A Powerful Transformation Rule, its Applications and Variants

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Abstract

This paper introduces, discusses and proves a transformation rule to convert specifications of set-valued functions defined by set comprehension into functional implementations. The power of the rule is illustrated by several examples, among them a Prolog interpreter. Also, variants of the rule for specifications involving existential quantification and arbitrary choice are presented and illustrated by representative examples.

Keywords

Transformational programming, backtracking, set comprehension, functional implementation

1 INTRODUCTION

Transformational programming is short for a methodology of constructing programs from formal problem specifications by stepwise application of formal, semantics-preserving transformation rules. In the following the reader is assumed to be basically familiar with the idea of transformational programming and its basic principles. A brief introduction and an overview can be found, e.g., in (Bauer et al., 1989) or (Boiten et al., 1992). For a comprehensive treatment of the subject, cf. (Partsch, 1990).

In this paper we mainly introduce a powerful transformation rule to convert specifications of set-valued functions defined by set comprehension into executable functional implementations.

The general form of such a specification reads

$f(X)$ where
 $f: m \rightarrow \text{set of } n$
 $f(x) =_{\text{def}} \{y: n \mid Q(x, y)\}.$

It asks for computing the value of a function f for some initial argument value X of type $\mathbf{m}$, where f computes the set of all y of type $\mathbf{n}$ such that the argument of f and y are related by some predicate Q .

Example:

A typical instance of such a specification is provided by the N-Queens problem which asks for finding all ways to place N mutually non-attacking queens on an $N \times N$ chessboard:

queens(N) where
queens: $\mathbf{nat} \rightarrow \mathbf{set\ of\ nsequ}$
queens(n) =_{def} { $sp: \mathbf{nsequ} \mid \#sp = n \wedge ispossequ(sp, n) \wedge nconf(sp)$ }.

Here

nsequ =_{def} sequ of nat

is the type of sequences of natural numbers used to represent positions of queens in the different columns of the chessboard;

$sp = n$

checks whether the length of sequence sp is equal to n ;

ispossequ: $\mathbf{nsequ} \times \mathbf{nat} \rightarrow \mathbf{bool}$
ispossequ(s, n) =_{def} $\forall i: \mathbf{nat} \mid 1 \leq i \leq \#s \Delta s[i] \leq n$

restricts the elements of $\mathbf{nsequ}$ to those less or equal to n , and

nconf: $\mathbf{nsequ} \rightarrow \mathbf{bool}$
**nconf(s) =_{def} $\forall (i, j: \mathbf{nat} \mid 1 \leq i, j \leq \#s \wedge i \neq j) \mid$
 $s[i] \neq s[j] \wedge \mathbf{abs}(s[i] - s[j]) \neq \mathbf{abs}(i - j)$**

guarantees that all positions in the different columns are “nonattacking”, i.e., do not mutually occupy the same row or the same diagonal.

The rule we will present deals with the above kind of specifications and can be profitably used for all kinds of problems that ask for a set of complex objects that may somehow be built up incrementally, possibly involving backtracking. The rule emerged when dealing with a formal derivation of a Prolog interpreter from a descriptive specification. It turned out that most transformation steps in this concrete exercise were independent of the particular problem and that the remaining ones could be abstracted appropriately. Meanwhile, the rule has been successfully applied to numerous non-trivial examples, some of which are presented in this

paper. More examples and a detailed treatment of those presented here can be found in (Achatz, Partsch, 1996).

1.1 Notational conventions

Within our rules and examples, we use the following notational conventions.

1.1.1 Fonts

Italic is used for all bound variables. Scheme parameters (for expressions in rules) are denoted by upright-uppercase letters (augmented with “arguments” to indicate potential dependencies). And for types (and type variables) we employ boldface-lowercase symbols.

1.1.2 Data types

Type variables (in program schemes) are denoted by single boldface letters (such as **m** or **n**). As basic types, we assume to have available natural numbers (with type **nat** and the usual operations) as well as booleans.

Based on basic types, more complex ones may be built by forming supertypes (e.g., **m** | **n** comprising all elements of types **m** and **n**) and record types (e.g., (*a*: **m**, *b*: **n**) denoting pairs of elements of types **m** and **n** respectively, where *a* and *b* are selectors for the first (second) argument).

More complex types we will use are sets, sequences, and maps. For respective formal definitions in terms of algebraic types we refer to (Partsch, 1990). The intuition of the basic operations of these types (appearing in the examples) is as follows:

sets (type **set of**) with operations

- $\emptyset$ the empty set
- $\cup$ set union
- $\subseteq$ subset relation
- $\in$ element relation

sequences (type **sequ of**) with operations

- $[]$ the empty sequence
- **hd** (**lst**) first (last) element of a (non-empty) sequence
- **tl** (**fst**) (non-empty) sequence without first (last) element
- **++** concatenation of sequences (also used to append single elements)
- $s[i]$ *i*-th element of a (non-empty) sequence *s*
- $\#s$ length of sequence *s*
- $\in$ element relation

maps (correspondences defined by an algebraic type **EMAP**, cf. [Partsch 90]) with operations

- $\emptyset$ the empty map
- $\circ$ composition of maps

- $m[i]$ the “value” of map m for “argument” i
- dom the domain of the map
- ran the range of the map
- $m[i] \leftarrow a$ map m with “value” of i -th “argument” updated to a .

1.1.3 Programs and program schemes

For programs and program schemes a mainly self-explanatory Pascal-like notation is used with a strict, call-by-value semantics. For a complete definition of the language used, we refer to (Bauer et al., 1985). Some of the less conventional notations (and their meaning) are:

- $\{a: m \mid P(a)\}$ set comprehension, denoting the set of all elements a (of type m) for which $P(a)$ holds
- $some\ a: m \mid P(a)$ (non-deterministic) choice, denoting some element a (of type m) which satisfies $P(a)$
- $(m \mid P(a))$ subtype of m , comprising all elements a (of type m) which fulfil $P(a)$; also used for assertions in the domain of functions
- $\#\{\dots\}$ cardinality of a set
- $\Delta (\nabla)$ sequential conjunction (disjunction)

As an abbreviation, we omit the type m where it is obvious.

1.1.4 Transformation rules

Transformation rules are denoted as $\begin{array}{c} I \\ \updownarrow \\ O \end{array} \vdash C$ resp. $\begin{array}{c} I \\ \downarrow \\ O \end{array} \vdash C$.

I is the input scheme of the rule, O its output scheme. The double arrow means semantic equivalence, the single arrow descendence. C is a set of applicability conditions, all of which are (implicitly) universally quantified clauses (consisting of a premise, $\vdash$ as an implicational symbol, and a semantic relation over program schemes – equivalence or descendence – as a consequence). Within applicability conditions, $\equiv$ is used to denote semantic equivalence, and $\sqsubseteq$ to denote descendence. If there is no premise, also $\vdash$ will be left out. The syntactic constraints (following a rule) give sufficient (but not necessarily minimal) information to infer complete typing of all expressions and functions involved. For details on the logical background of transformation rules, cf. (Partsch, 1990).

To shorten the presentation of the rules we furthermore use the following

General convention

In all transformation rules definedness and determinacy are assumed for all expression symbols and auxiliary functions. Also, all program schemes are supposed to be syntactically valid and context-correct.

1.2 Outline of the following sections

In Sect. 2 we formally develop our central transformation rule, give some intuition on how to use it, and illustrate the essential steps by means of our running example, the N-Queens problem. Section 3 deals with some applications of the rule. In this context, the treatment of the N-Queens problem will be completed by giving all proof obligations that result from instantiating the applicability conditions of our rule and by sketching how to satisfy them. As another example, the problem of a Prolog interpreter is treated in a condensed form. In Sect. 4 some variants of the rule from Sect. 2 are discussed. These variants deal with other specification constructs (such as existential quantification or non-deterministic choice) and are also illustrated by examples (from the area of parsing). As a particular variant we will also show in Sect. 4 that, under additional constraints, our rule from Sect. 2 indeed simplifies to a form one would intuitively expect. Section 5, finally, deals with related work and draws some conclusions.

2 A POWERFUL TRANSFORMATION RULE

In this section, we give a stepwise formal derivation which is based on certain assumptions (called **postulate** below). At the end, the starting point of this formal derivation, the final result, and all assumptions used in the derivation will be compiled into a corresponding transformation rule.

2.1 Derivation of the transformation rule

Our derivation is split up into several phases each of which is given in a calculational style. The single steps of the respective calculations are labelled for later reference. For the details of the basic transformations used in reasoning, we refer the reader to (Partsch, 1990).

The starting point of our derivation is the program scheme

$f(X)$ where
 $f: m \rightarrow \text{set of } n$
 $f(x) =_{\text{def}} \{y: n \mid Q(x, y)\}.$

Obviously, the definition of *queens* as given in Sect. 1 matches this scheme and yields the instantiations

f, X, x, y **by** *queens*, N, n, sp
 $Q(x, y)$ **by** $Q(n, sp) \equiv \#sp = n \wedge ispossequ(sp, n) \wedge nconf(sp).$

To be able to compute f we require

postulate (0) $(\#\{y: n \mid Q(x, y)\} < \infty) = \text{true}$

which guarantees the finiteness of the set to be computed.

(a) As a first step, we transform $f(X)$ into $g(X)$ for a suitably defined function g . To this end we define

$$g: m \rightarrow \text{set of } n$$

$$g(u) =_{\text{def}} g'(u, E, \emptyset)$$

and

$$g': (u: m \times v: r \times s: \text{set of } n \mid P(X, u, v)) \rightarrow \text{set of } n$$

$$g'(u, v, s) =_{\text{def}} s \cup \{y: n \mid Q'(u, v, y)\}.$$

The intuition behind this step is the idea of “embedding”, that is the introduction of a suitably initialized function with additional parameters. The purpose of the additional parameters in g' is as follows:

- The parameter v of type r basically serves to incrementally construct elements of the set aimed at with E being kind of a neutral element to start with. Type r is supposed to be a “supertype” of n – in most applications identical with n or a product type with a component of type n .
- The parameter s of type $\text{set of } n$ is used as an accumulator for the set to be constructed, and, obviously, initialized with $\emptyset$.

P is an invariant (for g') to couple the additional parameter of type r to the original one (of type m) for which

postulate (1) $P(X, X, E) = \text{true}$

is required; that is P holds for the initial values.

Predicate Q' is a “weaker” form of the original predicate Q that takes the incremental construction of elements of type r into account. In most applications, Q' will simply be of the form

$$Q'(u, v, y) = \exists y' \mid c(v, y') = y \Delta Q(u, y')$$

with c being a “constructor” for elements of type r .

To establish the relationship between Q and Q' we require

postulate (2) $Q(x, y) = Q'(x, E, y)$

To establish the equivalence of $f(X)$ with $g(X)$ we calculate as follows:

$$g(X)$$

$$= [(a1) \text{ unfold } g; \text{ postulate (1)}]$$

$$g'(X, E, \emptyset)$$

$$= [(a2) \text{ unfold } g'; \text{ neutrality of } \emptyset \text{ w.r.t. } \cup]$$

$$\{y \mid Q'(X, E, y)\}$$

$$= [(a3) \text{ postulate (2)}]$$

$$\{y \mid Q(X, y)\}$$

Since f has a defined value due to the general convention and **postulate** (0), so do g and g' according to the above reasoning.

and choose to instantiate

we obtain (as an intermediate form) for our running example:

(b) Next, we aim at transforming g' as defined above. To this end, we calculate as follows:

T is a termination condition that serves a two-fold purpose: for the very first call of g' it has to cover the "trivial case" (i.e., the case when E is already a solution for the initial input value X); for all other recursive calls of g' it has to "signal" that the incremental construction of an element of type r has been completed.

To be able to continue our derivation we need additional information. To this end we require

postulate (3) $P(X, u, v) \Delta T(u, v) = \text{true} \vdash H(u, v) = \{y \mid Q'(u, v, y)\}$
postulate (4) $P(X, u, v) \Delta \neg T(u, v) = \text{true} \vdash$
 $\{y \mid Q'(u, v, y)\} =$
 $\bigcup_{i=1..n(u, v)} \{y \mid B(i, u, v) \Delta Q'(K(i, u, v), R(i, u, v), y)\}$

Postulate (3) gives the information how to construct the solution upon termination using an expression H . In most applications H will be of the simple form $H(u, v) = \{v\}$, i.e. a singleton set.

Postulate (4) gives the information on how to split up the computation of the desired set into subcomputations the number of which, $n(u, v)$, may (dynamically) depend on the actual parameter values. The expressions K and R characterize the modification of the parameters within subcomputations.

Moreover, in order to be able to deal with the union in **postulate (4)** in an efficient way, we also require

postulate (5) $\text{next}(k, u, v) \subseteq$
 $\text{some } k': \text{nat} \mid k' \in \{i: \text{nat} \mid k+1 \leq i \leq n(u, v)+1 \Delta$
 $\forall i': \text{nat} \mid k < i' < i \Rightarrow \neg B(i', u, v)\}$

Postulate (5) describes an operation *next* which is characterized non-deterministically on purpose in order to be able to choose a descendant that allows pruning the computation by cutting off unsuccessful branches before they are entered. Obviously, the choice $\text{next}(k, u, v) =_{\text{def}} k+1$ is a valid descendant that satisfies this postulate.

Our calculation then proceeds as follows:

```

■ [ (b3) modification using postulate (3) and postulate (4)]
if T(u, v) then s ∪ H(u, v)
else s ∪ ⋃_{i=1..n(u, v)} {y ∣ B(i, u, v) Δ Q'(K(i, u, v), R(i, u, v), y)} fi
■ [ (b4) simplification using postulate (5)]
if T(u, v) then s ∪ H(u, v)
else s ∪ ⋃_{i=next(0, u, v)..n(u, v)} {y ∣ B(i, u, v) Δ Q'(K(i, u, v), R(i, u, v), y)} fi
fi
■ [ (b5) abstraction ]
if T(u, v) then s ∪ H(u, v) else g"(next(0, u, v), u, v, s) fi
where
g": (k: nat × u: m × v: r × s: set of n ∣ P(X, u, v)) → set of n
g"(k, u, v, s) =_{def}
s ∪ ⋃_{i=k..n(u, v)} {y ∣ B(i, u, v) Δ Q'(K(i, u, v), R(i, u, v), y)}.

```


Example (cont.):

For our running example we choose the instantiations

$T(u, v) \quad \text{by } T(n, p) \triangleq \#p = n$
 $H(u, v) \quad \text{by } H(n, p) \triangleq \{p\}$
 $n(u, v) \quad \text{by } n(n, p) \triangleq n$
 $B(i, u, v) \quad \text{by } B(k, n, p) \triangleq nc(p, k) \text{ where}$
 $\quad \quad \quad nc: \text{nsequ} \times \text{nat} \rightarrow \text{bool}$
 $\quad \quad \quad nc(p, k) \triangleq_{\text{def}} \forall (i: \text{nat} \mid 1 \leq i \leq \#p):$
 $\quad \quad \quad \quad \quad p[i] \neq k \wedge \text{abs}(p[i] - k) \neq \text{abs}(i - (\#p + 1))$
 $\quad \quad \quad (* \text{ checks whether } k \text{ does not "attack" any of the elements of } p *)$
 $K(i, u, v) \quad \text{by } K(k, n, p) \triangleq n$
 $R(i, u, v) \quad \text{by } R(k, n, p) \triangleq p++k$
 $next(k, u, v) \quad \text{by } next: \text{nat} \times \text{nat} \times \text{nsequ} \rightarrow \text{nat}$
 $\quad \quad \quad next(k, n, p) \triangleq_{\text{def}} \min \{i: \text{nat} \mid k+1 \leq i \leq n+1 \wedge$
 $\quad \quad \quad \quad \quad \forall i': \text{nat} \mid k < i' < i \Rightarrow i' \in p\}$

and obtain as the next intermediate version

$q': \text{nat} \times \text{nsequ} \times \text{set of nsequ} \rightarrow \text{set of nsequ}$
 $q'(n, p, s) \triangleq_{\text{def}} \text{if } \#p = n \text{ then } s \cup \{p\} \text{ else } q''(next(0, n, p), n, p, s) \text{ fi}$
 $q'': \text{nat} \times \text{nat} \times \text{nsequ} \times \text{set of nsequ} \rightarrow \text{set of nsequ}$
 $q''(k, n, p, s) \triangleq_{\text{def}} s \cup \bigcup_{i=k..n} \{sp \mid nc(p, i) \wedge$
 $\quad \quad \quad \exists s': \text{nsequ} \mid sp = (p++i)++s' \wedge \#sp = n \wedge \text{ispossequ}(sp, n) \wedge \text{nconf}(sp)\}.$

(c) Third, we transform g'' as defined in (b). We reason as follows:

$g''(k, u, v, s)$
 $\quad \quad \quad \equiv [(c1) \text{ unfold } g'']$
 $s \cup \bigcup_{i=k..n(u, v)} \{y \mid B(i, u, v) \wedge Q'(K(i, u, v), R(i, u, v), y)\}$
 $\quad \quad \quad \equiv [(c2) \text{ case-introduction; simplification of then-branch}]$
 $\text{if } k > n(u, v) \text{ then } s$
 $\text{else } s \cup \bigcup_{i=k..n(u, v)} \{y \mid B(i, u, v) \wedge Q'(K(i, u, v), R(i, u, v), y)\} \text{ fi.}$

For the **else-branch** we reason as follows:

$s \cup \bigcup_{i=k..n(u, v)} \{y \mid B(i, u, v) \wedge Q'(K(i, u, v), R(i, u, v), y)\}$
 $\quad \quad \quad \equiv [(c3) \cup\text{-split in else-branch}]$
 $s \cup \{y \mid B(k, u, v) \wedge Q'(K(k, u, v), R(k, u, v), y)\} \cup$
 $\quad \quad \quad \bigcup_{i=k+1..next(k, u, v)-1} \{y \mid B(i, u, v) \wedge Q'(K(i, u, v), R(i, u, v), y)\} \cup$

$$\bigcup_{i = \text{next}(k, u, v) \dots n(u, v)} \{y \mid B(i, u, v) \Delta Q'(K(i, u, v), R(i, u, v), y)\}$$

■ [(c4) simplification: def. $\text{next} \Rightarrow \bigcup_{i = k+1 \dots \text{next}(k, u, v)-1} \{\dots\} = \emptyset$]

$$s \cup \{y \mid B(k, u, v) \Delta Q'(K(k, u, v), R(k, u, v), y)\} \cup$$

$$\bigcup_{i = \text{next}(k, u, v) \dots n(u, v)} \{y \mid B(i, u, v) \Delta Q'(K(i, u, v), R(i, u, v), y)\}$$

■ [(c5) case introduction; simplification]

if $B(k, u, v)$
then $s \cup \{y \mid Q'(K(k, u, v), R(k, u, v), y)\} \cup$
 $\bigcup_{i = \text{next}(k, u, v) \dots n(u, v)} \{y \mid B(i, u, v) \Delta Q'(K(i, u, v), R(i, u, v), y)\}$
else $s \cup \bigcup_{i = \text{next}(k, u, v) \dots n(u, v)} \{y \mid B(i, u, v) \Delta Q'(K(i, u, v), R(i, u, v), y)\}$ **fi**

Our next steps aim at folding g' and g'' . Since both functions have the same invariant P , we have to make sure that the invariant is maintained. To this end, we require

postulate (6) $1 \leq i \leq n(u, v) \Delta P(X, u, v) \Delta \neg T(u, v) \Delta B(i, u, v) = \text{true} \vdash$
 $P(X, K(i, u, v), R(i, u, v)) = \text{true}$

Our calculation then continues as follows

■ [(c6) fold g' - **postulate (6)** guarantees assertion; termination below]

if $B(k, u, v)$
then $g'(K(k, u, v), R(k, u, v), s) \cup$
 $\bigcup_{i = \text{next}(k, u, v) \dots n(u, v)} \{y \mid B(i, u, v) \Delta Q'(K(i, u, v), R(i, u, v), y)\}$
else $s \cup \bigcup_{i = \text{next}(k, u, v) \dots n(u, v)} \{y \mid B(i, u, v) \Delta Q'(K(i, u, v), R(i, u, v), y)\}$ **fi**

■ [(c7) fold g'' - assertion remains valid; termination below]

if $B(k, u, v)$ **then** $g''(\text{next}(k, u, v), u, v, g'(K(k, u, v), R(k, u, v), s))$
else $g''(\text{next}(k, u, v), u, v, s)$ **fi**.

Altogether, we have:

$g''(k, u, v, s) =_{\text{def}}$
if $k > n(u, v)$ **then** s
elsif $B(k, u, v)$ **then** $g''(\text{next}(k, u, v), u, v, g'(K(k, u, v), R(k, u, v), s))$
else $g''(\text{next}(k, u, v), u, v, s)$ **fi**

Example (cont.)

Applying this to our running example we yield

```

q": nat × nat × nsequ × set of nsequ → set of nsequ
q"(k, n, p, s) =def
  if k > n then s
  elif nc(p, k) then q"(next(k, n, p), n, p, q'(n, p++k, s))
  else q"(next(k, n, p), n, p, s) fi.

```

- (d) It remains to prove termination of g' and g'' which is required for the correctness of the folding steps. By our general convention, all expressions are defined. In the result obtained so far, g' can be eliminated by unfolding its calls in g and g'' :

```

g(X) where
  g(x) =def if T(x, E) then ∅ ∪ H(x, E)
            else g"(next(0, x, E), x, E, ∅) fi

  g"(k, u, v, s) =def
    if k > n(u, v) then s
    elif B(k, u, v) then
      g"(next(k, u, v), u, v,
        if T(K(i, u, v), R(i, u, v))
        then s ∪ H(K(i, u, v), R(i, u, v))
        else g"(next(0, K(i, u, v), R(i, u, v)),
                  K(i, u, v), R(i, u, v), s) fi)
    else g"(next(k, u, v), u, v, s) fi

```

Simplification and distributivity of function call over conditional (in g'') then yields

```

g"(k, u, v, s) =def
  if k > n(u, v) then s
  elif B(k, u, v) then
    if T(K(i, u, v), R(i, u, v))
    then g"(next(k, u, v), u, v, s ∪ H(K(i, u, v), R(i, u, v)))
    else g"(next(k, u, v), u, v, g"(next(0, K(i, u, v), R(i, u, v)),
                                   K(i, u, v), R(i, u, v), s)) fi
  else g"(next(k, u, v), u, v, s) fi

```

Termination of g is obvious provided g'' terminates. To prove termination of g'' we define the following (lexicographic) termination ordering $\angle$ on $\text{nat} \times \mathbf{m} \times \mathbf{r} \times \text{set of n}$:

$$(k, u, v, s) \angle (k', u', v', s') \Leftrightarrow \begin{array}{l} v < v' \vee (v = v' \wedge n(u, v) - k < n(u, v) - k') \\ \vee \\ u < u' \vee (u = u' \wedge n(u, v) - k < n(u, v) - k') \end{array}$$

$f(X)$ where

$$f(x) =_{\text{def}} \{r(X, y): n \mid Q(x, y)\}$$

$g(X)$ where

$$g(x) =_{\text{def}} g'(x, E, \emptyset)$$

$$g'(u, v, s) =_{\text{def}}$$

if $T(u, v)$ then $s \cup H(u, v)$ else $g''(\text{next}(0, u, v), u, v, s)$ fi

$$g''(k, u, v, s) =_{\text{def}}$$

if $k > n(u, v)$ then s

elsif $B(k, u, v)$ then $g''(\text{next}(k, u, v), u, v, g'(K(k, u, v), R(k, u, v), s))$

else $g''(\text{next}(k, u, v), u, v, s)$ fi

Syntactic constraints

$$\text{KIND}[r] = m \times p \rightarrow n$$

$$\text{KIND}[f, g] = m \rightarrow \text{set of } n$$

$$\text{KIND}[g'] = (u: m \times v: r \times s: \text{set of } n \mid P(X, u, v)) \rightarrow \text{set of } n$$

$$\text{KIND}[g''] = (k: \text{nat} \times u: m \times v: r \times s: \text{set of } n \mid P(X, u, v)) \rightarrow \text{set of } n$$

$$\text{KIND}[\text{next}] = (\text{nat} \times m \times r) \rightarrow \text{nat}$$

Admittedly, the whole bunch of applicability conditions looks complicated at a first glance, due to the many expressions to be “invented” and the lot of resulting proof obligations. As to the first, one should keep in mind that in most applications the simple forms as commented above are sufficient - which substantially reduces the amount of work for finding suitable instantiations. As to the latter, it will be exemplified with the queens example below, that, whenever suitable instantiations have been found, all proofs of the instantiated applicability conditions are straightforward or even trivial.

3 SOME APPLICATIONS OF THE RULE

In this section we complete the treatment of the queens example and deal with another (non-trivial) problem to illustrate the power of our rule and how to use it.

3.1 The N-Queens Problem completed

In summary, the application of our rule to the queens problem (with r instantiated by the identity on its second argument, and $\text{WF-ORD}(r, <)$ instantiated by $(\text{nsequ}, a < b \Leftrightarrow n - \#a < n - \#b)$) results in the following program:

```

qu(N) where
  qu: nat  $\rightarrow$  set of nsequ
  qu(n) =def q'(n, [],  $\emptyset$ )

  q': nat  $\times$  nsequ  $\times$  set of nsequ  $\rightarrow$  set of nsequ
  q'(n, p, s) =def
    if #p = n then s  $\cup$  {p} else q"(next(0, n, p), n, p, s) fi

  q": nat  $\times$  nat  $\times$  nsequ  $\times$  set of nsequ  $\rightarrow$  set of nsequ
  q"(k, n, p, s) =def
    if k > n then s
      elif nc(p, k) then q"(next(k, n, p), n, p, q'(n, p++k, s))
      else q"(next(k, n, p), n, p, s) fi.

```

It remains to prove the instantiated applicability conditions (which is straightforward or even trivial):

$$(0) \quad (\# \{sp: \text{nsequ} \mid \#sp = n \wedge \text{ispossequ}(sp, n) \wedge \text{nconf}(sp)\}) < \infty) = \text{true}$$

Trivial, as already **nsequ** restricted by **ispossequ** has only finitely many elements (for any n).

$$(1) \quad (\#[] \leq n \wedge \text{ispossequ}([], n) \wedge \text{nconf}([])) = \text{true}$$

Obviously, $\#[] = 0 \leq n$, $\text{ispossequ}([], n)$, and $\text{nconf}([])$ are all true.

$$(2) \quad \#sp = n \wedge \text{ispossequ}(sp, n) \wedge \text{nconf}(sp) = \\ \exists s': \text{nsequ} \mid sp = [] ++ s' \wedge \#sp = n \wedge \text{ispossequ}(sp, n) \wedge \text{nconf}(sp)$$

Trivial, as for $s' = sp$ the righthand side expression simplifies to the one on the lefthand side.

$$(3) \quad (\#p \leq n \wedge \text{ispossequ}(p, n) \wedge \text{nconf}(p)) \Delta \#p = n = \text{true} \vdash \\ \{p\} = \{sp: \text{nsequ} \mid \exists s': \text{nsequ} \mid sp = p ++ s' \wedge \#sp = n \wedge \\ \text{ispossequ}(sp, n) \wedge \text{nconf}(sp)\}$$

From $\#p = n$, $\#sp = n$, and $sp = p++s'$, it follows that $\#s' = 0$, and, thus, $s' = []$. Hence, the set on the righthand side of the consequence simplifies to

$$\{sp: \mathbf{nsequ} \mid sp = p \wedge \#sp = n \wedge \mathbf{ispossequ}(sp, n) \wedge \mathbf{nconf}(sp)\}.$$

The equality of both sets then follows from the premise on p .

$$\begin{aligned} (4) \quad & (\#p \leq n \wedge \mathbf{ispossequ}(p, n) \wedge \mathbf{nconf}(p)) \Delta \#p < n \equiv \mathbf{true} \vdash \\ & \{sp: \mathbf{nsequ} \mid \exists s': \mathbf{nsequ} \mid sp = p++s' \wedge \#sp = n \wedge \\ & \quad \mathbf{ispossequ}(sp, n) \wedge \mathbf{nconf}(sp)\} \equiv \\ & \bigcup_{i=1..n} \{sp: \mathbf{nsequ} \mid \mathbf{nc}(p, i) \Delta \exists s': \mathbf{nsequ} \mid sp = (p++i)++s' \wedge \\ & \quad \#sp = n \wedge \mathbf{ispossequ}(sp, n) \wedge \mathbf{nconf}(sp)\} \end{aligned}$$

Due to the premise $\#p < n$ and associativity of $++$, the lefthand side of the consequence can be rewritten into

$$\{sp: \mathbf{nsequ} \mid \exists s': \mathbf{nsequ}, i: \mathbf{pos}(n) \mid sp = (p++i)++s' \wedge \#sp = n \wedge \mathbf{ispossequ}(sp, n) \wedge \mathbf{nconf}(sp)\}$$

where $i: \mathbf{pos}(n)$ abbreviates $(i: \mathbf{nat} \mid 1 \leq i \leq n)$.

Since $\neg \mathbf{nc}(p, i)$ implies $\neg \mathbf{nconf}((p++i)++s')$, the existential quantification on i can be restricted to those i that satisfy $\mathbf{nc}(p, i)$. The requested equality then follows from the general rule

$$\{x \mid \exists y, z \mid P(z) \wedge Q(x, y, z)\} \equiv \bigcup_{z \mid P(z)} \{x \mid \exists y \mid Q(x, y, z)\}.$$

$$\begin{aligned} (5) \quad & \mathbf{min} \{i: \mathbf{nat} \mid k+1 \leq i \leq n+1 \Delta \forall i': \mathbf{nat} \mid k < i' < i \Rightarrow i' \in p\} \subseteq \\ & \mathbf{some} \ k': \mathbf{nat} \mid k' \in \{i: \mathbf{nat} \mid k+1 \leq i \leq n+1 \Delta \forall i': \mathbf{nat} \mid \\ & \quad k < i' < i \Rightarrow \neg \mathbf{nc}(p, i')\} \end{aligned}$$

Obviously, $i' \in p$ implies $\neg \mathbf{nc}(p, i')$. $\mathbf{min} \ m$ is always a descendant of $\mathbf{some} \ x \mid x \in m$.

$$(6) \quad 1 \leq i \leq n \Delta (\#p \leq n \wedge \mathbf{ispossequ}(p, n) \wedge \mathbf{nconf}(p)) \Delta \#p < n \Delta \mathbf{nc}(p, i) \equiv \mathbf{true} \vdash \#(p++i) \leq n \wedge \mathbf{ispossequ}(p++i, n) \wedge \mathbf{nconf}(p++i) \equiv \mathbf{true}$$

$\#p < n$ implies $\#(p++i) \leq n$, $\mathbf{ispossequ}(p++i, n)$ follows immediately from the premise, and $\mathbf{nconf}(p++i)$ follows from $\mathbf{nconf}(p)$ and $\mathbf{nc}(p, i)$.

$$(7) \quad 1 \leq i \leq n \Delta \mathbf{nc}(p, i) \equiv \mathbf{true} \vdash n - \#(p++i) < n - \#p \equiv \mathbf{true}$$

The consequence is even true for arbitrary n, p , and i .

Remarks

- The choice of *next* as defined above (instead of the generally possible choice $next(k) =_{\text{def}} k+1$) is a simple optimization (according to the remark on *next* in Sect. 2).
- As a further improvement, we could have added another parameter to keep all possible candidate positions to be added to the current sequence. This would require modified instantiations, viz.

$$\begin{array}{ll}
 P(X, u, v) & \text{by } \#p \leq n \wedge ispossequ(p, n) \wedge nconf(p) \wedge \\
 & \qquad \qquad \qquad \forall (i: pos(n) \mid i \in c) \mid nc(p, i) \\
 E & \text{by } ([], \{i: pos(n)\}) \\
 B(i, u, v) & \text{by } c \neq \emptyset \\
 R(i, u, v) & \text{by } (p++i, \{k: pos(n) \mid nc(p++i, k)\}).
 \end{array}$$

Together with

$$next(k, c) =_{\text{def}} \text{if } c \neq \emptyset \text{ then } \min c \text{ else } n+1 \text{ fi}$$

these modified instantiations then would have resulted in

$$\begin{array}{l}
 qu(N) \text{ where} \\
 qu(n) =_{\text{def}} q'(n, [], \{i: pos(n)\}, \emptyset) \\
 q'(n, p, c, s) =_{\text{def}} \\
 \quad \text{if } \#p = n \text{ then } s \cup \{p\} \text{ else } q''(next(0, c), n, p, c, s) \text{ fi} \\
 q''(k, n, p, c, s) =_{\text{def}} \\
 \quad \text{if } k > n \text{ then } s \\
 \quad \text{elsif } c \neq \emptyset \text{ then} \\
 \quad \quad q''(next(k, c), n, p, c, q'(n, p++k, \{i: pos(n)\} \mid nc(p++k, i)), s) \\
 \quad \text{else } q''(next(k, c), n, p, c, s) \text{ fi.}
 \end{array}$$

As a further improvement – which is left to the interested reader – one could think of applying finite differencing to efficiently compute the new candidate set in the first recursive call of *g''*.

3.2 A Prolog Interpreter

A Prolog interpreter takes a Prolog program and a query and yields as a result either the set of all substitutions (for the variables in the query) such that the instantiated query is a logical consequence from the program, or it does not terminate. If there is no substitution (such that the instantiated query follows from the program) the result is the empty set.

To formally specify the problem, we first have to define the notions of program, query, and substitution. A program is a sequence of clauses (or rules) each of which consists of a literal as its lefthand side and a (possibly empty) sequence of literals as its righthand side. A literal in turn consists of a predicate symbol and a (possibly

empty) sequence of terms (as arguments). And a term is either a constant symbol, a variable symbol, or a function symbol together with a sequence of terms as arguments. All this can straightforwardly be formalized as follows:

```

program =def sequ of clause
clause =def (lhs: literal, rhs: sequ of literal)
literal =def (ps: predsymb, st: sequ of term)
term =def const | var | (functsymb, sequ of term).

```

A query is a sequence of literals, and a substitution is an association of variable symbols with terms. Again, formalization is straightforward:

```

query =def sequ of literal
subst =def EMAP(var, term, =).

```

To construct a substitution and simultaneously to prove that the instantiated query gl follows from the program p , a prolog interpreter uses sld-resolution which is captured by the following derivability relation:

```

 $\overline{p} : \text{query} \times \text{subst} \times \text{query} \times \text{subst} \rightarrow \text{bool}$ 
 $(gl, \Theta) \overline{p} (gl', \Theta \circ \Theta') =_{\text{def}}$ 
   $\exists c \in p \mid gl' = (rhs(c') ++ tlg l)\Theta' \wedge$ 
   $c' = rn(c, gl) \wedge unify(hdgl, lhs(c')) \neq \text{fail} \Delta$ 
   $\Theta' = unify(hdgl, lhs(c'))$ 

```

where

```

 $\Theta \circ \Theta'$  denotes the composition of the substitutions  $\Theta$  and  $\Theta'$ ,
 $a\Theta$  denotes the application of the substitution  $\Theta$  to  $a$ ,
rn: clause  $\times$  query  $\rightarrow$  clause
rn( $c, gl$ ) =def
  « clause  $c$  with all variables renamed such that they do not occur in  $gl$  »

```

and (with c_i and v_i denoting constant and variable symbols, respectively, f , g denoting function symbols, and r_i, s_i denoting terms)

```

struct =def term | literal

unify: struct  $\times$  struct  $\rightarrow$  subst | fail
unify( $c_1, c_2$ ) =def if  $c_1 = c_2$  then  $\emptyset$  else fail fi
unify( $c_1, v_2$ ) =def ( $c_1$  for  $v_2$ )
unify( $c_1, g(r_1, \dots, r_k)$ ) =def fail
unify( $v_1, c_2$ ) =def ( $c_2$  for  $v_1$ )
unify( $v_1, v_2$ ) =def if  $v_1 \neq v_2$  then ( $v_2$  for  $v_1$ ) else  $\emptyset$  fi

```

```

unify( $v_1, g(r_1, \dots, r_k)$ ) =def
  if notoccurs( $v_1, g(r_1, \dots, r_k)$ ) then ( $g(r_1, \dots, r_k)$  for  $v_1$ ) else fail fi
unify( $f(s_1, \dots, s_n), c_2$ ) =def fail
unify( $f(s_1, \dots, s_n), v_2$ ) =def
  if notoccurs( $v_2, f(s_1, \dots, s_n)$ ) then ( $f(s_1, \dots, s_n)$  for  $v_2$ ) else fail fi
unify( $f(s_1, \dots, s_n), g(r_1, \dots, r_k)$ ) =def
  if  $f = g \wedge n = k$  then unifylist( $[s_1, \dots, s_n], [r_1, \dots, r_k], \emptyset$ ) else fail fi

notoccurs: var  $\times$  struct  $\rightarrow$  bool
notoccurs( $v, s$ ) =def «  $v$  does not occur in  $s$  »

unifylist: sequ of struct  $\times$  sequ of struct  $\times$  subst  $\rightarrow$  subst | fail
unifylist( $[], [], \Theta$ ) =def  $\Theta$ 
unifylist( $g++gl, h++hl, \Theta$ ) =def
  if  $\Theta' \neq \text{fail}$  then unifylist( $gl\Theta', hl\Theta', \Theta \cdot \Theta'$ ) else fail fi
  where  $\Theta' = \text{unify}(g, h)$ 

```

Equipped with this environment, a formal specification of a prolog interpreter is as follows:

```

interpret( $P, A$ ) where
  interpret: program  $\times$  query  $\rightarrow$  set of subst
  interpret( $p, gl$ ) =def  $\{\Theta \upharpoonright_{\text{vars}(A)} \mid (gl, \emptyset) \xrightarrow{p}^* ([], \Theta)\}$ 

```

where

$\Theta \upharpoonright_{\text{vars}(A)}$ denotes Θ restricted to those variable symbols occurring in A

and

$\xrightarrow{p}^*$ denotes the reflexive transitive closure of $\xrightarrow{p}$.

Correct application of our transformation rule requires to satisfy the applicability condition

$$(\#\{\Theta \upharpoonright_{\text{vars}(A)} \mid (gl, \emptyset) \xrightarrow{p}^* ([], \Theta)\} < \infty) = \text{true}.$$

Whereas in the previous example, the corresponding proof obligation was straightforward to be shown, it is less trivial here. Indeed, it only can be proved if both the program and the query satisfy additional constraints the detailed treatment of which is beyond the scope of this paper. A similar remark holds for condition (7) about the existence of a suitable well-founded ordering. Therefore we simply take these applicability conditions for granted and yield as result of rule application (as to the respective instantiations, cf. (Achatz, Partsch, 1996)):

int(P, A) where

int: program $\times$ query $\rightarrow$ set of subst

int(p, gl) =_{def} int'(p, gl, $\emptyset$, $\emptyset$)

int': program $\times$ query $\times$ subst $\times$ set of subst $\rightarrow$ set of subst

int'(p, gl, Φ , s) =_{def}

if gl = [] then s $\cup$ { $\Phi \upharpoonright_{vars(A)}$ }

else int"(next(0, p, gl), p, gl, Φ , s) fi

**int": nat $\times$ program $\times$ query $\times$ subst $\times$ set of subst $\rightarrow$
set of subst**

int"(k, p, gl, Φ , s) =_{def}

if k > #p then s

elsif $\Phi' \neq \text{fail}$

then int"(next(k, p, gl), p, gl,

int'(p, (rhs(rn(p[k], gl)) ++ tlg) Φ' , $\Phi \circ \Phi'$, s))

else int"(next(k, p, gl), p, gl, Φ , s) fi

where $\Phi' = \text{unify}(\text{hdgl}, \text{lhs}(\text{rn}(p[k], \text{gl})))$

next: nat $\times$ program $\times$ query $\rightarrow$ nat

next(k, p, gl) =_{def}

min {i: nat | (k+1 $\leq$ i $\leq$ #p Δ ps(lhs(p[i])) = ps(hd gl)) ∇ i = #p+1}

3.3 Further examples

Of course, there are many more problems our rule might be applied to. Apart from the additional ones dealt with in (Achatz, Partsch, 1996) (i.e., all permutations of a sequence, all partitions of a natural number, all subsequences of a sequence, a knapsack-problem, all topological sortings, all maximal, cycle-free paths from a given node), these range, for example, from mathematical problems (e.g., all subsets of a set, all partitions of a set, all subsets of a set up to a fixed size), combinatorial problems (e.g., space-filling puzzles, other known problems on the chessboard), or further graph problems.

4 VARIANTS OF THE GENERAL RULE AND THEIR APPLICATIONS

In the following we discuss a couple of “variants” of our general rule and illustrate them by respective examples. These “variants” look very similar to our general rule, although they deal with different kinds of specifications and require (slightly) different applicability conditions. Using the word “variant” is simply motivated by the fact that the proofs of these rules only marginally deviate from the proof of our general rule.

4.1 Variant 1: Existential quantification

4.1.1 The rule

$f(X)$ where

$$f(x) \stackrel{\text{def}}{=} \exists y: \mathbf{p} \mid Q(x, y)$$

$\left[\begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} \right.$

(1), (2), (5), (6), (7) as in general rule

(3) $P(X, u, v) \Delta T(u, v) \stackrel{\text{def}}{=} \text{true} \vdash \text{true} \stackrel{\text{def}}{=} \exists y: \mathbf{p} \mid Q'(u, v, y)$

(4) $P(X, u, v) \Delta \neg T(u, v) \stackrel{\text{def}}{=} \text{true} \vdash$

$\exists y: \mathbf{p} \mid Q'(u, v, y) \stackrel{\text{def}}{=}$

$\exists_{i=1..n(u, v)} \exists y: \mathbf{p} \mid B(i, u, v) \Delta Q'(K(i, u, v), R(i, u, v), y)$

$g(X)$ where

$$g(x) \stackrel{\text{def}}{=} g'(x, E, \text{false})$$

$$g'(u, v, s) \stackrel{\text{def}}{=} \text{if } T(u, v) \text{ then true else } g''(\text{next}(0, u, v), u, v, s) \text{ fi}$$

$$g''(k, u, v, s) \stackrel{\text{def}}{=} \begin{array}{l} \text{if } s \vee k > n(u, v) \text{ then } s \\ \text{elsf } B(k, u, v) \text{ then } g''(\text{next}(k, u, v), u, v, g'(K(k, u, v), R(k, u, v), s)) \\ \text{else } g''(\text{next}(k, u, v), u, v, s) \text{ fi} \end{array}$$

Syntactic constraints

$$\begin{array}{l} \text{KIND}[f, g] = \mathbf{m} \rightarrow \text{bool} \\ \text{KIND}[g'] = (u: \mathbf{m} \times v: \mathbf{r} \times s: \text{bool} \mid P(X, u, v)) \rightarrow \text{bool} \\ \text{KIND}[g''] = (k: \text{nat} \times u: \mathbf{m} \times v: \mathbf{r} \times s: \text{bool} \mid P(X, u, v)) \rightarrow \text{bool} \\ \text{KIND}[\text{next}] = (\text{nat} \times \mathbf{m} \times \mathbf{r}) \rightarrow \text{nat} \end{array}$$

Proof (changes)

- a) (a2'): unfold g' ; neutrality of **false** w.r.t. v
- b) (b2'): case-introduction; distributivity $\vee$ over conditional
- c) (c3'): v -split in **else**-branch

4.1.2 An Application: Top-down recognition (for context-free grammars)

Given a context-free grammar $G = (N, T, Z, P)$ (where N is a non-empty set of nonterminal symbols, T a non-empty set of terminal symbols which is disjoint from N , $Z \in N$, and P is a non-empty set of context-free productions consisting of a nonterminal symbol as its lefthand side and a (possibly empty) sequence of (nonterminal or terminal) symbols as its righthand side) and a terminal word w , it is requested to check whether w is derivable from Z using P .

Using

nont =_{def} « nonterminal symbols »
term =_{def} « terminal symbols »
symb =_{def} **nont** | **term**
prod =_{def} (*lhs*: **nont**, *rhs*: **sequ of symb**)

the problem can be formalized by

recognize(*P*, *W*) where
recognize: **sequ of prod** × **sequ of term** → **bool**
recognize(*p*, *w*) =_{def} ∃ *y*: **sequ of term** | [*Z*] $\overrightarrow{p}^* y \wedge y = w$

where

$\overrightarrow{p}$: **sequ of symb** × **sequ of symb** → **bool**
 $x \overrightarrow{p} y$ =_{def} ∃ *l*, *r*: **sequ of symb**, *i*: **nat** |
 $x = l ++ lhs(p[i]) ++ r \wedge y = l ++ rhs(p[i]) ++ r$

and

$\overrightarrow{p}^*$ denotes the reflexive transitive closure of $\overrightarrow{p}$.

The result of rule application is

rec(*P*, *W*) where
rec: **sequ of prod** × **sequ of term** → **bool**
rec(*p*, *w*) =_{def} *rec'*(*p*, *w*, [*Z*], **false**)
rec': **sequ of prod** × **sequ of term** × **sequ of symb** × **bool** → **bool**
rec'(*p*, *w*, *v*, *s*) =_{def}
 if *w* = [] ∧ *v* = [] then **true** else *rec*"(*next*(1, *p*, *v*), *p*, *w*, *v*, *s*) fi
rec" : **nat** × **sequ of prod** × **sequ of term** × **sequ of symb** × **bool**
 → **bool**
rec"(*k*, *p*, *w*, *v*, *s*) =_{def}
 if *s* ∨ *k* > #*p*+1 then *s*
 elsif (*k* ≤ #*p* ∧ *v* ≠ [] Δ *hdv* = *lhs*(*p*[*k*])) ∨
 (*k* = #*p*+1 ∧ *w* ≠ [] ∧ *v* ≠ [] Δ *hdw* = *hdv*)
 then *rec*"(*next*(*k*, *p*, *v*), *p*, *w*, *v*,
 rec'(*p*, if *k* ≤ #*p* then *w* else *tlw* fi,
 if *k* ≤ #*p* then *rhs*(*p*[*k*]) ++ *tlv* else *tlv* fi, *s*)
 else *rec*"(*next*(*k*, *p*, *v*), *p*, *w*, *v*, *s*) fi
next: **nat** × **sequ of prod** × **sequ of symb** → **nat**
next(*k*, *p*, *v*) =_{def}
 min {*i*: **nat** | (*k*+1 ≤ *i* ≤ #*p* Δ *lhs*(*p*[*i*]) = *hdv*) ∇ *i* = #*p*+1 }

where rec can be further transformed into

```

rec"(k, p, w, v, s) =def
  if s v k > #p+1 then s
  elif k ≤ #p ∧ v ≠ [] Δ hdv = lhs(p[k])
    then rec"(next(k, p, v), p, w, v, rec'(p, w, rhs(p[k])+tlv, s))
  elif k = #p+1 ∧ w ≠ [] ∧ v ≠ [] Δ hdw = hdv
    then rec"(next(k, p, v), p, w, v, rec'(p, tlw, tlv, s))
  else rec"(next(k, p, v), p, w, v, s) fi.

```

Similar to the Prolog interpreter, proving termination is slightly more difficult than in most of the other examples. In fact, as is known, absence of left-recursion in the grammar is necessary for the proof of termination.

Of course, we could have chosen an even “stronger” definition for $next$, viz.

```

next(k, p, v) =def
  min {i: nat | (k+1 ≤ i ≤ #p Δ lhs(p[i]) = hdv Δ
    ∃ z: symb | hdv →p hdw++z) ∇ i = #p+1}

```

which obviously reflects the idea of LL(1) recognition.

4.2 Variant 2: Non-deterministic choice

4.2.1 The rule

$f(X)$ where

```

f(x) =def
  if {y: p | Q(x, y)} ≠ ∅ then some y: p | Q(x, y) else dummy fi
  +
  (1), (2), (5), (6), (7) as in general rule
  (3) P(X, u, v) Δ T(u, v) = true ⊢ H(u, v) ∈ {y | Q'(u, v, y)} = true
  (4) P(X, u, v) Δ ¬T(u, v) = true ⊢
    some y: p | Q'(u, v, y) = some y: p | ∃ i = 1..n(u, v) (B(i, u, v) Δ
    Q'(K(i, u, v), R(i, u, v), y))
  ↓

```

$g(X)$ where

```

g(x) =def g'(x, E, dummy)
g'(u, v, s) =def if T(u, v) then H(u, v) else g"(next(0, u, v), u, v, s) fi
g"(k, u, v, s) =def
  if s ≠ dummy v k > n(u, v) then s
  elif B(k, u, v) then g"(next(k, u, v), u, v, g'(K(k, u, v), R(k, u, v), s))
  else g"(next(k, u, v), u, v, s) fi

```

Syntactic constraints

$KIND[f, g] = m \rightarrow n \mid dummy$

$$\begin{aligned} \text{KIND}[g'] &= (u: \mathbf{m} \times v: \mathbf{r} \times s: \mathbf{n} \mid \mathbf{dummy} \mid P(X, u, v)) \rightarrow \mathbf{n} \mid \mathbf{dummy} \\ \text{KIND}[g''] &= (k: \mathbf{nat} \times u: \mathbf{m} \times v: \mathbf{r} \times s: \mathbf{n} \mid \mathbf{dummy} \mid P(X, u, v)) \rightarrow \\ &\quad \mathbf{n} \mid \mathbf{dummy} \\ \text{KIND}[next] &= (\mathbf{nat} \times \mathbf{m} \times \mathbf{r}) \rightarrow \mathbf{nat} \end{aligned}$$
Proof (changes)

- a) Define
 - $a \odot b =_{\text{def}} \text{if } \text{def}(a) \text{ then } a \mid \text{def}(b) \text{ then } b \text{ else } \mathbf{dummy} \text{ fi where}$
 - $\text{def}(\text{some } y \mid Q) =_{\text{def}} \{y \mid Q\} \neq \emptyset$
 - $\text{def}(x) =_{\text{def}} x \neq \mathbf{dummy}.$
 - (a2'): unfold g'
 - (a3'): cond. (2); unfold $\odot$
- b) (b2'): case-introduction
- (b3'): $\supseteq$ for $\equiv$
- c) (c2'): $\supseteq$ for $\equiv$
- (c3'): v -split in **else**-branch
- (c5'): case introduction in **else**-branch; simplification using **some** $y \mid (B \vee C)$
 $\equiv (\text{some } y \mid B) \odot (\text{some } y \mid C)$

4.2.2 An Application: Bottom-up parsing (of context-free grammars)

Given a context-free grammar G (as defined in the previous example) and a terminal word w , it is asked for computing a sequence of shift and reduce actions the application of which allows to reduce w to the axiom Z of G provided such a sequence exists; if not, the result should be **error**.

For a formal specification of this problem we define **nont**, **term**, **symb**, and **prod** as in the previous example. The sequence of shift and reduce actions is defined by

srsequ $=_{\text{def}}$ « sequences of S (= *shift*) and $R(X, \alpha)$ (= *reduce with* (X, α)) ».

The application of such a sequence to a terminal word is specified by

apply: **sequ of prod** $\times$ **sequ of symb** $\times$ **sequ of term** $\rightarrow$
sequ of symb $\times$ **sequ of term**
 $\text{apply}([], st, w) =_{\text{def}} (st, w)$
 $\text{apply}(S++srs, st, t++w) =_{\text{def}} \text{apply}(srs, st++t, w)$
 $\text{apply}(R(X, \alpha)++srs, st++\alpha, w) =_{\text{def}} \text{apply}(srs, st++X, w).$

With these prerequisites, the original problem may be specified as follows:

parse($P, [], W$) **where**
parse: **sequ of prod** $\times$ **sequ of symb** $\times$ **sequ of term** $\rightarrow$
srsequ $\mid$ **error**
 $\text{parse}(p, st, w) =_{\text{def}}$

if { y : srsequ | $apply(y, st, w) = ([Z], [])$ } $\neq \emptyset$
then some y : srsequ | $apply(y, st, w) = ([Z], [])$ **else error fi.**

The result of rule application is

$par(P, [], W)$ **where**
 par : sequ of prod $\times$ sequ of symb $\times$ sequ of term $\rightarrow$
 srsequ | error
 $par(p, st, w) =_{\text{def}} par'(p, st, w, [], \text{error})$

 par' : sequ of prod $\times$ sequ of symb $\times$ sequ of term $\times$
 srsequ $\times$ srsequ | error $\rightarrow$ srsequ | error
 $par'(p, st, w, sr, s) =_{\text{def}}$
 if $w = [] \wedge st = [Z]$ **then** sr **else** $par''(1, p, st, w, sr, s)$ **fi**

 par'' : nat $\times$ sequ of prod $\times$ sequ of symb $\times$ sequ of term $\times$
 srsequ $\times$ srsequ | error $\rightarrow$ srsequ | error
 $par''(k, p, st, w, sr, s) =_{\text{def}}$
 if $s \neq \text{error} \vee k > \#p+1$ **then** s
 elsif $(k \leq \#p \wedge \exists s' \mid st = s'++rhs(p[k])) \vee (k = \#p+1 \wedge w \neq [])$
 then $par''(k+1, p, st, w, sr,$
 $par'(p, \text{if } k \leq \#p \text{ then } (s'++lhs(p[k]), w)$
 else $(st++hdw, tlw)$ **fi,**
 if $k \leq \#p$ **then** $sr++R(p[k])$ **else** $sr++S$ **fi, s))
 else $par''(k+1, p, st, w, sr, s)$ **fi****

where par'' can be further transformed into

$par''(k, p, st, w, sr, s) =_{\text{def}}$
 if $s \neq \text{error} \vee k > \#p+1$ **then** s
 elsif $k \leq \#p \wedge \exists s' \mid st = s'++rhs(p[k])$
 then $par''(k+1, p, st, w, sr, par'(p, s'++lhs(p[k]), w, sr++R(p[k]), s))$
 elsif $k = \#p+1 \wedge w \neq []$
 then $par''(k+1, p, st, w, sr, par'(p, st++hdw, tlw, sr++S, s))$
 else $par''(k+1, p, st, w, sr, s)$ **fi.**

Again, as in the case of top-down recognition, additional constraints on the grammar are necessary for the proof of termination.

4.3 Variant 3: A special instance of the rule

4.3.1 The rule

Differently to the previous variants, we now consider a variant that is obtained by specializing the general rule by

- defining $n(u, v) =_{\text{def}} 1$ and $next(k, u, v) =_{\text{def}} k+1$

- instantiating these definitions and simplifying the resulting expressions
- skipping the dependencies (on i) of the expressions K , R , and B .

This leads to the following specialization of our general rule:

$f(X)$ **where**

$$f(x) =_{\text{def}} \{r(X, y): n \mid Q(x, y)\}$$

$g(X)$ **where**

$$g(x) =_{\text{def}} g'(x, E, \emptyset)$$

$$g'(u, v, s) =_{\text{def}} \text{if } T(u, v) \text{ then } s \cup H(u, v) \text{ else } g''(1, u, v, s) \text{ fi}$$

$$g''(k, u, v, s) =_{\text{def}}$$

if $k > 1$ **then** s

elsf $B(u, v)$ **then** $g''(k+1, u, v, g'(K(u, v), R(u, v), s))$

else $g''(k+1, u, v, s)$ **fi**

Syntactic constraints

as in general rule

Intuitively, under these particular constraints, one would expect a linear recursive output scheme. Indeed, this can be obtained by further transformation:

$$(a) \ g''(2, u, v, s)$$

$$\equiv [\text{instantiation}]$$

s

$$(b) \ g''(1, u, v, s)$$

$$\equiv [\text{unfold } g'']$$

if $1 > 1$ **then** s

elsf $B(u, v)$ **then** $g''(2, u, v, g'(K(u, v), R(u, v), s))$ **else** $g''(2, u, v, s)$ **fi**

$$\equiv [\text{simplification of conditional}]$$

if $B(u, v)$ **then** $g''(2, u, v, g'(K(u, v), R(u, v), s))$ **else** $g''(2, u, v, s)$ **fi**

$$\equiv [(a)]$$

if $B(u, v)$ **then** $g'(K(u, v), R(u, v), s)$ **else** s **fi**

- (c) $g'(u, v, s)$
 $\equiv [\text{unfold } g']$
if $T(u, v)$ **then** $s \cup H(u, v)$ **else** $g''(1, u, v, s)$ **fi**
 $\equiv [(b)]$
if $T(u, v)$ **then** $s \cup H(u, v)$
elsif $B(u, v)$ **then** $g'(K(u, v), R(u, v), s)$ **else** s **fi**
- (d) Define $g'''(u, v) =_{\text{def}} g'(u, v, \emptyset)$. Then
 $g'''(u, v)$
 $\equiv [\text{unfold } g''']$
 $g'(u, v, \emptyset)$
 $\equiv [\text{unfold } g' \text{ as obtained in (b)}]$
if $T(u, v)$ **then** $\emptyset \cup H(u, v)$
elsif $B(u, v)$ **then** $g'(K(u, v), R(u, v), \emptyset)$ **else** $\emptyset$ **fi**
 $\equiv [\text{simplification; fold } g''']$
if $T(u, v)$ **then** $H(u, v)$ **elsif** $B(u, v)$ **then** $g'''(K(u, v), R(u, v))$ **else** $\emptyset$ **fi**

Altogether, we have derived a new (linear recursive) output scheme:

$g(X)$ **where**
 $g: \mathbf{m} \rightarrow \text{set of } \mathbf{n}$
 $g(x) =_{\text{def}} g'''(x, E)$ **where**
 $g''': (u: \mathbf{m} \times v: \mathbf{r} \mid P(X, u, v)) \rightarrow \text{set of } \mathbf{n}$
 $g'''(u, v) =_{\text{def}}$
if $T(u, v)$ **then** $H(u, v)$
elsif $B(u, v)$ **then** $g'''(K(u, v), R(u, v))$ **else** $\emptyset$ **fi**.

4.3.2 An application: Prime factors of a natural number (≥ 2)

Given a natural number n (≥ 2), it is asked to compute the sequence of all prime factors of n in increasing order.

The output type may be formally specified by

$\text{psequ} =_{\text{def}} (s: \text{natsequ} \mid \forall 1 \leq i \leq \#s \Delta \text{prime}(s[i]))$

where,

$\text{prime}: \text{nat} \rightarrow \text{bool}$
 $\text{prime}(x) =_{\text{def}} \ll x \text{ is a prime number} \gg.$

Using, additionally,

$\text{prod}: \text{psequ} \rightarrow \text{nat}$
 $\text{prod}(sp) =_{\text{def}} \text{if } sp = [] \text{ then } 1 \text{ else } \text{hdsp} * \text{prod}(\text{tlsp}) \text{ fi}$

and

ordered: psequ $\rightarrow$ bool
ordered(sp) =_{def} « sp is increasingly ordered »

a complete formal specification of the problem is given by

all-primes(N) where
all-primes: nat $\rightarrow$ set of psequ
all-primes(n) =_{def} {sp: psequ | prod(sp) = n $\wedge$ ordered(sp)}.

The result of rule application is (where **div** denotes integer division, and **divides(c, n)** tests divisibility):

primes(N) where
primes: nat $\rightarrow$ set of psequ
primes(n) =_{def} prs(n, [], 2)

prs: (nat $\times$ psequ $\times$ nat) $\rightarrow$ set of psequ
prs(n, s, c) =_{def}
 if n = 1 then {s}
 elsif true then prs(if divides(c, n) then n div c else n fi,
 if divides(c, n) then (s++c, c)
 else (s, next(c)) fi)
 else $\emptyset$ fi

next: nat $\rightarrow$ nat
next(c) =_{def} if c=2 then 3 else c+2 fi

where **prs** can be further transformed (using simplification of conditional; distributivity of function call and conditional; simplification) into

prs(n, s, c) =_{def}
 if n = 1 then {s}
 elsif divides(c, n) then prs(n div c, s++c, c) else prs(n, s, next(c)) fi.

4.4 Further variants

Similar to variants 1 and 2 (where the initial specification asked for some expression over a set rather than the set itself), further variants could be given (and proved in an analogous way) to deal with other set expressions (e.g., **min**, **max**, or the like).

So far, we also have only considered variants where the alternatives resulting from the splitting of the computation according to applicability condition (4) are considered in an ordered way. Of course, we can also give analogous variants where the union

$$\bigcup_{i=1..n(u, v)} \{r(X, y) \mid B(i, u, v) \Delta Q'(K(i, u, v), R(i, u, v), y)\}$$

in condition (4) is replaced by

$$\bigcup_{x \in m(u, v)} \{r(X, y) \mid B(x, u, v) \Delta Q'(K(x, u, v), R(x, u, v), y)\}$$

and by using

$$\begin{array}{ll} m(u, v) \setminus \{x\} & \text{for } \text{next}(k, u, v) \\ u \setminus \{x\} & \text{for } K(x, u, v) \\ m(u, v) = \emptyset & \text{for } k > n(u, v). \end{array}$$

Since, however, efficient computation requires an ordering of these alternatives anyhow, we have refrained from giving these variants explicitly.

5. RELATED WORK AND CONCLUDING REMARKS

As its main contribution, this paper has presented a powerful transformation rule to convert specifications of set-valued functions defined by set comprehension into executable functional implementations that basically show the behaviour of backtrack algorithms. From this central transformation rule we have furthermore derived several variants to cope with related problems and their solution by backtracking.

5.1 Related work

Early papers that dealt with the problem of backtracking from a methodical point of view are (Gerhart, Yelowitz, 1976) and (Broy, Wirsing, 1980). Both papers give kind of functional program schemes to characterize and discuss the idea of backtracking, but do not investigate the problem of how to transform descriptive specifications into these functional schemes.

A treatment of backtracking as a solution to search problems which is fairly close to our approach can be found in the work of Doug Smith (cf., e.g., (Smith, 1986), (Smith, 1988), or (Smith, 1990) – also for further references). He gives various design strategies which achieve roughly the same effect as the application of our rule. Also the entirety of conditions required to apply these strategies is basically comparable to our applicability conditions. The main difference in the two approaches – apart from having a strategy there and a compact rule here – is in various technical aspects. Also, it is not quite clear how the variants we deal with fit into Smith's approach.

Of course, our rules presented in this paper also show many relationships to the rules presented in (Partsch, 1990). E.g., the rule “ $\exists$ -elimination by backtracking II” may be considered a specialized form of variant 2 presented here, and the rule “computing qualifiers for a finite set” is but a particular special case of our central rule. Moreover, a difference in methodically looking at the problem is worth mentioned here. Whereas in (Partsch, 1990) we primarily concentrate on solutions for the existence problem and present tactics to extend these solutions to also cover

the “some” or the “set-of-all” problem, we started here the other way round which considerably reduces the amount of work for concrete applications.

5.2 Concluding remarks

In some sense, the problems our rule is able to deal with, can be viewed as generate-and-test problems to produce sets the elements of which are characterized by some predicate. Although the only real restriction for the application of the rule is provided by the applicability conditions, best use of the rule can be made for those problems where the elements of the solution are “complex” (i.e., not directly computable from the input) and where the search space is large (maybe infinite) and not easily enumerable. For “simpler” problems (not showing these characteristics) application of the rule is well possible, but usually will require additional effort to simplify the result.

It is obvious, that our rule is primarily aimed at languages with the initially mentioned semantic characteristics. For these, the rule offers a possibility to convert non-operational, descriptive specifications into operational ones – maybe as a first step towards further improving efficiency or switching to an imperative implementation. In the context of a non-strict, lazy functional language (such as Gofer), the value of our rule may be doubted, since in such a language our initial specifications may be directly executed. For all our examples, however, corresponding comparisons (with Gofer implementations of the respective program versions) have shown a substantial increase in efficiency (through application of our rule) also for a non-strict, lazy functional language.

The main problem in using the given transformation rules for concrete applications is in finding the appropriate instantiations required by the applicability conditions. Although some guidance (in finding instantiations) is given by the conditions themselves as well as by the remarks in Sect. 2, three major creative activities remain to be done:

- finding a suitable way of splitting the problem into subproblems (condition (4))
- inventing an appropriate invariant P (conditions (2) and (5)), and
- defining a suitable ordering for the proof of termination (condition (6)).

Once a suitable instantiation has been found, the verification of the resulting proof obligations is straightforward and even gives potential for automation using a theorem prover. Experience (when dealing with all the examples given here and in (Achatz, Partsch, 1996)) has confirmed the assumption that the reverse of this statement also holds: Whenever complicated proof obligations resulted from some instantiation, it turned out that the real problem was with the instantiation rather than the proof obligation.

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