NEW MEANING OF EXACT CONTROLLABILITY OF LINEAR SYSTEMS IN HILBERT SPACES

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Abstract: It is showed that under reasonable conditions, exact controllability of a linear system in a Hilbert space with bounded control implies time reversibility and spectral completeness of the system. The theory is applied to the beam and plate equations with boundary damping to establish their backward-time wellposedness and completeness of associated generalized eigenfunctions.

Key Words: exact controllability, C_0 group, generalized eigenfunctions, completeness, elastic system, damping

1 INTRODUCTION

There is an extensive literature on exact controllability for both abstract systems and concrete systems governed by partial differential equations (cf. Russell [18], Pritchard and Zabczyk [17] Lions [6, 7], Komornik [5], Liu [8] and references cited therein). So far, exact controllability has been established almost in the systems which are reversible in time. Consider the wave and plate equations with boundary damping and distributed controls:

$$\begin{cases} \ddot{w} - \Delta w = u & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial w}{\partial \nu} + \alpha(x)\dot{w} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases} \quad (1.1)$$

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$$\begin{cases} \ddot{w} + \Delta^2 w = u & \text{in } \Omega \times \mathbb{R}^+ \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \quad \frac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_0 \times \mathbb{R}^+ \\ \Delta w = -\frac{\partial \dot{w}}{\partial \nu} & \text{on } \Gamma_1 \times \mathbb{R}^+ \end{cases} \quad (1.2)$$

where Ω is a bounded open subset in $\mathbb{R}^n$ with the boundary $\partial\Omega$ smooth enough, ν is the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω , $\alpha(x) \geq 0$ is a smooth function on $\partial\Omega$, $u \in L^2(\Omega \times \mathbb{R})$ is control, $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. It has not been known whether the system (1.1) / (1.2) is exactly controllable in the finite energy state space, even in the one-dimensional case ($n = 1$). Majda [15] proved that the backward mixed problem of (1.1) is wellposed in the energy norm if and if $\alpha \equiv 0$. He [14] also showed that for certain $\alpha(x)$ the span of the generalized eigenfunctions associated with (1.1) has infinite codimension.

In this note, we announce the main results in Liu and Russell [12] (without the proofs) and apply them to establish backward-time wellposedness of (1.2), completeness of generalized eigenfunctions associated with (1.2), as well as lack of exact controllability of (1.1) when $\alpha(x_0) \neq 0$ for some $x_0 \in \partial\Omega$.

2 EXACT CONTROLLABILITY AND THE C_0 GROUP

Let H be a Hilbert space over the field $\mathbb{C}$ of complex numbers with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let A generate a C_0 semigroup e^{tA} on H . Consider the control system (A, B) :

$$y(u, t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}Bu(s)ds \quad (2.1)$$

where $y_0 \in H$, $B \in \mathcal{L}(U; H)$, U is another Hilbert space.

Definition 2.1 *The system (A, B) is said to be exactly controllable if for every $y_0, y_1 \in H$ there exist $T > 0$, $u(\cdot) \in L^2(0, T; U)$ such that $y(u, 0) = y_0$, $y(u, T) = y_1$. It is said to be reversible in time if A generates a C_0 group.*

Let us recall a fundamental result.

Proposition 2.1 (Zabczyk [20]) *The following conditions are equivalent:*

- (1) *The system (A, B) is exactly controllable.*
- (2) *There is a $T > 0$ such that for every $y_0, y_1 \in H$ there exists $u(\cdot) \in L^2(0, T; U)$ for which $y(u, 0) = y_0$, $y(u, T) = y_1$.*
- (3) *(observability inequality) There exist $T, \delta > 0$ such that*

$$\int_0^T \|B^* e^{tA^*} y\|_U^2 dt \geq \delta \|y\|^2 \quad \forall y \in H. \quad (2.2)$$

In infinite-dimensional spaces, the class of systems (A, B) which are exactly controllable is quite restricted. Pazy [16] proved that the observability inequality (2.2) with $U = H$ and $B = I$ holds if and only if for any $T > 0$, there exist $\delta_T > 0$ such that

$$\|e^{tA^*}y\| \geq \delta_T \|y\| \quad \forall t \in [0, T], y \in H. \quad (2.3)$$

As an immediate consequence of Pazy's result, the system (A, B) is never exactly controllable if e^{tA} is either compact or differentiable for $t \geq t_0 > 0$. Triggiani [19] proved that the system (A, B) is never exactly controllable if B is a compact operator. Louis and Wexler [13] proved that (A, I) is exactly controllable if and only if there exists a C_0 semigroup $\hat{S}(t)$ on H such that $e^{tA}\hat{S}(t) = I$ for all $t \geq 0$.

Let L be a densely defined linear operator in H . We denote its defined domain, range, null space, adjoint operator, resolvent set, point spectrum by $D(L)$, $R(L)$, $\text{Ker}L$, L^* , $\rho(L)$, $\sigma_p(L)$, respectively. Let $\mathcal{L}(H)$ be the linear space of bounded linear operators on H . We define

$$\mathcal{S} = \{L \in \mathcal{L}(H) \mid L^* = L\}, \quad (2.4)$$

$$\mathcal{S}_d^+ = \{L \in \mathcal{S} \mid \exists a > 0 \text{ such that } \langle Ly, y \rangle \geq a\|y\|^2 \quad \forall y \in H\}. \quad (2.5)$$

$$\sigma_p^*(A) = \{\lambda \in \sigma_p(A) \mid R(\lambda I - A) = H\}. \quad (2.6)$$

An operator $L \in \mathcal{L}(H)$ is said to be *isometric* if $\|Ly\| = \|y\|$ for all $y \in H$; it is said to be *unitary* if $L^* = L^{-1}$. Now, we are in a position to present our results.

Theorem 1 (Liu and Russell [12]) *The following statements are equivalent:*

- (a) *The system (A, I) is exactly controllable.*
- (b) *There exist $P \in \mathcal{S}_d^+$, $Q \in \mathcal{S}$ such that $Pe^{t(A^*+Q)}P^{-1}$ is a C_0 semigroup of isometric operators on H .*
- (c) *There exist $P \in \mathcal{S}_d^+$, $Q \in \mathcal{S}$ such that*

$$e^{tA}Pe^{t(A^*+Q)}P^{-1} = I \quad \forall t \geq 0. \quad (2.7)$$

Theorem 2 (Liu and Russell [12]) *The following statements are equivalent:*

- (a) *A generates a C_0 group on H .*
- (b) *There exist $P \in \mathcal{S}_d^+$, $Q \in \mathcal{S}$ such that $Pe^{t(A+Q)}P^{-1}$ is a C_0 group of unitary operators on H , i.e., $P(A+Q)P^{-1}$ is skew-adjoint.*
- (c) *The system (A, I) is exactly controllable and there exists a sequence $\{\lambda_n\}$ satisfying $\lambda_n \notin \sigma_p^*(A)$ and $\text{Re}\lambda_n \rightarrow -\infty$.*

Liu [8] also proved that e^{tA} can be embedded in a C_0 group on H if and only if

- (i) There exists $\sigma > 0$ such that $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq \sigma\} \subset \rho(-A)$.
- (ii) There exists a nonincreasing function $\psi: [\sigma, \infty) \rightarrow (0, \infty)$ satisfying $\lim_{\tau \rightarrow +\infty} \psi(\tau) = 0$, such that

$$\|(\lambda + A)^{-1}\| \leq \psi(\operatorname{Re} \lambda) \quad \forall \operatorname{Re} \lambda \geq \sigma. \quad (2.8)$$

Corollary 2.1 *If the system (A, B) with some bounded B is exactly controllable and A has compact resolvent, then A generates a C_0 group on H .*

Combination of Majda's result [15] and our corollary here yields the following:

Theorem 3 *The system (1.1) is never exactly controllable if $\alpha(x_0) \neq 0$ for some $x_0 \in \partial\Omega$.*

Remark 2.1 From Theorem 2.2, if (A, I) is exactly controllable but A does not generate a C_0 group, then there exists $\sigma_0 \in \mathbb{R}$ such that $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq \sigma_0\} \subset \sigma_p^*(A)$. This case occurs when e^{tA} is the semigroup of left translations on $L^2(\mathbb{R}^+)$ (see Zabczyk [20]).

3 THE COMPLETENESS OF GENERALIZED EIGENVECTORS

Let A be a closed linear operator defined in H . A nonzero vector y is called a generalized eigenvector of A , corresponding to the eigenvalue $\lambda \in \sigma_p(A)$, if $(\lambda I - A)^n y = 0$ for some positive integer n . Denote by $\operatorname{Sp}(A)$ the set of all finite linear combinations of the generalized eigenvectors. A is said to have a complete system of generalized eigenvectors if $\operatorname{Sp}(A)$ is dense in H . The completeness of generalized eigenvectors is a desirable property in spectral analysis of linear operators. In the present section we will present this property for the infinitesimal generator of a C_0 group, by the theory in Gokhberg and Krein [3].

Let $\mathcal{M}_\infty$ be the set of all compact operators on H . For $L \in \mathcal{M}_\infty$, we have $0 \leq F \equiv (L^* L)^{\frac{1}{2}} \in \mathcal{M}_\infty \cap \mathcal{S}$. Following [3, §II.2], we enumerate the nonzero eigenvalues of F in decreasing order, taking account of their multiplicities,

$$s_j(L), \quad j = 1, 2, \dots, r(F) = \dim R(F).$$

If $r(F) < \infty$, we put $s_j(L) = 0$ for $j = r(F) + 1, \dots$. For $1 \leq p < \infty$, we define

$$\mathcal{M}_p = \{L \in \mathcal{M}_\infty \mid \sum_{j=1}^{\infty} s_j^p(L) < +\infty\}.$$

By the discussions in [3, Chap.III], we know that $\mathcal{M}_p$ is a two-sided ideal of the ring $\mathcal{L}(H)$. A compact operator is said to be of finite order if it is in $\mathcal{M}_p$ for some $1 \leq p < \infty$.

Now, we are ready to present our completeness result.

Theorem 1 (Liu and Russell [12]) *If A is the infinitesimal generator of a C_0 group on a Hilbert space and $(\lambda_0 I + A)^{-1}$ is a compact operator of finite order for some $\lambda_0 \in \rho(A)$, then A has a complete system of generalized eigenvectors.*

We will apply our completeness theorem to the following variational evolution equation:

$$\mathbf{c}(\ddot{w}(t), v) + \mathbf{b}(\dot{w}(t), v) + \mathbf{a}(w(t), v) = 0, \quad \forall v \in V, \quad t > 0. \quad (3.1)$$

A damped linear elastic system can be written in the form (3.1), where $\mathbf{a}(v, v)$ and $\mathbf{c}(v, v)$ are the strain energy and kinetic energy quadratic forms, respectively, $\operatorname{Re} \mathbf{b}(v, v)$ and $\operatorname{Im} \mathbf{b}(v, v)$ represent damping and gyroscopic forces, respectively. See K. Liu and Z. Liu [11], S. Chen, K. Liu and Z. Liu [2] for various properties of the C_0 semigroup associated with (3.1). We assume that

(H1) V and W are Hilbert spaces with inner products $\mathbf{a}(\cdot, \cdot)$ and $\mathbf{c}(\cdot, \cdot)$, respectively;

(H2) $V \hookrightarrow W$ is a compact and dense embedding;

Assumptions (H1) and (H2) imply that there exists a self-adjoint, positive definite operator A_0 in W with compact resolvent such that ([4])

$$D(A_0^{\frac{1}{2}}) = V, \quad \mathbf{a}(u, v) = \mathbf{c}(A_0^{\frac{1}{2}} u, A_0^{\frac{1}{2}} v), \quad \forall u, v \in V. \quad (3.2)$$

It is clear that for every $0 < \beta \leq 1$, $D(A_0^\beta)$ is also a Hilbert space with the inner product $\mathbf{c}(A_0^\beta \cdot, A_0^\beta \cdot)$. We further assume that

(H3) $\mathbf{b}(\cdot, \cdot)$ is a continuous sesquilinear form on $D(A_0^\theta)$ for some $0 \leq \theta < \frac{1}{2}$.

By the Riesz representation theorem, assumption (H3) implies that there exists a compact operator $B_1 \in \mathcal{L}(D(A_0^\theta))$ such that

$$\mathbf{b}(u, v) = \mathbf{c}(A_0^\theta B_1 u, A_0^\theta v), \quad \forall u, v \in D(A_0^\theta). \quad (3.3)$$

Denote by $B = A_0^\theta B_1$. Then $B \in \mathcal{L}(D(A_0^\theta); W)$. Let $\mathcal{H} = V \times W$ with the naturally induced inner product

$$\langle (u, v), (\hat{u}, \hat{v}) \rangle_{\mathcal{H}} = \mathbf{a}(u, \hat{u}) + \mathbf{c}(v, \hat{v}); \quad (3.4)$$

$\mathcal{H}$ may be regarded as the *finite energy state space* of the system (3.1). We define the operator $\mathcal{A}$ in $\mathcal{H}$ as follows:

$$\begin{cases} D(\mathcal{A}) = \{(u, v) \mid u \in D(A_0^{1-\theta}), v \in V, A_0^{1-\theta} u + Bv \in D(A_0^\theta)\}, \\ \mathcal{A}(u, v) = (v, -A_0^\theta[A_0^{1-\theta} u + Bv]). \end{cases} \quad (3.5)$$

It is easy to check that

$$\mathcal{A}^{-1}(u, v) = \begin{bmatrix} -A_0^{\theta-1} B & -A_0^{-1} \\ I & 0 \end{bmatrix} (u, v) \quad (3.6)$$

for $(u, v) := (u, v)^T \in \mathcal{H}$. Obviously, $\mathcal{A}^{-1}$ is a compact operator in $\mathcal{L}(\mathcal{H})$. We will show the connection between the finite-order property of $\mathcal{A}^{-1}$ and the asymptotic distribution of eigenvalues of A_0 .

Theorem 2 *Let the assumptions (H1)-(H3) hold. Then the compact operator $\mathcal{A}^{-1}$ is of finite order if A_0^{-1} is so.*

Enumerating the eigenvalues of A_0 in increasing order, counting the multiple eigenvalues according to their multiplicity, we have

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \lambda_{n+1} \leq \dots \quad (3.7)$$

We know that $\lambda_n \rightarrow +\infty$ (since $\dim W = \infty$). We define

$$N(\lambda) = \max\{j \mid \lambda_j \leq \lambda\} \quad \forall \lambda > 0. \quad (3.8)$$

It is easy to see that A_0^{-1} is finite order if

$$N(\lambda) = \mathcal{O}(\lambda^p), \quad \lambda \rightarrow +\infty \quad (3.9)$$

for some $p \geq 1$. We point out that the condition (3.9) always holds for a general self-adjoint elliptic operator A_0 on a bounded region (cf. Agmon [1] and the references therein).

4 BACKWARD WELLPOSEDNESS AND SPECTRAL COMPLETENESS

To write (1.2) with $u = 0$ in the form (3.1), we can introduce

$$W = L^2(\Omega), \quad V = \{v \in H^2(\Omega) \mid v = 0 \text{ on } \partial\Omega, \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_0\},$$

$$\mathbf{c}(v, v) = \int_{\Omega} |v|^2 dx, \quad \mathbf{a}(v, v) = \int_{\Omega} |\Delta v|^2 dx, \quad \mathbf{b}(v, v) = \int_{\Gamma_1} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma.$$

Then, the assumptions (H1)-(H3) with $\theta = \frac{1-\epsilon}{2}$ for $0 < \epsilon < \frac{1}{4}$ and the condition (3.9) are valid.

Theorem 1 *If $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, then $\mathcal{A}$ corresponding to (1.2) generates a C_0 group on $V \times W$ and has a complete system of generalized eigenvectors.*

Sketch of the Proof. We know that $\mathcal{A}$ generates a C_0 semigroup $S(t)$ of contractions on $\mathcal{H}$ and $\mathcal{A}(w, v) = (v, -\Delta^2 w)$,

$$D(\mathcal{A}) = \{(w, v) \mid w \in H^4(\Omega), w, v \in V, w = 0 \text{ on } \partial\Omega, \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0, \Delta w = -\frac{\partial v}{\partial \nu} \text{ on } \Gamma_1\}. \quad (4.1)$$

For $(\xi, \eta) \in D(\mathcal{A})$, set $(w(t), w_1(t)) = S(t)(\xi, \eta)$. Then the function $w(\cdot)$ is in

$$C([0, \infty); H^4(\Omega)) \cap C^1([0, \infty); V) \cap C^2([0, \infty); W). \quad (4.2)$$

It satisfies $\dot{w}(\cdot) = w_1(\cdot)$ and (1.2) with $u = 0$. Let

$$E(t) = \frac{1}{2} \|S(t)(\xi, \eta)\|_{\mathcal{H}}^2 = \frac{1}{2} \int_{\Omega} (|\Delta w|^2 + |\dot{w}|^2) dx. \quad (4.3)$$

We only need to prove the observability inequality

$$\int_0^T E(t) dt \geq \delta E(0) \quad (4.4)$$

for some $T, \delta > 0$ and all real-valued $(\xi, \eta) \in D(\mathcal{A})$. Multiplying the first equation of (1.2)($u = 0$) by $\dot{w}$ and integrating by parts, we obtain

$$TE(0) - \int_0^T E(t) dt = \int_0^T (T-t) \int_{\Gamma_1} \left(\frac{\partial \dot{w}}{\partial \nu} \right)^2 d\sigma dt. \quad (4.5)$$

We choose $h \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that $h = 0$ on Γ_0 and $h = \nu$ on Γ_1 . Then we multiply the first equation of (1.2)($u = 0$) by $(T-t)h \cdot \nabla w$ to get

$$\begin{aligned} & \int_0^T (T-t) \int_{\Gamma_1} \left[\left(\frac{\partial \dot{w}}{\partial \nu} \right)^2 + \frac{1}{2} \operatorname{div} \frac{\partial \dot{w}}{\partial \nu} \frac{\partial w}{\partial \nu} \right] d\sigma dt \\ & \leq - \int_{\Omega} \eta h \cdot \nabla \xi dx + C_T \int_0^T E(t) dt. \end{aligned} \quad (4.6)$$

where C_T is some positive number independent of (ξ, η) . By a compactness argument, we can now conclude that the observability inequality (4.4) is valid. $\square$

See Liu [10] for the details of proof and more examples dealing with backward wellposedness and spectral completeness of PDEs with boundary dissipation.

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