarrow \infty} \langle\langle \hat{A}x_n, x_n \rangle\rangle \leq \langle\langle b, x \rangle\rangle$,
 imply
 $\hat{A}x = b$.

Proof. One can easily verify that (a) and (b) hold true. Now we show (c)(i).

Let $x_n \xrightarrow{w} x$ in W_{pq} and $\overline{\lim}_{n \rightarrow \infty} \langle\langle \hat{A}x_n, x_n - x \rangle\rangle \leq 0$.

Thanks to (b), we have

$$\begin{aligned} c \overline{\lim}_{n \rightarrow \infty} \|x_n - x\|_V^p &\leq d \overline{\lim}_{n \rightarrow \infty} \|x_n - x\|_{\mathcal{H}}^2 \\ &+ \overline{\lim}_{n \rightarrow \infty} \langle\langle \hat{A}x_n, x_n - x \rangle\rangle \\ &- \lim_{n \rightarrow \infty} \langle\langle \hat{A}x, x_n - x \rangle\rangle \leq 0. \end{aligned}$$

Since $W_{pq} \hookrightarrow \mathcal{H} = L^2(I, H)$ compactly, we get that $x_n \xrightarrow{s} x$ in $\mathcal{H}$. Hence, it follows that $x_n \xrightarrow{s} x$ in $V = L^p(I, X)$. Thus, there exists a subsequence of (x_n) , denoted by (x_{n_k}) , such that $x_{n_k}(t) \xrightarrow{s} x(t)$ in X , a.e. on I , and there exists a real function $v \in L^p(0, T)$ satisfying

$$\|x_{n_k}(t)\| \leq v(t), \text{ for a.e. } t \in I$$

(See pp. 579, [3]).

By virtue of Hypothesis (A)(c), we can show that

$$\lim_{k \rightarrow \infty} \langle A(t, x_{n_k}(t)), x_{n_k}(t) - x(t) \rangle = 0, \text{ a.e. on } I.$$

Since A is pseudomonotone, we have

$$\langle A(t, x(t)), x(t) - w(t) \rangle \leq \underline{\lim}_{k \rightarrow \infty} \langle A(t, x_{n_k}(t)), x_{n_k}(t) - w(t) \rangle dt, \\ \text{a.e. on } I, \text{ for all } w \in V. \text{ And consequently,}$$

$$\langle\langle \hat{A}x, x - w \rangle\rangle \leq \int_I \underline{\lim}_{k \rightarrow \infty} \langle A(t, x_{n_k}(t)), x_{n_k}(t) - w(t) \rangle dt.$$

Invoking Hypothesis (A) and Theorem 7 of [7], we obtain that

$$\begin{aligned} \int_I \underline{\lim}_{k \rightarrow \infty} \langle A(t, x_{n_k}(t)), x_{n_k}(t) - w(t) \rangle dt \\ \leq \underline{\lim}_{k \rightarrow \infty} \int_I \langle A(t, x_{n_k}(t)), x_{n_k}(t) - w(t) \rangle dt. \end{aligned}$$

Hence,

$$\langle\langle \hat{A}x, x - w \rangle\rangle \leq \underline{\lim}_{k \rightarrow \infty} \langle\langle \hat{A}x_{n_k}, x_{n_k} - w \rangle\rangle.$$

Next we show that $\hat{A}|_{W_{pq}}$ satisfies the condition(M). Let $x_n \xrightarrow{w} x$ in W_{pq} , $\hat{A}x_n \xrightarrow{w} b$ in V^* and $\overline{\lim}_{n \rightarrow \infty} \langle\langle \hat{A}x_n, x_n \rangle\rangle \leq \langle\langle b, x \rangle\rangle$.

Due to (c)(i) and

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \langle\langle \hat{A}x_n, x_n - x \rangle\rangle &\leq \overline{\lim}_{n \rightarrow \infty} \langle\langle \hat{A}x_n, x_n \rangle\rangle \\ &- \lim_{n \rightarrow \infty} \langle\langle \hat{A}x_n, x \rangle\rangle \leq 0, \end{aligned}$$

there exists a subsequence of (x_n) , denoted by (x_{n_k}) , such that

$$\langle\langle \hat{A}x, x - w \rangle\rangle \leq \underline{\lim}_{k \rightarrow \infty} \langle\langle \hat{A}x_{n_k}, x_{n_k} - w \rangle\rangle$$

$$\begin{aligned} & \leq \overline{\lim}_{k \rightarrow \infty} \langle \hat{A}x_{n_k}, x_{n_k} \rangle - \lim_{k \rightarrow \infty} \langle \hat{A}x_{n_k}, w \rangle \\ & \leq \langle b, x - w \rangle, \end{aligned}$$

for all $w \in V$. Thus,

$$\hat{A}x = b.$$

Similar to the proofs of Theorem 23.A and Lemma 2.5.1 of [4], one can verify that the following Lemmas.

Lemma 2. Under Hypothesis (F), $\hat{f} : Z \rightarrow V^*$ is bounded and satisfies

- (a) $\|\hat{f}x\|_{V^*} \leq \text{const}(\|a_2\|_q + b_2\|x\|_Z^{p-1})$;
- (b) $\langle \hat{f}x, x \rangle \leq \|\phi\|_1$, for all $x \in V$;
- (c) $\hat{f} : Z \rightarrow V^*$ is continuous.

Lemma 3. If Hypotheses (A), (F), (H_0) are satisfied, and $x \in W_{pq}$ is a solution of (3.1), then there exist positive constants M_1, M_2, M_3 , such that

$$\begin{aligned} \max_{t \in I} |x(t)| & \leq M_1; \\ \|x\|_{W_{pq}} & \leq M_2; \\ \|\hat{A}x\|_{V^*} & \leq M_3. \end{aligned}$$

Note that (3.1) is equivalent to the following operator equation

$$\begin{cases} \dot{x} + \hat{A}x = \hat{f}x \\ x(0) = x_0 \end{cases} \quad (3.2)$$

(See pp. 767-770, [3]).

Theorem 1. Suppose that Hypotheses of (A), (F), and (H_0) hold, then the problem (3.1) admits a solution $x \in W_{pq} \subseteq C(I, H)$.

Proof. Let $\{w_1, w_2, \dots, w_n, \dots\}$ be a basis in X . We set $X_n = \text{span}\{w_1, w_2, \dots, w_n\}$, and introduce on the n -dimensional space X_n the scalar product of the H -space H . Note that $X_n \subseteq X \subseteq H$.

Suppose that $x_{n_0} \xrightarrow{s} x_0$ in H . The Galerkin equation of (3.1) can be constructed by

$$\begin{cases} \langle \dot{x}_n(t), w_j \rangle + \langle A(t, x_n(t)), w_j \rangle = \langle f(t, x_n(t)), w_j \rangle, & \text{a.e. on } I \\ x_n(0) = x_{n_0}, & j = 1, 2, \dots, n \\ w_j \in X_n, x_n \in X_n \end{cases} \quad (3.3)$$

By using Lemma 3, Hypotheses (A)(a)-(d), (F)(a)-(c), Proposition 27.7 of [3] and Carathéodory theorem ([7]), the Galerkin equation (3.3) must have a solution x_n ($n=1, 2, \dots$).

By Lemma 3, we know that there exists $M^* > 0$ s.t.

$$\max_{t \in I} |x_n(t)| + \|x_n\|_{W_{pq}} + \|\hat{A}x_n\|_{V^*} \leq M^*.$$

Hence, we can find a subsequence, again denoted by $\{x_n\}$, such that $x_n \xrightarrow{w} x$ in W_{pq} , $\hat{A}x_n \xrightarrow{w} b$ in V^* , and $x_n(T) \xrightarrow{w} z$ in H . Using the same procedure of the proof of Lemma 30.4 of [3], we can verify that the limit elements x, w , and z satisfy

$$\begin{cases} \dot{x} + b = \hat{f}x \\ x(0) = x_0, \quad x(T) = z, \quad x \in W_{pq} \end{cases}$$

It follows from the Galerkin equation that

$$\langle \hat{A}x_n, x_n \rangle = \langle \hat{f}x_n, x_n \rangle - \frac{1}{2}(|x_n(T)|^2 - |x_{n0}|^2).$$

Since $x_n(T) \xrightarrow{w} x(T)$ in H , we have

$$|x(T)| \leq \liminf_{n \rightarrow \infty} |x_n(T)|.$$

Hence

$$\overline{\lim}_{n \rightarrow \infty} \langle \hat{A}x_n, x_n \rangle \leq \langle b, x \rangle.$$

Using Lemma 1, we assert that

$$\hat{A}x = b.$$

This shows that x is a solution of (3.2), and hence a solution of (3.1).

4. Existence of Admissible Trajectories and Optimal Controls

We now consider the following controlled evolution system:

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = f(t, x(t), u(t)), & a.e. on I \\ x(0) = x_0, & u \in \mathcal{U}_{ad} \end{cases}$$

We impose some hypotheses.

Hypothesis (F*). $f : I \times Y \times E \rightarrow X^*$ satisfies:

- (a) $(t, x, u) \rightarrow f(t, x, u)$ is measurable;
- (b) $(x, u) \rightarrow f(t, x, u)$ is continuous from $Y \times E_w$ into X^* ,

here E_w denotes the space E with its weak topology;

- (c) $\|f(t, x, u)\|_* \leq a_3(t) + b_3(\|x\|_Y^{p-1} + \|u\|_E)$,
for all $x \in Y$ and $u \in E$, with $a_3 \in L_+^q(0, T)$ and $b_3 > 0$;

- (d) $\langle f(t, x, u), x \rangle \leq \tilde{\phi}(t) + k\|u\|_E^{\frac{p}{p-1}}$,

for all $x \in X$ and $u \in E$ with $\tilde{\phi} \in L^1(0, T)$ and $k \in R$.

Hypothesis (U). $U : I \rightarrow P_f(E)$ is L^q -bounded multifunction.

By virtue of Theorem 1, we can conclude the existence of admissible trajectories of (4.1).

Theorem 2. Assume that Hypotheses (A), (F*), (H₀), and (U) hold, then the problem (4.1) admits a solution corresponding to each date pair $(x_0, u) \in H \times \mathcal{U}_{ad}$.

By Hypothesis (U), $\mathcal{U}_{ad}$ is a bounded subset in $L^q(I, E)$. Set $P(x_0) = \{x_u \in W_{pq}, x_u \text{ is a solution of (4.1) corresponding to } u \in \mathcal{U}_{ad} \text{ and } x_0 \in H\}$, and $K(T) = \{y = x(T) : x \in P(x_0)\}$. Utilizing the similar method of Lemma 3, we obtain

Lemma 5. Under Hypotheses of Theorem 2, there exists positive constants M_1, M_2 , and M_3 , such that for all $x \in P(x_0)$,

$$\begin{aligned} \max_{t \in I} |x(t)| &\leq M_1; \\ \|x\|_{W_{pq}} &\leq M_2. \end{aligned}$$

Consider the optimal control problem (P):

$$J(x, u) = \phi_0(x(T)) + \int_0^T L(t, x(t), u(t)) dt \longrightarrow \inf$$

$$\begin{cases} \text{s.t.} \\ \dot{x}(t) + A(t, x(t)) = f(t, x(t), u(t)), & a.e. on I \\ x(0) = x_0, & u \in \mathcal{U}_{ad} \end{cases}$$

Suppose

Hypothesis (U*). $U : I \rightarrow P_{fc}(E)$ is L^q -bounded multi-function.

Hypothesis (F).** $f(\cdot, x_n(\cdot), u_n(\cdot)) \xrightarrow{s} f(\cdot, x(\cdot), u(\cdot))$ in V^* as $x_n \xrightarrow{s} x$ in $Z = L^p(I, Y)$ and $u_n \xrightarrow{w} u$ in $L^q(I, E)$.

Hypothesis (L). $L : I \times X \times E \rightarrow R \cup \{+\infty\}$ s.t.

- (a) $(t, x, u) \rightarrow L(t, x, u)$ is measurable;
- (b) $(x, u) \rightarrow L(t, x, u)$ is l.s.c.;
- (c) $u \rightarrow L(t, x, u)$ is convex;
- (d) $\phi(t) - c(\|x\| + \|u\|_E) \leq L(t, x, u)$, a.e. on I ,

with $\phi \in L^1(0, T)$ and $c \geq 0$.

Hypothesis (P). $P \equiv K(T) \cap Q \neq \emptyset$, where Q denotes a weakly closed subset of H -space H .

Hypothesis (Φ_0). $\phi_0 : P \subseteq H \rightarrow R$ is w.l.s.c.

Hypothesis (J). There exists an admissible state-control pair (x, u) such that

$$J(x, u) < +\infty.$$

Theorem 3. If Hypotheses (A), (F*), (F**), (H₀), (U*), (L), (P), (Φ_0), and (J) hold, then the problem (P) has an optimal pair (x, u) .

Proof Suppose that (x_n, u_n) is the minimizing sequence, then it satisfies the following initial value problem:

$$\begin{cases} \dot{x}_n(t) + A(t, x_n(t)) = f(t, x_n(t), u_n(t)) \\ x_n(0) = x_0 \end{cases} \quad (4.2)$$

Recalling Lemma 5 and boundedness of U_{ad} in $L^q(I, E)$, by passing to a subsequence, if necessary, we may assume that $x_n \xrightarrow{w} x$ in W_{pq} , $\hat{A}x_n \xrightarrow{w} b$ in V^* , $x_n(T) \xrightarrow{w} z \in Q$ in H , and $u_n \xrightarrow{w} u$ in $L^q(I, E)$. By virtue of Mazur Lemma and Hypothesis (U*), we have

$$u(t) \in \overline{co} w - \lim_n \sup \{u_n(t)\}_{n \geq 1} \subset U(t), \text{ a.e. on } I.$$

Hence, $u \in U_{ad}$. As in the proof of Theorem 1, we can prove that x satisfies

$$\begin{cases} \dot{x} + \hat{A}x = f(\cdot, x(\cdot), u(\cdot)) \\ x(0) = x_0 \\ x(T) = z \in Q \end{cases}$$

Therefore, $x(T) \in P \subseteq H$. Next, it follows from (4.2) that $\langle \dot{x}_n - \dot{x}, x_n - x \rangle + \langle \hat{A}x_n - \hat{A}x, x_n - x \rangle = \langle f(\cdot, x_n(\cdot), u_n(\cdot)) - f(\cdot, x(\cdot), u(\cdot)), x_n(\cdot) - x(\cdot) \rangle$.

By Lemma 1, we obtain that

$$c\|x_n - x\|_V^p \leq d\|x_n - x\|_H^2 + \langle f(\cdot, x_n(\cdot), u_n(\cdot)) - f(\cdot, x(\cdot), u(\cdot)), x_n(\cdot) - x(\cdot) \rangle.$$

Therefore, by Hypothesis (F**), $W_{pq} \hookrightarrow L^p(I, Y)$ compactly, we claim that $x_n \xrightarrow{s} x$ in V . Since $x_n(T) \xrightarrow{w} x(T)$ in H , Hypothesis (Φ_0) infers that

$$\phi_0(x(T)) \leq \liminf_{n \rightarrow \infty} \phi_0(x_n(T)).$$

By applying Hypothesis (L) and Balder theorem 2.1 ([6]), we conclude that

$$\int_0^T L(t, x(t), u(t)) dt \leq \liminf_{n \rightarrow \infty} \int_0^T L(t, x_n(t), u_n(t)) dt$$

Consequently,

$$J(x, u) \leq \lim_{n \rightarrow \infty} J(x_n, u_n) = \lim_{n \rightarrow \infty} J(x_n, u_n).$$

This shows that (x, u) is the desired optimal pair.

5. Examples

Our results can be used to the following the initial boundary value problem of parabolic system of order $2m$

$$\begin{cases} \frac{\partial}{\partial t} x(t, z) + L(t)x(t, z) = f(t, z, x(t, z)), & \text{on } I \times G \\ D^\beta x = 0 & \text{on } I \times \partial G, \quad |\beta| \leq m-1 \\ x(0, z) = x_0(z), & \text{on } G \end{cases}$$

where $L(t)x(t, z) = \sum_{|\beta| \leq m} (-1)^{|\beta|} D^\beta A_\beta(t, z, \theta(x(t, z)))$,

$f(t, z, x(t, z)) = \sum_{|\beta| \leq k} (-1)^{|\beta|} D^\beta f_\alpha(t, z, \eta(x(t, z)))$, $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is n -tuple of nonnegative integers, $|\beta| = \sum_{i=1}^n \beta_i$, $D^\beta = D_1^{\beta_1} D_2^{\beta_2} \dots D_n^{\beta_n}$, $D_i = \frac{\partial}{\partial x_i}$, $\theta(x) = \{D^\beta x : |\beta| \leq m\}$, $\eta(x) = \{D^\beta x, |\beta| \leq k\}$, and $k < m$. Take

$$X = W_0^{m,p}(G), Y = W_0^{k,p}(G), H = L^2(G).$$

Their dual spaces are

$$X^* = W^{-m,q}(G), Y = W^{-k,q}(G), H^* = H,$$

respectively. From Sobolev embedding theorem, we know that

$$X \hookrightarrow Y \hookrightarrow H \hookrightarrow Y^* \hookrightarrow X^*,$$

with all embeddings being continuous, dense, and compact([1]). Define,

$$a_1(t, x, y) = \int_G \sum_{|\beta| < m} \xi_\beta(t, z, \theta(x)) D^\beta y dz,$$

$$a_2(t, x, y) = \int_G \sum_{|\beta|=m} \xi_\beta(t, z, \theta(x)) D^\beta y dz, \text{ for all } x, y \in X,$$

$$b(t, x, y) = \int_G \sum_{|\beta| \leq k} f_\beta(t, z, \eta(x)) D^\beta y dz.$$

Under some reasonable assumptions similar to Section 30.4 of [3], one can verify that there exist operators $A_i(\cdot, \cdot) : I \times X \rightarrow X^* (i = 1, 2)$ and $f(\cdot, \cdot) : I \times Y \rightarrow X^*$ satisfying

$$\langle A_i(t, x), y \rangle = a_i(t, x, y), i = 1, 2;$$

$$\langle f(t, x), y \rangle = b(t, x, y).$$

Define

$$A(t, x) = A_1(t, x) + A_2(t, x).$$

By Proposition 26.16 of [3] and Corollary 26.14 of [3], we can check that the operators A and f satisfy the assumptions of Theorem 1. Hence, the existence of weak solutions can be proved. Further, under some assumptions of Example 1 of [1], we can solve the following optimal control problem:

$$\begin{aligned} J(x, u) = & \int_G |x(T, z)| dz + \int_0^T \int_G \frac{1}{2} |x(t, z) - y_0(z)|^2 dt dz \\ & + \frac{\theta}{2} \int_0^T \int_G |u(t, z)|^2 dt dz \longrightarrow \inf, \end{aligned}$$

s. t.

$$\begin{cases} \frac{\partial}{\partial t} x(t, z) + L(t)x(t, z) = f(t, z, x(t, z)) + b(t)u(t, z), & \text{on } I \times Q \\ D^\beta x = 0, & \text{on } I \times \partial G, \quad |\beta| \leq m-1 \\ x(0, z) = x_0(z) & \text{on } G \end{cases}$$

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Part II. Stochastic Systems