Additive Schwarz method with strip substructures

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Abstract

Domain Decomposition Method for finite element discretization of elliptic problems with discontinuous coefficients is analyzed. The domain is divided into overlapping strip shaped subdomains for 2-D problems and slice or pillar shape subdomains in 3-D case. The coefficients of the problem are discontinuous and constant on these subdomains. The approximate solution is obtained iteratively by solving local problems associated with each subdomain and the global problem associated with the coarse triangulation.

The framework for Schwarz methods, as developed by Dryja and Widlund (1990), is used. The estimates of convergence are established for strips in 2-D and slices or pillars in 3-D, which are independent of the jumps of coefficients and discretization parameters.

Keywords

Elliptic problems, domain decomposition, iterative methods, Schwarz method

1 INTRODUCTION

Domain Decomposition Methods are very powerful iterative technique for solving linear systems arising from discretization of partial differential equations. We apply this technique in order to solve elliptic equations with discontinuous and piecewise constant coefficients. A two level nested triangulation is defined. The original problem is discretized with respect to the fine triangulation. The domain is divided into a number of subdomains with boundaries matching the fine triangulation. The arising linear system is solved by an iterative method. Each iteration step consists of solving a number of linear systems, which correspond to restrictions of the original problem to subdomains and a small global problem defined on the coarse triangulation.

There are two classes of Domain Decomposition Methods: Schwarz methods (for overlapping subdomains) and Substructuring methods (for non overlapping subdomains). In this paper we construct and analyze the Additive Schwarz Algorithms in the case when the domain is divided into strip shape subdomains (i.e. strips in 2-D and slices or pillars in 3-D).

The partition of the domain into strips has several advantages. The bandwidth of local matrices is narrow, which minimizes computations and memory requirements. Also the structure of local problems is useful for vectorization of an algorithm. We also have better estimates on convergence than for partition into boxes, since less number of subdomains intersect with each other.

The special attention is paid to the problems with coefficients constant on these subdomains, thus these algorithms may be used for solving real problems arising in mechanics, fluid dynamics and similar areas.

For construction of Additive Schwarz Algorithms we use the framework for Schwarz methods, as developed by Dryja and Widlund (1990). The convergence of algorithms is independent of jumps of the coefficients.

Optimal estimates on convergence with respect to parameters of discretization is established in the case of strips in 2-D and slices in 3-D. The analysis of these cases is very similar to analysis for boxes cf. Dryja, Sarkis and Widlund (1994). In the case of pillars in 3-D the convergence is bounded by $(1 + \ln(H/h))$. This result is better then known results for boxes, cf. Dryja, Sarkis and Widlund (1994).

Results of the related numerical experiments can be found in Mróz (1995).

2 MODEL PROBLEM

We consider the problem of finding an approximate solution of the following elliptic boundary value problem.

For given a bilinear form $a(\cdot, \cdot)$ and a linear functional $l(\cdot)$ on $H_0^1(\Omega)$ we want to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = l(v) \quad \forall v \in H_0^1(\Omega), \quad (1)$$

where Ω is a Lipschitz bounded domain in R^2 or R^3 . For simplicity of presentation we assume that Ω is a polygon in R^2 or polyhedron in R^3 .

The bilinear form $a(\cdot, \cdot)$ is defined as follows

$$a(u, v) = \int_{\Omega} \rho(x) \sum_{i=1}^2 \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx. \quad (2)$$

The function $\rho(x)$ is piecewise constant i.e.

$$\rho(x) = \rho_i > 0, \quad x \in \Omega_i, \quad (3)$$

where Ω_i denote the strip shape subdomain consisting of elements of coarse triangulation (see Section 4). The jumps of coefficients between subdomains may be large. This model problem can be applied to the case when the function $\rho(x)$ varies moderately on each subdomain and is discontinuous between subdomains. If $\rho \equiv 1$ we have the case of Poisson's equation.

Let $l(v)$ denote the linear form defined by

$$l(v) = (f, v)_{L^2(\Omega)} = \int_{\Omega} f v dx.$$

The bilinear forms $a(\cdot, \cdot)$ define scalar product on $H_0^1(\Omega)$.

3 FINITE ELEMENT APPROXIMATION

A two level triangulation is defined on the domain Ω . First, we construct a coarse triangulation Ω_H that consists of shape regular, quasi uniform (cf. Ciarlet (1978)), non overlapping simplices Ω^j of diameter of order H . In second step, we further divide each element of the triangulation Ω_H into smaller, shape regular, quasi uniform simplices of diameter $O(h)$. They form the fine triangulation Ω_h .

Spaces of piecewise linear, continuous functions on Ω_H and Ω_h are denoted by $V^H(\Omega)$ and $V^h(\Omega)$. The restriction to subspaces of functions vanishing on $\partial\Omega$ is denoted by $V_0^H(\Omega)$ and $V_0^h(\Omega)$ respectively.

A finite element formulation is obtained by projecting the space $H_0^1(\Omega)$ onto its finite dimensional subspace $V_0^h(\Omega)$. The corresponding approximate problem for (1) is then:

Find $u^* \in V_0^h(\Omega)$ such that

$$a(u^*, v) = l(v) \quad \forall v \in V_0^h(\Omega). \quad (4)$$

Let $\{\phi_j^h\}$ be the set of standard, piecewise linear, nodal basis functions, thus $V_0^h(\Omega) = \text{span}\{\phi_j^h\}$. In this basis, the discrete variational problem (4) can be rewritten as a system of linear equations

$$Au = f, \quad (5)$$

where coefficients of the matrix A and the vector f are given by

$$A_{ji} = a(\phi_j^h, \phi_i^h), \quad f_j = l(\phi_j^h).$$

The matrix A is positive definite and symmetric. The condition number of A is proportional to $\frac{\max_j \rho_j^h}{\min_j \rho_j^h} h^{-2}$.

4 ADDITIVE SCHWARZ METHOD

In this section we present a construction of the Additive Schwarz Method with overlaps for model problem (1). We use the abstract framework of Schwarz method developed by Dryja and Widlund (cf. Dryja and Widlund (1990) or Dryja and Widlund (1994)).

The domain Ω is divided into n subdomains Ω_i , $i = 1, \dots, n$. In 2-D case the subdomains are called strips. We assume that the boundary of each strip consists only of sides of elements from the coarse triangulation Ω_H . The strip Ω_i has common boundary with at most two neighboring strips.

In similar way we define slices in 3-D case. The boundary of each slice consists only of faces and edges of elements from the coarse triangulation Ω_H . Each slice has a common boundary with at most two neighboring slices.

The partition into 3-D pillars is defined in the following way. We first divide the domain Ω into slices. Each slice is further divided into pillars according to the same rules as for

division into slices. We assume that the interface between two pillars is empty or consists of an edge of pillar (a number of adjacent edges of the elements of Ω_H) or face of a pillar (a number of adjacent sides of elements of Ω_H). The common interface between more than two pillars consists of edge or is empty. We assume that the boundary points of each edge lies on $\partial\Omega$.

Each subdomain Ω_i is extended to Ω'_i such that the distance between $\partial\Omega_i$ and $\partial\Omega'_i$ is positive and equal to $\delta_i < H$. We now restrict each Ω'_i to Ω . The boundaries of these subdomains match with the fine triangulation Ω_h . The parameter δ denotes the minimal overlap,

$$\delta = \min_i \delta_i .$$

The Additive Schwarz Method is described in the terms of subspaces, variational forms and projections. The space $V_0^h(\Omega)$ is decomposed into a sum of $N + 1$ subspaces V_i .

$$V_0^h(\Omega) = \sum_{i=0}^N V_i .$$

The subspace $V_0 = V_0^H(\Omega)$ is defined in Section 3. Each of the remaining subspaces V_i is related to the subdomain Ω'_i ,

$$V_i = V_0^h(\Omega) \cap H_0^1(\Omega'_i) ,$$

and extended by zero off Ω'_i .

Let P_i , $i = 1, \dots, N$, denote the orthogonal global projections from $V_0^h(\Omega)$ to V_i with respect to bilinear form $a(\cdot, \cdot)$, i.e.

$$a(P_i u, v) = a(u, v) \quad \forall v \in V_i .$$

The mapping P from $V_0^h(\Omega)$ into $V_0^h(\Omega)$ is defined as the sum of projections P_i ,

$$P = \sum_{i=0}^N P_i . \tag{6}$$

The operator P is invertible (see Theorems 1 and 2), thus (4) is equivalent to the auxiliary problem of finding $u^* \in V_0^h(\Omega)$ which satisfies

$$Pu^* = g . \tag{7}$$

if $g = \sum_{i=0}^N g_i$, where $g_i = P_i w$ is the solution of the equation

$$a(g_i, v) = l(v) \quad \forall v \in V_i$$

since

$$a(g_i, v) \equiv a(P_i u^*, v) = a(u^*, v) \equiv l(v) \quad \forall v \in V_i .$$

It should be pointed out that g can be found without knowing the solution of (1).

The operator P is well conditioned (see Theorems 1 and 2) and the right hand side is known, thus one can use the preconditioned conjugate gradient (PCG) method (cf. Concus, Golub, O'Leary (1976)) to solve (7).

5 IMPLEMENTATION

In each step of the PCG method a vector is multiplied by the matrix P corresponding to the operator P from (6). We describe the algorithm that determines the function $v \in V_0^h(\Omega)$

$$v = Pu,$$

where $u \in V_0^h(\Omega)$ is a given function.

Algorithm

1. Find projections $P_i u$, $i = 1, \dots, N$ by solving

$$a(P_i u, \phi_j^h) = a(u, \phi_j^h) \quad \forall \phi_j^h \in V_i = V_0^h(\Omega'_i).$$

These systems correspond to solving the subproblems individually for each extended subdomain Ω'_i with homogeneous Dirichlet boundary conditions on $\partial\Omega'_i$.

2. Find the projection P_0 by solving the global system

$$a(P_0 u, \phi_j^H) = a(u, \phi_j^H) \quad \forall \phi_j^H \in V_0 = V_0^H(\Omega).$$

3. Compute the vector v from

$$v = \sum_{i=0}^N P_i u.$$

This algorithm admits easy parallelization since the systems from Step 1 and 2 are independent. If we assign $N+1$ processors to the algorithm, then we achieve the maximal degree of parallelism. If the number of unknowns in the global system (Step 2) is of the same order as in the strip, then we have balanced loads on each processor.

Let us now describe the matrix representation of the operator P defined by (6). Computing the projection $P_i u$, $i = 1, \dots, N$ of an arbitrary function u involves the solution of linear system defined only for nodes of interiors of Ω'_i . Let us denote by A_i the matrix that corresponds the variational problem (4) defined on the subdomain Ω'_i . The matrix representation of local projection P_i , $i = 1, \dots, N$ is given by

$$P_i = R_i^T A_i^{-1} R_i A = B_i A$$

where R_i is the restriction operator which returns only those unknowns which are associated with the interior nodes of Ω'_i . The matrix representation of projection P_0 involves

solving the global system defined on the coarse triangulation Ω_H . Let us assume that A_0 denotes the stiffness matrix for the original problem constructed by the basis functions of $V_0^H(\Omega)$. The matrix R_0^T represents the linear interpolation from $V^H(\Omega)$ to $V^h(\Omega)$, thus the projection matrix P_0 is of the form

$$P_0 = R_0^T A_0^{-1} R_0 A = B_0 A.$$

The matrix representation of the problem (7) is the following system

$$BAu = Bf \tag{8}$$

where

$$B = \sum_{i=0}^N B_i.$$

Thus the matrix B can be interpreted as a inverse of a preconditioner for matrix A .

6 CONVERGENCE ESTIMATES

In this section we formulate theorems, that gives us estimates on condition number of the operator P , see (6). The proof of these theorems can be found in Mróz (1995). We first establish estimates in the case of 2-D strips or 3-D slices.

Theorem 1 *The operator $P : V_0^h(\Omega) \rightarrow V_0^h(\Omega)$ defined by (6) (in the case of 2-D strips or 3-D slices) is symmetric and positive definite in the scalar product $a(\cdot, \cdot)$, and hold*

$$m \left(1 + \frac{H}{\delta}\right)^{-1} a(u, u) \leq a(Pu, u) \leq M a(u, u) \quad \forall u \in V_0^h(\Omega), \tag{9}$$

where positive constants m and M are independent of H , h , overlap δ , and the jumps of coefficient.

As a consequence of this theorem, the condition number of BA , see (8), is proportional to

$$\text{cond}(BA) \sim \left(1 + \frac{H}{\delta}\right).$$

Note that if δ is bounded from below by fixed fraction of H , then the algorithm is optimal.

In the case of partitioning into 3-D pillars we have the following estimate

Theorem 2 *The operator $P : V_0^h(\Omega) \rightarrow V_0^h(\Omega)$ defined by (6) (in the case of 3-D pillars) is symmetric and positive definite in scalar product $a(\cdot, \cdot)$, and hold*

$$m \left(1 + \frac{H}{\delta}\right)^{-1} \left(1 + \ln \frac{H}{h}\right)^{-1} a(u, u) \leq a(Pu, u) \leq M a(u, u) \quad \forall u \in V_0^h(\Omega), \tag{10}$$

where constants m and M are independent of H , h , the overlap δ and the jumps of coefficient.

The condition number of BA defined by (8) in the case of pillars is proportional to

$$\text{cond}(\text{BA}) \sim \left(1 + \frac{H}{\delta}\right) \left(1 + \ln \frac{H}{h}\right).$$

7 REMARKS

The algorithm presented in this paper has one disadvantage. The coefficients of local matrices A_i , see Section 5 are not constant. In order to omit such disadvantage we can define the extended subdomain Ω'_i as a union of two subdomains Ω_i and Ω_{i+1} . Now the local problem corresponds to two adjacent subdomains. Now we can use any substructuring methods for two subdomains, cf. Mróz (1995). Similar algorithms are described in Nepomnyaschikh (1992).

The algorithm presented in this paper can be easily extended for parabolic problems.

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