Relaxation of optimal control problems coercive in L^p -spaces

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Abstract

This contribution surveys a relaxation method applicable for optimal control problems whose admissible controls are not a priori bounded so that the coercivity can be expected only in L^p - but not L^∞ -spaces. Correct relaxed problem is constructed by a continuous extension of the original one and proper optimality conditions are derived and analyzed. For this, the relative L^1 -weak compactness of energy of minimizing sequences is essential. The method is illustrated on an elliptic optimal control problem.

Keywords

Nonconvex optimal-control problems, unbounded control, oscillations, relaxation, nonconcentration, Young measures, Pontryagin maximum principle, elliptic problems

1 THE ORIGINAL PROBLEM

Optimization problems involving controls from Lebesgue spaces usually impose control constraints ensuring the set of admissible controls to be bounded in an L^∞ -space. If the control constraints have a more general structure, one can expect only a boundedness of the set of admissible controls in an L^p -space provided a suitable coercivity of the problem is supposed. The aim of this contribution is to show that, in this more general case, basically the standard results can be obtained but more advanced techniques must be used. Though such phenomena can be pursued also in nonconvex variational calculus or game theory, we want to illustrate them on a rather simpler situation.

Let us consider a model optimal control problem in a general form

$$(P) \quad \begin{cases} \text{Minimize} & \int_{\Omega} \varphi_0(x, y(x), u(x)) dx & (\text{cost functional}) \\ \text{subject to} & Ay = \varphi(x, y(x), u(x)) \text{ on } \Omega, & (\text{state equation}) \\ & u(x) \in S(x) \text{ on } \Omega, & (\text{control constraints}) \\ & y \in L^q(\Omega; \mathbb{R}^n), \quad u \in L^p(\Omega; \mathbb{R}^m), \end{cases}$$

where u is a control distributed on a sufficiently regular domain $\Omega \subset \mathbb{R}^{n_1}$, y is a state, $\varphi_0 : \Omega \times (\mathbb{R}^n \times \mathbb{R}^m) \rightarrow \mathbb{R}$ and $\varphi : \Omega \times (\mathbb{R}^n \times \mathbb{R}^m) \rightarrow \mathbb{R}^{m_1}$ are Carathéodory mappings, the later one satisfying the growth condition

$$|\varphi(x, r, s)| \leq a(x) + b|r|^{q/p_1} + c|s|^{p/p_1} \quad (1)$$

with some $a \in L^{p_1}(\Omega)$ and $b, c \in \mathbb{R}$ so that $\varphi(y, u)$ lives in $L^{p_1}(\Omega; \mathbb{R}^{m_1})$, $p_1 > 1$. Moreover, we suppose that $S : \Omega \rightrightarrows \mathbb{R}^m$ is a measurable multivalued mapping, $A : \mathcal{D}_A \rightarrow L^{p_1}(\Omega; \mathbb{R}^{m_1})$ is a (for simplicity linear) mapping whose domain $\mathcal{D}_A$ is a subspace of $L^q(\Omega; \mathbb{R}^n)$, and the state equation $Ay = \varphi(x, y(x), u(x))$ has, for any $u \in L^p(\Omega; \mathbb{R}^m)$, precisely one solution $y = \pi(u) \in \mathcal{D}_A$. A natural growth condition for φ_0 is

$$|\varphi_0(x, r, s)| \leq a_0 + b_0|r|^q + c_0|s|^p \quad (2)$$

with some $a_0 \in L^1(\Omega)$ and $b_0, c_0 \in \mathbb{R}$ so that $\varphi_0(y, u)$ lives in $L^1(\Omega)$.

Usually, the control-constraint mapping S is assumed bounded to ensure coercivity of the problem in $L^\infty(\Omega; \mathbb{R}^m)$ but here we want to put off such assumption. Instead of this, we suppose the coercivity of φ_0 in the sense

$$\varphi_0(x, r, s) \geq c_1|s|^p \quad (3)$$

for some $c_1 > 0$. This ensures that every sequence of controls $\{u_k\}_{k \in \mathbb{N}}$ minimizing for (P) is inevitably bounded in $L^p(\Omega; \mathbb{R}^m)$ (but not in $L^\infty(\Omega; \mathbb{R}^m)$, however!).

2 THE RELAXED PROBLEM

As neither convexity of $\varphi_0(x, r, \cdot)$ and of $S(x)$ nor linearity of $\varphi(x, r, \cdot)$ is supposed, the original optimal control problem (P) need not have any solution, and therefore its relaxation is urgent. We will make it by a continuous extension of (P). For this reason, let us consider a suitable separable normed linear space H of Carathéodory integrands $\Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$ with the growth at most p (i.e. each $h \in H$ fulfils $|h(x, s)| \leq a_h(x) + c_h|s|^p$ with some $a_h \in L^1(\Omega)$ and $c_h \in \mathbb{R}$) which is sufficiently rich, namely $\varphi_0 \circ y \in H$ and $\varphi \circ y \in H^{m_1}$ for any $y \in L^q(\Omega; \mathbb{R}^n)$, where $[\varphi \circ y](x, s) = \varphi(x, y(x), s)$. In concrete cases, there is usually enough freedom in the choice of H .

A natural imbedding of $L^p(\Omega; \mathbb{R}^m)$ into H^* , denoted by i_H , is defined by $\langle i_H(u), h \rangle = \int_{\Omega} h(x, u(x)) dx$. It is known [13] that the weak* closure of $i_H(L^p(\Omega; \mathbb{R}^m))$, denoted by $Y_H^p(\Omega; \mathbb{R}^m)$, is a convex, locally compact, locally sequentially compact, and σ -compact subset of H^* . These properties of the full of $Y_H^p(\Omega; \mathbb{R}^m)$ enables us to employ techniques like if the relaxed problems were defined on a finite-dimensional space.

Let us still impose a slight requirement on H , namely that $gh \in H$ and $\|gh\|_H \leq C\|g\|_{C(\bar{\Omega})}\|h\|_H$ for any $g \in C(\bar{\Omega})$ and any $h \in H$, where $[gh](x, s) = g(x)h(x, s)$. Then we can define the bilinear mapping $(h, \eta) \mapsto h \bullet \eta$ from $H \times H^*$ to the space of Radon measures $\mathcal{M}(\bar{\Omega}) \cong C(\bar{\Omega})^*$ by $\langle h \bullet \eta, g \rangle = \langle \eta, gh \rangle$ for any $g \in C(\bar{\Omega})$. Note that $\eta \mapsto h \bullet \eta : Y_H^p(\Omega; \mathbb{R}^m) \rightarrow \mathcal{M}(\bar{\Omega})$ represents the affine (weak*, weak*)-continuous extension of the Nemytskii mapping $u \mapsto h \circ u : L^p(\Omega; \mathbb{R}^m) \rightarrow L^1(\Omega)$. Also note that, as φ satisfies the growth condition (1), $(\varphi \circ y) \bullet \eta$ lives not only in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^{m_1})$ but even in $L^{p_1}(\Omega; \mathbb{R}^{m_1})$.

Supposing still that, for some $a_2 \in L^{q/(q-1)}(\Omega)$ and $b_2, c_2 \geq 0$,

$$|\varphi(x, r_1, s) - \varphi(x, r_2, s)| \leq (a_2(x) + b_2|r_1|^{q-1} + b_2|r_2|^{q-1} + c_2|s|^{p(q-1)/q})|r_1 - r_2| \quad (4)$$

$$|\varphi_0(x, r_1, s) - \varphi_0(x, r_2, s)| \leq (a_2(x) + b_2|r_1|^{q-1} + b_2|r_2|^{q-1} + c_2|s|^{p(q-1)/q})|r_1 - r_2| \quad (5)$$

the two-argument Nemytskii mappings $(y, u) \mapsto \varphi(\cdot, y(\cdot), u(\cdot)) : L^q(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m) \rightarrow L^{p_1}(\Omega; \mathbb{R}^{m_1})$ and $(y, u) \mapsto \varphi_0(\cdot, y(\cdot), u(\cdot)) : L^q(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m) \rightarrow L^1(\Omega)$ admit jointly (norm $\times$ weak*, weak*)-continuous extensions $(y, \eta) \mapsto (\varphi \circ y) \bullet \eta : L^q(\Omega; \mathbb{R}^n) \times Y_H^p(\Omega; \mathbb{R}^m) \rightarrow L^{p_1}(\Omega; \mathbb{R}^{m_1})$ and $(y, \eta) \mapsto (\varphi_0 \circ y) \bullet \eta : L^q(\Omega; \mathbb{R}^n) \times Y_H^p(\Omega; \mathbb{R}^m) \rightarrow \mathcal{M}(\bar{\Omega})$, respectively.

Then the relaxed problem, created by a continuous extension of (P), takes the form

$$(RP) \quad \begin{cases} \text{Minimize} & \int_{\Omega} (\varphi_0 \circ y) \bullet \eta \, dx \\ \text{subject to} & Ay = (\varphi \circ y) \bullet \eta \text{ on } \Omega, \\ & y \in L^q(\Omega; \mathbb{R}^n), \quad \eta \in \bar{U}_{ad} = w^*\text{-cl } i_H(U_{ad}) \subset Y_H^p(\Omega; \mathbb{R}^m), \end{cases}$$

where $U_{ad} = \{u \in L^p(\Omega; \mathbb{R}^m); u(x) \in S(x) \text{ for a.a. } x \in \Omega\}$ denotes the set of admissible controls for (P). We will suppose that the extended state equation $Ay = (\varphi \circ y) \bullet \eta$ has, for any $\eta \in Y_H^p(\Omega; \mathbb{R}^m)$, precisely one solution $y = \bar{\pi}(\eta) \in \mathcal{D}_A$ and

$$\bar{\pi} : Y_H^p(\Omega; \mathbb{R}^m) \rightarrow L^q(\Omega; \mathbb{R}^n) \text{ is continuous, } \bar{\pi} \circ i_H = \pi. \quad (6)$$

Then the coercivity (3) together with the local compactness of $Y_H^p(\Omega; \mathbb{R}^m)$, density of U_{ad} (imbedded via i_H) in $\bar{U}_{ad}$ and the continuity of all data involved in (RP) enable us to build in a standard way an existence, a stability, and an approximation theories for the relaxed problem, as well as to show that (RP) is actually a correct relaxation (P). In particular:

Proposition 1. *Let (1)–(6) be fulfilled. Then:*

1. $\inf(P) = \min(RP)$.
2. *If $\{u_k\}_{k \in \mathbb{N}}$ is a minimizing sequence for (P), then every weak* cluster point of $\{i_H(u_k)\}_{k \in \mathbb{N}}$ is an optimal relaxed control for (RP).*
3. *For every optimal relaxed control η for (RP) there is a minimizing sequence $\{u_k\}_{k \in \mathbb{N}}$ for (P) such that $\eta = w^*\text{-}\lim_{k \rightarrow \infty} i_H(u_k)$.*

An important property of elements of $Y_H^p(\Omega; \mathbb{R}^m)$ is their possible “nonconcentration”. More precisely, we say that $\eta \in Y_H^p(\Omega; \mathbb{R}^m)$ is p -nonconcentrating if it is attainable by a

sequence $\{u_k\}_{k \in \mathbb{N}}$ (i.e. $\eta = w^* \text{-}\lim_{k \rightarrow \infty} i_H(u_k)$) such that the set $\{|u_k|^p; k \in \mathbb{N}\}$ is not only bounded in $L^1(\Omega)$ but even relatively weakly compact in $L^1(\Omega)$. Using Ball's theorem [2], it is possible to show [13] that every p -nonconcentrating η admits an L^p -Young-measure representation [14] in the sense that there is a weakly measurable mapping $x \mapsto \nu_x$ from Ω to the set of all probability Radon measures on $\mathbb{R}^m$ such that $x \mapsto \int_{\mathbb{R}^m} |s|^p \nu_x(ds)$ belongs to $L^1(\Omega)$ and

$$\forall h \in H: \quad \langle \eta, h \rangle = \int_{\Omega} \int_{\mathbb{R}^m} h(x, s) \nu_x(ds) dx. \quad (7)$$

Then also $[h \bullet \eta](x) = \int_{\mathbb{R}^m} h(x, s) \nu_x(ds)$ for a.a. $x \in \Omega$.

We say that $\hat{\eta} \in Y_H^p(\Omega; \mathbb{R}^m)$ is a p -nonconcentrating modification of a given $\eta \in Y_H^p(\Omega; \mathbb{R}^m)$ if $\hat{\eta}$ is p -nonconcentrating and $\langle \hat{\eta}, h \rangle = \langle \eta, h \rangle$ for any $h \in H$ such that $|h(x, s)| \leq a(x) + o(|s|^p)$ with some $a \in L^1(\Omega)$ and $o: \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying $\lim_{r \rightarrow \infty} o(r)/r = 0$. It is a nontrivial result, based on Chacon's biting lemma [7] and the Dunford-Pettis theorem, that every $\eta \in Y_H^p(\Omega; \mathbb{R}^m)$ possesses precisely one p -nonconcentrating modification. Also one can prove that $\langle \eta - \hat{\eta}, h \rangle > 0$ provided $\eta \neq \hat{\eta}$ and $h \in H$ is coercive in the sense $h(x, s) \geq a_0(x) + b|s|^p$ with some $a_0 \in L^1(\Omega)$ and $b > 0$.

Let us assume, for a moment, that an optimal relaxed control η for (RP) is not p -nonconcentrating. This means $\eta \neq \hat{\eta}$. Due to our special form of the set of the admissible controls U_{ad} , one can show that $\eta \in \bar{U}_{ad}$ implies $\hat{\eta} \in \bar{U}_{ad}$. Thanks to (1), $|\langle \varphi \circ y \rangle(x, \cdot)|$ has the growth lesser than p , namely p/p_1 . Therefore $(\varphi \circ y) \bullet \eta = (\varphi \circ y) \bullet \hat{\eta}$, which yields $\bar{\pi}(\eta) = \bar{\pi}(\hat{\eta})$. Due to the coercivity (3), we have $\int_{\bar{\Omega}} (\varphi_0 \circ y) \bullet \eta \, dx = \langle \eta, \varphi_0 \circ y \rangle > \langle \hat{\eta}, \varphi_0 \circ y \rangle = \int_{\bar{\Omega}} (\varphi_0 \circ y) \bullet \hat{\eta} \, dx$. Therefore, $\hat{\eta}$ is an admissible relaxed control that achieves a strictly lower cost than η , which demonstrates that η cannot be optimal, a contradiction. This shows:

Proposition 2. *Let (1)–(6) be valid and η be an optimal relaxed control for (RP). Then η is p -nonconcentrating.*

This assertion also implies that the minimizing sequences for the original problems (P) do not concentrate “energy”:

Proposition 3. *Let (1)–(6) be valid and $\{u_k\}_{k \in \mathbb{N}} \subset U_{ad}$ be a minimizing sequence for (P), i.e. $\lim_{k \rightarrow \infty} \int_{\bar{\Omega}} \varphi_0(x, \pi(u_k), u_k) dx = \inf(P)$. Then the set $\{|u_k|^p; k \in \mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$.*

3 FIRST-ORDER OPTIMALITY CONDITIONS

The relaxed problem consists, in fact, in minimization of the cost functional $J(\eta) = \int_{\bar{\Omega}} (\varphi_0 \circ \bar{\pi}(\eta)) \bullet \eta \, dx$ over the set $\bar{U}_{ad}$ assumed now to be convex, which is basically a slight requirement of H and S . If J is Gâteaux differentiable, the first-order necessary optimality condition for $\eta \in \bar{U}_{ad}$ to be an optimal relaxed control for (RP) is

$$\nabla J(\eta) \in -N_{\bar{U}_{ad}}(\eta), \quad (8)$$

where ∇J denotes the Gâteaux differential of J and $N_{\bar{U}_{ad}}(\eta)$ the normal cone to $\bar{U}_{ad}$ at the point η . To ensure the smoothness of the extended problem, we have to suppose that both

$\varphi(x, \cdot, s) : \mathbb{R}^n \rightarrow \mathbb{R}^{m_1}$ and $\varphi_0(x, \cdot, s) : \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable and the derivatives, denoted by $\varphi' = (\frac{\partial \varphi_k}{\partial r_i})_{i=1, \dots, n}^{k=1, \dots, m_1}$ and $\varphi'_0 = (\frac{\partial \varphi_0}{\partial r_i})_{i=1, \dots, n}$, satisfy $g \cdot (\varphi' \circ y) \cdot \tilde{y} \in H$ and $g_0 \cdot (\varphi'_0 \circ y) \cdot \tilde{y} \in H$ for any $g \in L^{p_1/(p_1-1)}(\Omega; \mathbb{R}^{m_1})$, $g_0 \in C(\bar{\Omega})$ and $\tilde{y} \in L^q(\Omega; \mathbb{R}^n)$. Moreover, for some $a \in L^{q p_1/(q-p_1)}(\Omega)$, $a_0 \in L^{q/(q-1)}(\Omega)$, $a' \in L^{q p_1/(q-2 p_1)}(\Omega)$, $a'_0 \in L^{q/(q-2)}(\Omega)$ and $b, c < +\infty$, the derivatives φ' and φ'_0 satisfy

$$|\varphi'(x, r, s)| \leq a(x) + b|r|^{(q-p_1)/p_1} + c|s|^{p(q-p_1)/q p_1}, \quad (9)$$

$$|\varphi'_0(x, r, s)| \leq a_0(x) + b|r|^{q-1} + c|s|^{p(q-1)/q}, \quad (10)$$

$$|\varphi'(x, r_1, s) - \varphi'(x, r_2, s)| \leq (a'(x) + b|r_1|^{(q-2 p_1)/p_1} + b|r_2|^{(q-2 p_1)/p_1} + c|s|^{p(q-2 p_1)/q p_1})|r_1 - r_2| \quad (11)$$

$$|\varphi'_0(x, r_1, s) - \varphi'_0(x, r_2, s)| \leq (a'_0(x) + b|r_1|^{q-2} + b|r_2|^{q-2} + c|s|^{p(q-2)/q})|r_1 - r_2|. \quad (12)$$

If $q \geq 2 p_1$ and (9)–(12) is valid, the extended Nemytskii mappings $(y, \eta) \mapsto (\varphi \circ y) \cdot \eta$ and $(y, \eta) \mapsto (\varphi_0 \circ y) \cdot \eta$ possess Gâteaux differentials which are, if H is normed suitably, locally Lipschitz continuous. Then J is Gâteaux differentiable and, evaluating (8) by the adjoint equation technique, one obtains the following maximum principle:

Proposition 4. *Let $q \geq 2 p_1$, (1)–(6) and (9)–(12) be valid, let (y, η) solve (RP), and let there is $\lambda \in L^{p_1/(p_1-1)}(\Omega; \mathbb{R}^{m_1})$ solving the adjoint equation*

$$A^* \lambda = [(\varphi' \circ y) \cdot \eta]^T \lambda - (\varphi'_0 \circ y) \cdot \eta. \quad (13)$$

Then the integral maximum principle

$$\int_{\bar{\Omega}} [h_{y, \lambda} \cdot \eta](dx) = \sup_{u \in U_{ad}} \int_{\Omega} h_{y, \lambda}(x, u(x)) dx \quad (14)$$

holds, where the “Hamiltonian” $h_{y, \lambda} \in H$ is defined by $h_{y, \lambda} = -\varphi_0 \circ y + \lambda \cdot (\varphi \circ y)$. Moreover, if also the coercivity (3) holds, then (14) can be localized to the point-wise (also called Pontryagin) maximum principle

$$h_{y, \lambda} \cdot \eta = \max_{s \in \mathbb{R}^m} h_{y, \lambda}(\cdot, s) \quad \text{in the sense of } L^1(\Omega). \quad (15)$$

We saw in Proposition 2 that (RP) possesses typically a p -nonconcentrating solutions which have, due to (7), a Young-measure representation. We say that η is k -atomic if it has the Young-measure representation ν which is k -atomic, i.e. for a.a. $x \in \Omega$ the probability measure ν_x is a convex combination of at most k Dirac masses. The nonconcentration principle enables us to exploit in detail the maximum principle (15):

Proposition 5. *Let $q \geq 2 p_1$, (1)–(6) and (9)–(12) be valid. Then (RP) always possesses some solution such that η is $(m_1 + 1)$ -atomic. Moreover, if the Hamiltonian $h_{y, \lambda}(x, \cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$ attains its maximum at no more than k -points, then every solution to (RP) is k -atomic.*

Let us remark that information about a finite number of atoms can be exploited in numerical implementation of (RP).

4 ILLUSTRATION: AN ELLIPTIC OPTIMAL CONTROL PROBLEM

The general problem (P) covers, in particular, the following optimal control problem for a system of elliptic equations with homogeneous Neumann boundary conditions:

$$(P_1) \left\{ \begin{array}{l} \text{Minimize} \quad \int_{\Omega} \varphi_0(x, y(x), u(x)) dx \\ \text{subject to} \quad \sum_{j,l=1}^{n_1} \sum_{i=1}^n \frac{\partial}{\partial x_l} (c_{ijkl}(x) \frac{\partial y_i}{\partial x_j}) = \varphi_k(x, y(x), u(x)) \quad \text{on } \Omega, \quad k = 1, \dots, n \\ \sum_{j,l=1}^{n_1} \sum_{i=1}^n \nu_l c_{ijkl}(x) \frac{\partial y_i}{\partial x_j} = 0 \quad \text{on } \partial\Omega, \quad k = 1, \dots, n \\ u(x) \in S(x) \quad \text{on } \Omega, \\ y \in W^{1,2}(\Omega; \mathbb{R}^n), \quad u \in L^p(\Omega; \mathbb{R}^m), \end{array} \right.$$

where $\nu = (\nu_1, \dots, \nu_{n_1})$ denotes the unit outward normal to the boundary $\partial\Omega$. We will suppose the following symmetry condition:

$$c_{ijkl}(x) = c_{klij}(x), \quad \frac{\partial \varphi_k(x, r, s)}{\partial r_i} = \frac{\partial \varphi_i(x, r, s)}{\partial r_k}. \quad (16)$$

Under this condition, y solves the state problem in (P_1) (in the weak sense) if it minimizes the functional Φ_u defined by

$$\Phi_u(y) = \int_{\Omega} \left(\frac{1}{2} \sum_{j,l=1}^{n_1} \sum_{i,k=1}^n c_{ijkl}(x) \frac{\partial y_i}{\partial x_j} \frac{\partial y_k}{\partial x_l} + \hat{\varphi}(x, y(x), u(x)) \right) dx$$

over the Sobolev space $W^{1,2}(\Omega; \mathbb{R}^n)$, where $\hat{\varphi} : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is given by the formula $\hat{\varphi}(x, r, s) = \int_0^1 \sum_{k=1}^n r_k \varphi_k(x, \tau r, s) d\tau$. Let us still suppose that, for some ε positive and every matrix $\xi = [\xi_{ij}] \in \mathbb{R}^{n \times n_1}$, the coefficients $c_{ijkl} \in L^\infty(\Omega)$ satisfy

$$\sum_{j,l=1}^{n_1} \sum_{i,k=1}^n c_{ijkl} \xi_{ij} \xi_{kl} \geq \varepsilon \|\xi\|^2 \quad (17)$$

and that φ satisfies

$$\sum_{k=1}^n r_k \varphi_k(x, r, s) \geq \varepsilon |r|^2, \quad \sum_{k=1}^n (\varphi_k(x, r, s) - \varphi_k(x, \tilde{r}, s))(r_k - \tilde{r}_k) \geq \varepsilon |r - \tilde{r}|^2. \quad (18)$$

Then Φ is coercive and uniformly convex, which ensures uniqueness and continuous dependence on u of the solution $y = \pi(u)$ to the state problem in (P_1) provided $p_1 > 2n/(n+2)$ which just ensures the imbedding $L^{p_1}(\Omega; \mathbb{R}^n) \subset W^{-1,2}(\Omega; \mathbb{R}^n)$.

It is clear that (P_1) takes the form (P) with $n = m_1$, $q < 2n_1/(n_1 - 2)$, $\mathcal{D}_A = W^{1,q}(\Omega; \mathbb{R}^n)$, and $p_1 > 2n_1/(n_1 + 2)$ so that we can immediately apply the preceding results to (P_1) . Then, under the assumptions (1)–(2) and (4)–(5), the relaxed problem (RP_1) looks as

$$(RP_1) \quad \left\{ \begin{array}{ll} \text{Minimize} & \int_{\Omega} (\varphi_0 \circ y) \bullet \eta \, dx \\ \text{subject to} & \sum_{j,l=1}^{n_1} \sum_{i=1}^n \frac{\partial}{\partial x_l} (c_{ijkl} \frac{\partial y_i}{\partial x_j}) = (\varphi_k \circ y) \bullet \eta \quad \text{on } \Omega, \quad k = 1, \dots, n \\ & \sum_{j,l=1}^{n_1} \sum_{i=1}^n \nu_l c_{ijkl} \frac{\partial y_i}{\partial x_j} = 0 \quad \text{on } \partial\Omega, \quad k = 1, \dots, n \\ & u(x) \in S(x) \quad \text{on } \Omega, \\ & y \in W^{1,2}(\Omega; \mathbb{R}^n), \quad \eta \in \bar{U}_{ad} = w^*\text{-cl } i_H(U_{ad}) \subset Y_H^p(\Omega; \mathbb{R}^m), \end{array} \right.$$

We can apply Proposition 1 to see that (RP_1) is actually a correct relaxation scheme for (P_1) . The nonconcentration of every optimal relaxed control for (RP_1) can be obtained by Proposition 2, while Proposition 3 yields nonconcentration of energy of every minimizing sequence for (P_1) . We will additionally suppose $2p_1 \leq q$; note that this restriction basically enables us to treat at most five-dimensional problems, i.e. $n_1 \leq 5$ (if boundary controls were considered, even $n_1 \leq 3$ would have to be supposed). Then Proposition 4 justifies the Pontryagin maximum principle (15) involving the Hamiltonian $h_{y,\lambda} \in H$ given by

$$h_{y,\lambda}(x, s) = -\varphi_0(x, y(x), s) + \sum_{k=1}^n \lambda_k(x) \varphi_k(x, y(x), s)$$

with $\lambda \in W^{1,2}(\Omega; \mathbb{R}^n)$ solving the adjoint elliptic system

$$\left. \begin{array}{ll} \sum_{j,l=1}^{n_1} \sum_{i=1}^n \frac{\partial}{\partial x_l} (c_{ijkl} \frac{\partial \lambda_i}{\partial x_j}) = \sum_{i=1}^n [(\frac{\partial \varphi_k}{\partial r_i} \circ y) \bullet \eta] \lambda_i - (\frac{\partial \varphi_0}{\partial r_i} \circ y) \bullet \eta & \text{on } \Omega, \\ \sum_{j,l=1}^{n_1} \sum_{i=1}^n \nu_l c_{ijkl} \frac{\partial \lambda_i}{\partial x_j} = 0 & \text{on } \partial\Omega. \end{array} \right\} \quad (19)$$

with $k = 1, \dots, n$. Finally, by Proposition 5 we can claim that at least one optimal relaxed control for (RP_1) is $(n + 1)$ -atomic. Let us emphasize that our assumptions here, namely (16)–(18), ensures automatically (6) as well as the existence of the adjoint state λ which had to be supposed in Propositions 1–5.

The above outlined example reproduces (and sometimes generalizes) standard results, e.g. by Alibert and Raymond (preprint), Bonnans and Casas (1991, 1992, 1995), Bonnans and Tiba (1991), Casas (1993, to appear), Casas and Fernández (1991, 1995), Mackenroth (1986), Raitums (1986)], or Zolezzi (1972) where only scalar case (i.e. $n = 1$) has been considered without any relaxation, i.e. the maximum principle has been obtained only for ordinary optimal controls whose existence is, however, guaranteed only if the problem has a convex/linear structure. The presented example can be also modified for a boundary control (like in Alibert and Raymond (preprint), Bonnans and Casas (1992), Casas and Fernandez (1995) or nonlinear operator A (like in Casas and Fernandez (1995)).

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