Strong Pontryagin's principle for state-constrained control problems governed by parabolic equations

Eduardo Casas

*Dpto. de Matemática Aplicada y Ciencias de la Computación,
E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria,
Av. Los Castros s/n, 39071 Santander, Spain. Phone: 34-42-201427.
Fax: 34-42-201829. E-mail: casas@macc.unican.es*

Abstract

This paper deals with state-constrained optimal control problems governed by semilinear parabolic equations. We establish a minimum principle of Pontryagin's type in a qualified form, the so-called strong principle. To achieve this result we formulate two different assumptions, any of them leading to the desired result. One of the assumptions is an extension of Slater's condition. Since no hypothesis of differentiability with respect to the control, even continuity, of the functions involved in the control problem is made, the generalization of Slater's conditions is not obvious. The second assumption is announced in terms of a certain Lipschitz dependence of the optimal cost functional with respect to small perturbations of the set of feasible states.

Keywords

Pontryagin's principle, Slater's condition, state constraints, parabolic equations

1 SETTING OF THE CONTROL PROBLEM

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open and bounded set, with Lipschitz boundary Γ . Given $0 < T < +\infty$, we set $\Omega_T = \Omega \times (0, T)$ and $\Sigma_T = \Gamma \times (0, T)$. Let $(\mathcal{K}, d)$ be a metric space and let us consider a function $f : \Sigma_T \times \mathbb{R} \times \mathcal{K} \rightarrow \mathbb{R}$ of class C^1 with respect to the second variable and satisfying the following assumptions:

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial y}(x, t, y, u) \leq 0 \quad \forall (x, t, y, u) \in \Sigma_T \times \mathbb{R} \times \mathcal{K}; \\ \forall M > 0 \exists C_M > 0 \text{ such that } \forall (x, t, u) \in \Sigma_T \times \mathcal{K} \text{ and } |y| \leq M \\ |f(x, t, 0, u)| + \left| \frac{\partial f}{\partial y}(x, t, y, u) \right| \leq C_M. \end{array} \right. \quad (1)$$

The state equation is as follows

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) + Ay(x, t) + a_0(x, t, y(x, t)) = 0 & \text{in } \Omega_T, \\ \partial_{\nu_A} y(x, t) = f(x, t, y(x, t), u(x, t)) & \text{on } \Sigma_T, \\ y(x, 0) = y_0(x) & \text{in } \Omega, \end{cases} \quad (2)$$

where $y_0 \in C(\bar{\Omega})$, A is a linear operator with bounded coefficients and satisfying the usual ellipticity condition

$$\begin{aligned} Ay = & - \sum_{j=1}^n \partial_{x_j} \left\{ \sum_{i=1}^n [a_{ij}(x, t) \partial_{x_i} y(x, t)] + b_j(x, t) y(x, t) \right\} \\ & + \sum_{j=1}^n d_j(x, t) \partial_{x_j} y(x, t) + c(x, t) y(x, t) \end{aligned} \quad (3)$$

$$\partial_{\nu_A} y(x, t) = \sum_{j=1}^n \left\{ \sum_{i=1}^n [a_{ij}(x, t) \partial_{x_i} y(x, t)] + b_j(x, t) y(x, t) \right\} \nu_j(x), \quad (4)$$

$\nu(x)$ being the outward unit normal vector to Γ at the point x . Function $a_0 : \Omega_T \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function of class C^1 with respect to the second variable and satisfies the following assumptions

$$\begin{cases} \exists \psi_0 \in L^{\hat{p}}([0, T], L^{\hat{q}}(\Omega)) \text{ and } C_1 > 0 \text{ such that} \\ a_0(x, t, y) y \geq \psi_0(x, t) - C_1 y^2 \quad \forall (x, t, y) \in \Omega_T \times \mathbb{R}; \end{cases} \quad (5)$$

$$\begin{cases} a_0(\cdot, \cdot, 0) \in L^{\hat{p}}([0, T], L^{\hat{q}}(\Omega)) \text{ and } \forall M > 0 \exists C_M > 0 \text{ such that} \\ \left| \frac{\partial a_0}{\partial y}(x, t, y) \right| \leq C_M \quad \forall (x, t) \in \Omega_T, |y| \leq M; \end{cases} \quad (6)$$

where $\hat{q}, \hat{p} \in [1, +\infty]$ and $\frac{1}{\hat{p}} + \frac{n}{2\hat{q}} < 1$.

Once given the state equation, we introduce the cost functional

$$J(u) = \int_{\Omega_T} L(x, t, y_u(x, t)) dx dt + \int_{\Sigma_T} l(x, t, y_u(x, t), u(x, t)) d\sigma(x) dt,$$

where y_u is the solution of (2) associated to u ; $L : \Omega_T \times \mathbb{R} \rightarrow \mathbb{R}$ and $l : \Sigma_T \times \mathbb{R} \times \mathcal{K} \rightarrow \mathbb{R}$ are of class C^1 with respect to the second variable, measurable with respect to the first one, and satisfying

$$\begin{cases} \forall M > 0 \exists \psi_{dM} \in L^1(\Omega_T) \text{ such that } \forall (x, t) \in \Omega_T, |y| \leq M \\ |L(x, t, 0)| + \left| \frac{\partial L}{\partial y}(x, t, y) \right| \leq \psi_{dM}(x, t), \end{cases} \quad (7)$$

$$\begin{cases} \forall M > 0 \exists \psi_{bM} \in L^1(\Sigma_T) \text{ such that } \forall (x, t, u) \in \Sigma_T \times \mathcal{K}, |y| \leq M \\ \left| l(x, t, 0, u) + \left| \frac{\partial l}{\partial y}(x, t, y, u) \right| \right| \leq \psi_{bM}(x, t). \end{cases} \quad (8)$$

The space of controls $\mathcal{U}$ is formed by the measurable functions $u : \Sigma_T \rightarrow \mathcal{K}$ such that the mapping $(x, t) \in \Sigma_T \rightarrow (f(x, t, y, u(x, t)), l(x, t, y, u(x, t))) \in \mathbb{R}^2$ is measurable for every $y \in \mathbb{R}$. In $\mathcal{U}$ we consider Ekeland's distance

$$d_E(u, v) = m_{\Sigma_T}(\{(x, t) \in \Sigma_T : u(x, t) \neq v(x, t)\}), \quad (9)$$

where m_{Σ_T} is the measure on Σ_T obtained as the product of σ and the Lebesgue measure in the interval $(0, T)$. It is known (see Ekeland (1979)) that $(\mathcal{U}, d_E)$ is a complete metric space. Finally we formulate the optimal control problem as follows

$$(P_\delta) \text{ Minimize } \{J(u) : u \in \mathcal{U}, g(x, t, y_u(x, t)) \leq \delta \forall (x, t) \in \bar{\Omega}_T\}.$$

Before concluding this section, let us state some results concerning the state equation.

Theorem 1 *Under assumptions (1)–(6), problem (2) has a unique solution in $Y = C(\bar{\Omega}_T) \cap L^2([0, T], H^1(\Omega))$ for every control $u \in \mathcal{U}$. Moreover there exists a constant $M > 0$ such that*

$$\|y_u\|_\infty + \|y_u\|_{L^2([0, T], H^1(\Omega))} \leq M \quad \forall u \in \mathcal{U}. \quad (10)$$

Finally, if $\{u_k\}_{k=1}^\infty \subset \mathcal{U}$ is a sequence converging to u in $\mathcal{U}$, i.e. $d_E(u_k, u) \rightarrow 0$, then $\{y_{u_k}\}_{k=1}^\infty$ converges to y_u strongly in Y .

Theorem 2 *Let $u, v \in \mathcal{U}$. Given $\rho \in (0, 1)$, there exist m_{Σ_T} -measurable sets $E_\rho \subset \Sigma_T$, with $m_{\Sigma_T}(E_\rho) = \rho m_{\Sigma_T}(\Sigma_T)$, such that if we define $u_\rho = u + [v - u]\chi_{E_\rho}$, χ_{E_ρ} being the characteristic function on the set E_ρ , and if we denote by y_ρ and y the states corresponding to u_ρ and u , respectively, then the following equalities hold*

$$y_\rho = y + \rho z + r_\rho, \quad \lim_{\rho \rightarrow 0} \frac{1}{\rho} \|r_\rho\|_Y = 0, \quad (11)$$

$$J(u_\rho) = J(u) + \rho z^0 + r_\rho^0, \quad \lim_{\rho \rightarrow 0} \frac{1}{\rho} r_\rho^0 = 0, \quad (12)$$

where $z \in Y$ and $z_0 \in \mathbb{R}$ satisfy

$$\begin{cases} \frac{\partial z}{\partial t} + Az + \frac{\partial a_0}{\partial y}(x, t, y(x, t))z = 0 & \text{in } \Omega_T \\ \partial_{\nu_A} z = \frac{\partial f}{\partial y}(x, t, y(x, t), u(x, t))z \\ + f(x, t, y(x, t), v(x, t)) - f(x, t, y(x, t), u(x, t)) & \text{on } \Sigma_T, \\ z(x, 0) = 0 & \text{in } \Omega \end{cases} \quad (13)$$

$$\begin{aligned}
z^0 &= \int_{\Omega_T} \frac{\partial L}{\partial y}(x, t, y(x, t)) z(x, t) dx dt + \int_{\Sigma_T} \frac{\partial l}{\partial y}(x, t, y(x, t), u(x, t)) z(x, t) d\sigma(x) dt \\
&+ \int_{\Sigma_T} [l(x, t, y(x, t), v(x, t)) - l(x, t, y(x, t), u(x, t))] d\sigma(x) dt.
\end{aligned} \tag{14}$$

The reader is referred to Casas (1995) for a proof of these theorems.

2 PONTRYAGIN'S PRINCIPLE

Given $\alpha \geq 0$, we define the Hamiltonian $H_\alpha : \Sigma_T \times \mathbb{R} \times \mathcal{K} \times \mathbb{R} \longrightarrow \mathbb{R}$ as follows

$$H_\alpha(x, t, y, u, \varphi) = \alpha l(x, t, y, u) + \varphi f(x, t, y, u).$$

Theorem 3 *If $\bar{u} \in \mathcal{U}$ is a solution of (P_δ) , then there exist $\bar{\alpha} \geq 0$, $\bar{y} \in C(\bar{\Omega}_T) \cap L^2([0, T], H^1(\Omega))$, $\bar{\varphi} \in L^r([0, T], W^{1,p}(\Omega))$, for all $p, r \in [1, 2)$ with $(2/r) + (n/p) > n + 1$, $\bar{\mu} \in Z'$ and $\bar{\lambda} \in R'$ such that*

$$\bar{\alpha} + \|\bar{\mu}\|_{Z'} + |\bar{\lambda}| > 0; \tag{15}$$

$$\begin{cases} \frac{\partial \bar{y}}{\partial t} + A\bar{y} + a_0(x, t, \bar{y}(x, t)) = 0 & \text{in } \Omega_T, \\ \partial_{\nu_A} \bar{y}(x, t) = f(x, t, \bar{y}(x, t), \bar{u}(x, t)) & \text{on } \Sigma_T, \\ \bar{y}(0) = y_0 & \text{in } \Omega; \end{cases} \tag{16}$$

$$\begin{cases} -\frac{\partial \bar{\varphi}}{\partial t} + A^* \bar{\varphi} + \frac{\partial a_0}{\partial y}(x, t, \bar{y}) \bar{\varphi} = \bar{\alpha} \frac{\partial L}{\partial y}(x, t, \bar{y}) + \frac{\partial g}{\partial y}(x, t, \bar{y}) \bar{\mu}|_{\Omega_T} & \text{in } \Omega_T, \\ \partial_{\nu_A} \bar{\varphi} = \frac{\partial f}{\partial y}(x, t, \bar{y}, \bar{u}) \bar{\varphi} + \bar{\alpha} \frac{\partial l}{\partial y}(x, t, \bar{y}, \bar{u}) + \frac{\partial g}{\partial y}(x, t, \bar{y}) \bar{\mu}|_{\Sigma_T} & \text{on } \Sigma_T, \\ \bar{\varphi}(T) = \frac{\partial g}{\partial y}(x, t, \bar{y}) \bar{\mu}|_{\bar{\Omega} \times \{T\}} & \text{in } \bar{\Omega}; \end{cases} \tag{17}$$

$$\int_{\bar{\Omega}_T} [z(x, t) - g(x, t, \bar{y}(x, t))] d\bar{\mu}(x, t) \leq 0 \quad \forall z \in C(\bar{\Omega}_T), \text{ with } z(x, t) \leq \delta; \tag{18}$$

$$\begin{aligned}
&\int_{\Sigma_T} H_{\bar{\alpha}}(x, t, \bar{y}(x, t), \bar{u}(x, t), \bar{\varphi}(x, t)) d\sigma(x) dt \\
&= \min_{u \in \mathcal{U}} \int_{\Sigma_T} H_{\bar{\alpha}}(x, t, \bar{y}(x, t), u(x, t), \bar{\varphi}(x, t)) d\sigma(x) dt;
\end{aligned} \tag{19}$$

where A^* denotes the formal adjoint operator of A . Moreover, if one of the following assumptions is satisfied:

A1) Functions f and l are continuous with respect to the third variable on $(\mathcal{K}, d)$ and this space is separable;

A2) There exists a set $\Sigma_T^0 \subset \Sigma_T$, with $m_{\Sigma_T}(\Sigma_T^0) = m_{\Sigma_T}(\Sigma_T)$, such that the function

$$(x, t) \in \Sigma_T \longrightarrow (f(x, t, y, u), l(x, t, y, u)) \in \mathbb{R}^2$$

is continuous in Σ_T^0 for every $(y, u) \in \mathbb{R} \times \mathcal{K}$;

then the following pointwise relation holds a.e. $[\sigma]$ $x \in \Gamma$ and a.e. $t \in [0, T]$

$$H_{\bar{\alpha}}(x, t, \bar{y}(x, t), \bar{u}(x, t), \bar{\varphi}(x, t)) = \min_{u \in \mathcal{K}} H_{\bar{\alpha}}(x, t, \bar{y}(x, t), u, \bar{\varphi}(x, t)). \quad (20)$$

See Casas (1995) for the proof of this theorem. Our aim is to formulate some conditions under which the theorem remains to be true with $\bar{\alpha} = 1$. In this section we introduce a Slater's type hypothesis that leads to the desired result. In the next section, we will develop another method to achieve the same goal.

Slater Assumption: There exists $u_0 \in \mathcal{U}$ such that

$$g(x, t, \bar{y}(x, t)) + \frac{\partial g}{\partial y}(x, t, \bar{y}(x, t))z_0(x, t) < \delta \quad \forall (x, t) \in \bar{\Omega}_T, \quad (21)$$

$z_0 \in C(\bar{\Omega}_T) \cap L^2([0, T], H^1(\Omega))$ is the solution of (13), with $u_0 = v$, $\bar{u} = u$ and $\bar{y} = y$.

Corollary 1 *If $\bar{u}$ is a solution of (P_{δ}) and Slater's assumption holds, then Theorem 3 remains to be true with $\bar{\alpha} = 1$.*

Proof. If $\bar{\alpha} > 0$, then we can divide relations (16)–(18) by $\bar{\alpha}$ and rename $\bar{\varphi}/\bar{\alpha}$ and $\bar{\mu}/\bar{\alpha}$ as $\bar{\varphi}$ and $\bar{\mu}$, respectively, obtaining the searched result. Let us prove that Slater's assumption implies the strict positivity of $\bar{\alpha}$.

Let us assume that $\bar{\alpha} = 0$, then (14) implies that $\bar{\mu} \neq 0$. Since $\bar{\mu} \neq 0$, the inequality (18) is strict for every $z \in C(\bar{\Omega}_T)$ with $z(x, t) < \delta \quad \forall (x, t) \in \bar{\Omega}_T$, in particular (21) leads to

$$\begin{aligned} \int_{\bar{\Omega}_T} \frac{\partial g}{\partial y}(x, t, \bar{y}(x, t))z_0(x, t)d\bar{\mu}(x, t) = \\ \int_{\bar{\Omega}_T} \left\{ \left[g(x, t, \bar{y}(x, t)) + \frac{\partial g}{\partial y}(x, t, \bar{y}(x, t))z_0(x, t) \right] - g(x, t, \bar{y}(x, t)) \right\} d\bar{\mu}(x, t) < 0. \end{aligned} \quad (22)$$

On the other hand, (19) along with (13) and (16) gives

$$0 \geq \int_{\Sigma_T} [H_{\bar{\alpha}}(x, t, \bar{y}(x, t)u_0(x, t), \bar{\varphi}(x, t)) - H_{\bar{\alpha}}(x, t, \bar{y}(x, t)\bar{u}(x, t), \bar{\varphi}(x, t))]d\sigma(x)dt =$$

$$\int_{\Sigma_T} \bar{\varphi}(x, t)[f(x, t, \bar{y}(x, t), u_0(x, t)) - f(x, t, \bar{y}(x, t), \bar{u}(x, t))]d\sigma(x)dt =$$

$$\int_{\bar{\Omega}_T} \frac{\partial g}{\partial y}(x, t, \bar{y}(x, t)) z_0(x, t) d\bar{\mu}(x, t). \quad (23)$$

Thus (22) and (23) prove that $\bar{\alpha}$ can not be zero. $\square$

A similar assumption to the above one was used by Fattorini and Frankowska (1991) to derive the optimality conditions in a qualified form.

3 THE STABILITY ASSUMPTION

In this section we introduce a stability notion which allows to prove a qualified Pontryagin's principle. Here the qualification is not a consequence of Theorem 3, but it requires a new proof of this theorem with $\bar{\alpha} = 1$.

Definition 1 *We say that (P_δ) is strongly stable if there exist $\epsilon > 0$ and $C > 0$ such that*

$$\inf(P_\delta) - \inf(P_{\delta'}) \leq C(\delta' - \delta) \quad \forall \delta' \in [\delta, \delta + \epsilon]. \quad (24)$$

This concept was first introduced in relation with optimal control problems by Bonnans (1991); see also Bonnans and Casas (1995). A weaker stability concept was used by Casas (1992) to analyze the convergence of the numerical discretizations of optimal control problems. The following proposition states that almost all problems (P_δ) are strongly stable.

Proposition 1 *Let $\delta_0 \geq 0$ be the smallest number such that (P_δ) has feasible controls for every $\delta > \delta_0$. Then (P_δ) is strongly stable for all $\delta > \delta_0$ except at most a zero Lebesgue measure set.*

Proof. It is enough to consider the function $h : (\delta_0, +\infty) \rightarrow \mathbb{R}$ defined by $h(\delta) = \inf(P_\delta)$ and remark that it is a nonincreasing monotone function and, consequently, differentiable at every point of $(\delta_0, +\infty)$ except at a zero measure set. Now it is easy to check that (P_δ) is strongly stable at every point where h is differentiable. $\square$

Theorem 4 *If (P_δ) is strongly stable and $\bar{u}$ is a solution of this problem, then Theorem 3 holds with $\bar{\alpha} = 1$.*

The proof of this theorem is carried out by showing that the stability assumption implies the possibility of an exact penalization of the state constraints. Indeed, let us define

$$Q_\delta = \{z \in C(\bar{\Omega}_T) : z(x, t) \leq \delta \quad \forall (x, t) \in \bar{\Omega}_T\}.$$

Since $C(\bar{\Omega}_T)$ is separable, we can take in $C(\bar{\Omega}_T)$ a norm $\|\cdot\|$ equivalent to the usual such that the distance function $d_{Q_\delta} : C(\bar{\Omega}_T) \rightarrow \mathbb{R}$, $d_{Q_\delta}(z) = \inf\{\|y - z\| : y \in Q_\delta\}$ is convex, Lipschitz and Gâteaux differentiable at every point $z \notin Q_\delta$; see Li and Yong (1995).

Proposition 2 *If (P_δ) is strongly stable and $\bar{u}$ is a solution of this problem, then there exists $q_0 > 0$ such that for every $q \geq q_0$ $\bar{u}$ is also a solution of*

$$\inf_{u \in \mathcal{U}} J_q(u) = J(u) + q d_{Q_\delta}(g(\cdot, y_u)) \quad (25)$$

Proof. Let us suppose that it is false. Then there exists a sequence $\{q_k\}_{k=1}^\infty$ of real numbers, with $q_k \rightarrow +\infty$, and elements $\{u_k\}_{k=1}^\infty \subset \mathcal{U}$ such that

$$J(u_k) + q_k d_{Q_\delta}(g(\cdot, y_k)) < J(\bar{u}) \quad \forall k \geq 1,$$

where y_k is the state corresponding to u_k . From here we obtain that

$$d_{Q_\delta}(g(\cdot, y_k)) < \frac{J(\bar{u}) - J(u_k)}{q_k} \rightarrow 0 \quad \text{when } k \rightarrow +\infty$$

and $g(\cdot, y_k) \notin Q_\delta$. Let $\delta_k > \delta$ be the smallest number such that $g(\cdot, y_k) \in Q_{\delta_k}$. Since $\delta_k \rightarrow \delta$, we can use (24) to deduce

$$C(\delta_k - \delta) \geq \inf(P_\delta) - \inf(P_{\delta_k}) \geq J(\bar{u}) - J(u_k) > q_k d_{Q_\delta}(g(\cdot, y_k)) = q_k(\delta_k - \delta) \quad \forall k \geq k_\epsilon,$$

which is not possible. $\square$

Since J_q is not Gâteaux differentiable on Q_δ , we are going to modify slightly this functional to attain the differentiability necessary for the proof.

Proposition 3 *Let us take $q \geq q_0$ and for every $\epsilon > 0$ let us consider the problem*

$$(P_{\delta, \epsilon}) \quad \inf_{u \in \mathcal{U}} J_{q, \epsilon}(u) = J(u) + q \left\{ d_{Q_\delta}(G(y_u))^2 + \epsilon^2 \right\}^{1/2}.$$

Then $\inf(P_{\delta, \epsilon}) \rightarrow \inf(P_\delta)$ when $\epsilon \rightarrow 0$.

Proof. It is an immediate consequence of the inequality $J_q(u) \leq J_{q, \epsilon}(u) \leq J_q(u) + q\epsilon$ $\forall u \in \mathcal{U}$. $\square$

Proof of Theorem 4. Propositions 2 and 3 imply that $\bar{u}$ is a σ_ϵ^2 -solution of $(P_{\delta, \epsilon})$, with $\sigma_\epsilon \rightarrow 0$ when $\epsilon \rightarrow 0$, i.e. $J_{q, \epsilon}(\bar{u}) \leq \inf(P_{\delta, \epsilon}) + \sigma_\epsilon^2$. Then we can apply again Ekeland's principle and deduce the existence of an element $u^\epsilon \in \mathcal{U}$ such that

$$d(u^\epsilon, \bar{u}) \leq \sigma_\epsilon, \quad J_{q, \epsilon}(u^\epsilon) \leq J_{q, \epsilon}(\bar{u}) + \sigma_\epsilon d_E(u^\epsilon, \bar{u}) \quad \forall u \in \mathcal{U}.$$

Now we can argue as in the proof of Theorem 3; see Casas (1995). $\square$

ACKNOWLEDGMENT

The author would like to thank Dirección General de Investigación Científica y Técnica (Spain) for its support to this research.

REFERENCES

- Bonnans, J. (1991) Pontryagin's principle for the optimal control of semilinear elliptic systems with state constraints, in *30th IEEE Conference on Control and Decision*, Brighton, England, 1976–1979.
- Bonnans, J. and Casas, E. (1995) An extension of Pontryagin's principle for state-constrained optimal control of semilinear elliptic equations and variational inequalities. *SIAM J. Control Optim.*, **33**, 274–298.
- Casas, E. (1992) Finite element approximations for some state-constrained optimal control problems, in *Mathematics of the Analysis and Design in Process Control* (eds. P. Borne, S. Tzafestas, and N. Radhy), Amsterdam, North Holland, 293–301.
- Casas, E. (1995) Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations. Technical Report 1299, IMA Preprint Series.
- Ekeland, I. (1975) Nonconvex minimization problems. *Bull. Amer. Math. Soc.*, **1**, 76–91.
- Fattorini, H. and Frankowska, H. (1991) Necessary conditions for infinite dimensional control problems. *Math. Control Signals Systems*, **4**, 41–67.
- Li, X. and Yong, J. (1995) *Optimal Control Theory for Infinite Dimensional Systems*. Birkhäuser, Boston.