13 A genetic way to conic sections and planetary motion with Cabri-Géomètre

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Abstract

There is a need to diminish the discrepancy between the high quality of tools and programs offered by information technology and the orientation of mathematics curricula to general abstract concepts. An example here is how the power of Cabri-Géomètre can be used to support students in their investigation and discovery of the most important known and two unknown properties of non degenerate conic sections (conics) which finally lead to an adequate representation of planetary orbits.

Keywords

Learning models, creativity, geometry.

Introduction

The growing quality of tools like the hand-held computer TI-92, and of programs like DERIVE and Cabri-Géomètre, does not automatically enhance the situation in school mathematics. Science-oriented curricula and the 'sociological law of inertia' are well-known obstacles. Therefore the development of suitable conceptions and the construction of teaching material using information technology are the most challenging tasks in the near future.

Our work is based on two conceptions called 'Spatial Algebra' and 'Elementary Analysis'. The didactical framework following the model of 'guided discovery learning' and some technical references are described in Möller (1995) and a typical treatment of conics and planetary motion before the advent of information technology is given in Castelnuevo *et al.* (1979).

Well-known properties of conics

In this first phase Cabri-Géomètre is used for static drawings. We start with a table-tennis ball lying on a sheet of paper and a torchlight which gives a cone of light turned on the ball. Examining the frontiers of the shadows on the paper and on the ball, the *focal properties* of the central conics and the representation of all

Velocity vectors are tangent vectors of the orbit. Therefore we search for a relation between tangent vectors and the corresponding radius vector of a conic in the vector space which has a focus as origin and which is spanned by three orthogonal unit vectors, the first of it ($\mathbf{e}_1$) directed towards the nearest vertex of the conic and the third of it perpendicular to the plane of the section.

Figure 2

Now calculating the length of FH in terms of the magnitude r of $\mathbf{r}$ and of the eccentricity ε we find that

$$(1) \quad \mathbf{n} = r^{-1} \mathbf{r} + \varepsilon \cdot \mathbf{e}_1$$

is a normal vector belonging to $\mathbf{r}$. Varying $\mathbf{r}$ and multiplying (1) by a positive constant c we obtain the equation

$$(2) \quad c \cdot r^{-1} \mathbf{r} = c \cdot \mathbf{n} - c \cdot \varepsilon \cdot \mathbf{e}_1$$

which describes a circle with the radius c where $c \cdot \mathbf{n}$ is always a normal vector belonging to $\mathbf{r}$ (see figure 3). Therefore we call (2) a 'track circle' representation of a conic. This equation will enable the genetic representation of the planetary orbits in the next section.

Figure 3

With Cabri-Géomètre using the track circle representation we let the students discover that the foot of the perpendicular from a focus to a tangent lies on the smallest circle through the vertices in the case of a central conic respectively on the tangent at the vertex when the conic is a parabola. This property prepares the discovery of the above-mentioned representation of planetary orbits.

Simulation of planetary motion

From Newton's laws we get the differential equation

$$(3) \quad \mathbf{r}'' = -g.M.r^{-3}\mathbf{r}$$

where g is the gravitational constant and M is the mass of the sun. Using Cabri-Géomètre to simulate motions that have become discrete in a central force field on screen we find successive triangles with equal area. Since the magnitude of the vector product $\mathbf{r} \wedge \mathbf{r}'$ equals the double of the area of the triangle generated by $\mathbf{r}$ and $\mathbf{r}'$ it seems reasonable to consider the derivative of this vector product. We find $(\mathbf{r} \wedge \mathbf{r}') = \mathbf{r} \wedge \mathbf{r}'' = \mathbf{0}$. With the theorem of constant functions which can be derived without integration and even without limits in elementary analysis we get that $\mathbf{d} = \mathbf{r} \wedge \mathbf{r}'$ is a constant vector.

The inverse tool of Cabri-Géomètre enables the impressive simulation of motions becoming discrete in gravitational fields on screen. Motivated by the equality of triangular areas obtained by the first simulation we also construct the foot of the perpendicular to each difference vector (figure 4). The conjecture that these feet lie on a circle, is only approximately correct. Nevertheless, it leads to another conjecture connected with the track circle representation, namely that $(\mathbf{d} \wedge \mathbf{r}')'$ is a constant multiple of $(r^{-1}\mathbf{r})'$ where $\mathbf{d} \wedge \mathbf{r}'$ is a normal vector belonging to $\mathbf{r}$. With simple calculations we get

Figure 4

$$(g.M.r^{-1}\mathbf{r} + \mathbf{d} \wedge \mathbf{r}')' = \mathbf{0}.$$

Therefore the theorem of constant functions gives the track circle representation of a conic which in the case of planetary orbits must be an ellipse.

References

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