Relation between invariant subspaces of the Hamiltonian and the algebraic Riccati equation

H.J. Zwart and C.R. Kuiper

University of Twente, Faculty of Applied Mathematics

PO Box 217, 7500 AE Enschede, The Netherlands

Abstract

In this paper we study the relationship between solutions of the algebraic Riccati equation and invariant subspaces of the corresponding Hamiltonian. We show that there is a one-to-one relation between solutions of the algebraic Riccati equation and a class of invariant subspaces of the Hamiltonian. This class of subspaces is shown to be invariant under the inverse of the Hamiltonian as well. With this we generalize the results obtained by Potter (1966), Mårtensson (1971) and Willems (1971) to infinite-dimensional systems. As an example we characterize all solutions of the ARE for a class of unitary systems.

Keywords

Algebraic Riccati equation, Hamiltonian operator, invariant subspace

1 INTRODUCTION

For finite-dimensional systems it is standard to calculate solutions of the algebraic Riccati equation (ARE) via the eigenvectors of the corresponding Hamiltonian. It is well known that for a minimal system the stabilizing solution of the ARE is in one-to-one relationship with the stable eigenvectors of the Hamiltonian. The span of all these stable eigenvectors form the stable subspace of the Hamiltonian, i.e., it is an invariant subspace and the restriction of the Hamiltonian to this subspace is stable. For an overview of the finite-dimensional results we refer to Kučera (1991). In this paper we shall investigate the relation between the ARE and the Hamiltonian for infinite-dimensional systems. We study a general ARE, thus not only the one related to optimal control theory. The organization of the paper is as follow. In the next section, we shall see that there is a one-to-one correspondence between solutions of the ARE and certain invariant subspaces of the Hamiltonian. For infinite dimensional systems the Hamiltonian is an unbounded operator, and it is known that for unbounded operators invariant subspaces are hard to characterize. So the invariance result may seem not that useful. However, for our class of invariant subspaces we can show that they are invariant under spectral projections as well. Hence if the Hamiltonian has compact resolvent, then they are spanned by eigenvectors. With

this we can in the last section generalize a famous result of Willems (1971) to infinite dimensional systems.

2 GENERAL RELATIONS

In this section, we shall investigate the relation between the algebraic Riccati equation (ARE) and invariant subspaces of the Hamiltonian. We begin by introducing some notation and definitions.

Let Z be a Hilbert space. By $\mathcal{L}(Z)$ we denote the set of bounded, linear operators on Z . If Q is a linear operator on Z , then Q^* denotes its adjoint. A is a closed operator on Z with domain $D(A) \subset Z$. Furthermore, by Q_1 and Q_2 we denote two bounded, self-adjoint operators on Z . With this notation we can define the ARE and the Hamiltonian operator.

We say that $X \in \mathcal{L}(Z)$ the Algebraic Riccati equation, ARE, if

$$A^*X + XA - XQ_1X + Q_2 = 0 \quad \text{on } D(A). \quad (1)$$

For the same A, Q_1 and Q_2 , the corresponding Hamiltonian operator is given by

$$\mathcal{H} := \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix} \quad \text{on } D(\mathcal{H}) = D(A) \oplus D(A^*)$$

In the next lemma, we shall present one of the important relation between solutions of the ARE and invariant subspaces of the Hamiltonian.

Lemma 1 *The following relation holds between the Hamiltonian operator and the ARE. $X \in \mathcal{L}(Z)$ is a solution of the ARE if and only if $\text{ran} \begin{pmatrix} I \\ X \end{pmatrix}$ is an invariant subspace of $\mathcal{H}$ and $XD(A) \subset D(A^*)$.*

Proof. Let X be a solution of the ARE, then from (1) it follows easily that $XD(A) \subset D(A^*)$. Hence

$$\text{ran} \begin{pmatrix} I \\ X \end{pmatrix} \cap D(\mathcal{H}) = \left\{ \begin{pmatrix} z \\ Xz \end{pmatrix} \mid z \in D(A) \right\} \quad (2)$$

and

$$\begin{aligned} \mathcal{H} \begin{pmatrix} z \\ Xz \end{pmatrix} &= \begin{pmatrix} A & -Q_1 \\ -Q_2 & A^* \end{pmatrix} \begin{pmatrix} z \\ Xz \end{pmatrix} = \begin{pmatrix} (A - Q_1X)z \\ (-Q_2 - A^*X)z \end{pmatrix} \\ &\stackrel{(1)}{=} \begin{pmatrix} (A - Q_1X)z \\ (XA - XQ_1X)z \end{pmatrix} = \begin{pmatrix} (A - Q_1X)z \\ X(A - Q_1X)z \end{pmatrix} \subset \text{ran} \begin{pmatrix} I \\ X \end{pmatrix}. \end{aligned}$$

Assume next that

$$\mathcal{H} \left(\begin{pmatrix} z \\ Xz \end{pmatrix} \cap D(\mathcal{H}) \right) \subset \text{ran} \begin{pmatrix} I \\ X \end{pmatrix} \quad (3)$$

Since $XD(A) \subset D(A^*)$, we have that (2) holds. This together with (3) implies that for any $z \in D(A)$, there exists an y such that

$$\mathcal{H} \begin{pmatrix} z \\ Xz \end{pmatrix} = \begin{pmatrix} (A - Q_1X)z \\ (-Q_2 - A^*X)z \end{pmatrix} = \begin{pmatrix} y \\ Xy \end{pmatrix}$$

Combining both equalities gives $X(A - Q_1X)z = (-Q_2 - A^*X)z$, or equivalently

$$(XA + A^*X - XQ_1X + Q_2)z = 0. \quad (4)$$

Since this holds for all $z \in D(A)$, we conclude that (1) holds. $\square$

So we see that the ARE has a solution if the Hamiltonian operator has an invariant subspace $\text{ran} \begin{pmatrix} I \\ X \end{pmatrix}$. From Example I.6 of Zwart (1989), we know that invariant subspaces of unbounded operator can be very strange. For instance, even for a simple operator as the Laplacian there are invariant subspaces such that the Laplacian restricted to this subspace is nowhere invertible. It also follows from Zwart (1989) that if the subspace is invariant for the inverse of the operator as well, then the situation is more standard. The following theorem shows that our invariant subspaces of the Hamiltonian have this property. To simplify the proof we introduce a transformation of the Hamiltonian. Let $X \in \mathcal{L}(Z)$ be such that $XD(A) \subset D(A^*)$, then $\mathcal{H}_X$ is defined by

$$\mathcal{H}_X := \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \mathcal{H} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} A - Q_1X & -Q_1 \\ -XA + XQ_1X - Q_2 - A^*X & -A^* + XQ_1 \end{pmatrix}. \quad (5)$$

Theorem 2 Define for $X \in \mathcal{L}(Z)$ the subspace $V_X := \text{ran} \begin{pmatrix} I \\ X \end{pmatrix}$. Then the following assertions are equivalent:

1. X is a solution of the ARE;
2. $XD(A) \subset D(A^*)$, and the subspace V_X is an invariant subspace of $\mathcal{H}$;
3. $XD(A) \subset D(A^*)$, and for all $\lambda \in \rho(\mathcal{H}) \cap \rho(A - Q_1X)$, we have that

$$(\lambda I - \mathcal{H})^{-1}V_X \subset V_X. \quad (6)$$

Proof. The equivalence between 1. and 2. is already proved in Lemma 1. We shall first show that 2. implies 3.

First we remark the following. A subspace W is S -invariant if and only if TW is TST^{-1} -invariant, where T is a boundedly invertible operator. Using this we see that we can replace the condition in 2. by $\mathcal{H}_X \begin{pmatrix} D(A) \\ 0 \end{pmatrix} \subset \begin{pmatrix} Z \\ 0 \end{pmatrix}$. Furthermore it is easy to see that $\rho(\mathcal{H}) = \rho(\mathcal{H}_X)$. So we know that

$$\mathcal{H}_X \begin{pmatrix} D(A) \\ 0 \end{pmatrix} \subset \begin{pmatrix} Z \\ 0 \end{pmatrix} \quad (7)$$

For $\lambda \in \rho(A - Q_1X)$ we have that $(\lambda - A + Q_1X)D(A) = Z$. Hence with (7), it follows that for $\lambda \in \rho(A - Q_1X)$

$$(\lambda I - \mathcal{H}_X) \begin{pmatrix} D(A) \\ 0 \end{pmatrix} = \begin{pmatrix} Z \\ 0 \end{pmatrix} \quad (8)$$

Let $\lambda \in \rho(A - Q_1X) \cap \rho(\mathcal{H}_X)$ and $z \in Z$, then

$$(\lambda I - \mathcal{H}_X)^{-1} \begin{pmatrix} z \\ 0 \end{pmatrix} \stackrel{(8)}{=} (\lambda I - \mathcal{H}_X)^{-1} (\lambda I - \mathcal{H}_X) \begin{pmatrix} z_A \\ 0 \end{pmatrix} = \begin{pmatrix} z_A \\ 0 \end{pmatrix} \in \begin{pmatrix} Z \\ 0 \end{pmatrix},$$

where $z_A \in D(A)$. So we have shown 3. Now we shall prove that 3. implies 2.

Assume that 3. holds. By the remarks made at the beginning of this proof, we see that we may as well assume that

$$(\lambda I - \mathcal{H}_X)^{-1} \begin{pmatrix} Z \\ 0 \end{pmatrix} \subset \begin{pmatrix} Z \\ 0 \end{pmatrix}.$$

We write $(\lambda I - \mathcal{H}_X)^{-1} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ with $P_{ij} \in \mathcal{L}(Z)$ $i, j = 1, 2$.

Since $(\lambda I - \mathcal{H}_X)^{-1} \begin{pmatrix} Z \\ 0 \end{pmatrix} \subset \begin{pmatrix} Z \\ 0 \end{pmatrix}$ we have that $P_{21} = 0$.

Furthermore, we write $(\lambda I - \mathcal{H}_X) = \begin{pmatrix} \lambda - A + Q_1X & Q_1 \\ Q_3 & \lambda + A^* - XQ_1 \end{pmatrix}$ for some Q_3 .

So we have that

$$\begin{pmatrix} \lambda I - A + Q_1X & +Q_1 \\ Q_3 & \lambda + A^* - XQ_1 \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} = \begin{pmatrix} I_Z & 0 \\ 0 & I_Z \end{pmatrix} \quad (9)$$

Thus in particular, there holds $(\lambda I - A + Q_1X)P_{11} = I_Z$ and since $\lambda \in \rho(A - Q_1X)$, this implies that

$$P_{11} = (\lambda I - A + Q_1X)^{-1}$$

Furthermore, we see from (9) that $Q_3P_{11} = 0$, and so $Q_3 = 0$ on $D(A)$. Hence

$$(\lambda I - \mathcal{H}_X) = \begin{pmatrix} \lambda - A + Q_1X & +Q_1 \\ 0 & \lambda + A^* + XQ_1 \end{pmatrix}$$

and so $Z \oplus \{0\}$ is $\mathcal{H}_X$ -invariant. This implies that 2. holds. $\square$

It may seem strange that whereas the left-hand side of equation (6) has meaning for every $\lambda \in \rho(\mathcal{H})$, the equality is only valid on a smaller set. In the next example, we shall show that (6) need not to hold on the whole $\rho(\mathcal{H})$. Note that for finite-dimensional systems, $\rho(\mathcal{H}) \subset \rho(A - Q_1X)$. Thus in that case, one can just write $\lambda \in \rho(\mathcal{H})$ in part 3.

Example 3 We take A to be the shift on $\ell_2(N)$, i.e.

$$A \begin{pmatrix} z_1 \\ z_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ z_1 \\ \vdots \end{pmatrix} \quad (10)$$

Furthermore, we take $Q_1 = I$, $Q_2 = 0$ and $X = 0$. Then it is clear that X satisfies the ARE. So

$$\mathcal{H} \begin{pmatrix} Z \\ 0 \end{pmatrix} \subset \begin{pmatrix} Z \\ 0 \end{pmatrix}$$

Next we shall show that

$$\mathcal{H}^{-1} = \begin{pmatrix} A^* & -I \\ -P_1 & -A \end{pmatrix}, \quad (11)$$

where $P_1 \begin{pmatrix} z_1 \\ z_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} z_1 \\ 0 \\ \vdots \end{pmatrix}$. From this it is easy to see that $Z \oplus \{0\}$ is not $\mathcal{H}^{-1}$ invariant.

An easy calculation gives that $A^* \begin{pmatrix} z_1 \\ z_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \end{pmatrix}$. Hence

$$AA^* = I - P_1 \quad \text{and} \quad A^*A = I. \quad (12)$$

Using this we see that

$$\begin{pmatrix} A & -I \\ 0 & -A^* \end{pmatrix} \begin{pmatrix} A^* & -I \\ -P_1 & -A \end{pmatrix} = \begin{pmatrix} AA^* + P_1 & -A + A^* \\ A^*P_1 & A^*A \end{pmatrix} \stackrel{(12)}{=} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Furthermore

$$\begin{pmatrix} A^* & -I \\ -P_1 & -A \end{pmatrix} \begin{pmatrix} A & -I \\ 0 & -A^* \end{pmatrix} = \begin{pmatrix} A^*A & -A^* + A^* \\ -P_1A & P_1 + AA^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Hence (11) holds. Note that $0 \in \sigma(A) \cap \sigma(A^*)$.

So the previous example shows in particular that in general the resolvent set of $\mathcal{H}$ need not to be equal to the union of the resolvent set of $A - Q_1X$ and of $-A^* + XQ_1$. However, for an important subclass of systems there is equality. Without proof we state the following result.

Lemma 4 Let X be a solution of the ARE, if the spectra of $\mathcal{H}$, $A - Q_1X$, and $-A^* + XQ_1$ are all point spectra, then $\sigma(\mathcal{H}) = \sigma(A - Q_1X) \cup \sigma(-A^* + XQ_1)$.

3 ALL SOLUTIONS OF THE ARE FOR SKEW-ADJOINT GENERATORS

In this section we shall assume that A is skew-adjoint, i.e., $A^* = -A$, and has compact resolvent. Furthermore, we assume that $Q_2 = Q_1 \geq 0$, and $\Sigma(A, -, Q_1)$ is approximately observable (see Curtain and Zwart 1995). Since A has compact resolvent, so has every bounded perturbation of A , or of A^* , see Kato (1984), Theorem 1.16 of chapter IV. Furthermore, it follows easily that the Hamiltonian operator has compact resolvent as well. Hence, from Theorem 6.29 of chapter III of Kato (1984), $A - Q_1X$, $-A^* + XQ_1$, and $\mathcal{H}$ have pure point spectrum. The following results are standard and easy to prove.

Lemma 5 *Let A, Q_1 and Q_2 have the properties as stated above, then*

1. I and $-I$ are solutions of the ARE;
2. $A - Q_1$ is stable, i.e., all eigenvalues have real part negative;
3. $-A^* + Q_1 = A + Q_1$ is completely unstable, i.e., all eigenvalues have real part positive.

Associated with the solutions I and $-I$ of the ARE, we have the invariant subspaces $V_I := \text{ran} \begin{pmatrix} I \\ I \end{pmatrix}$ and $V_{-I} := \text{ran} \begin{pmatrix} I \\ -I \end{pmatrix}$, respectively. It is easy to see that

$$Z \oplus Z = V_I \oplus V_{-I} \quad (13)$$

Let P_I and P_{-I} denote the projection on V_I and V_{-I} , respectively. Note that (13) is equivalent with $I = P_I + P_{-I}$. From this it is easy to see that

$$P_I \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} z_1 + z_2 \\ z_1 + z_2 \end{pmatrix} \quad \text{and} \quad P_{-I} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} z_1 - z_2 \\ -z_1 + z_2 \end{pmatrix} \quad (14)$$

We shall show that these projections can also be seen as spectral projections. We recall that for a closed curve Γ which has empty intersection with the spectrum, the spectral projection is defined as

$$S_\Gamma x = \frac{1}{2\pi i} \int_\Gamma (\gamma I - \mathcal{H})^{-1} x d\gamma.$$

To define the spectral projection associated with the stable eigenvalues of $\mathcal{H}$ we take a sequence of curves, Γ_n , all lying in the left-half plane, such that the interior of Γ_n becomes the whole left half plane for n going to infinity. Now the spectral projection associated with the left half plane is defined as

$$S_{\text{stable}} x = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\Gamma_n} (\gamma I - \mathcal{H})^{-1} x d\gamma. \quad (15)$$

Similarly we define S_{unstable} as the spectral projection associated with the right-half plane. Now we can prove the following lemmas.

Lemma 6 Assume S_{stable} and S_{unstable} are well-defined, and that X is a solution of the ARE, then

$$S_{\text{stable}}V_X \subset V_X \quad \text{and} \quad S_{\text{unstable}}V_X \subset V_X, \quad (16)$$

where V_X is defined in Theorem 2.

Proof. Let $v \in V_X$, then by equation (6) there holds that $(\gamma I - \mathcal{H})^{-1}v \in V_X$. Thus since V_X is a closed linear subspace, $S_\Gamma v \in V_X$. Since this holds for any closed curve Γ , the result follows. $\square$

Lemma 7 If S_{stable} and S_{unstable} are well-defined, and $I = S_{\text{stable}} + S_{\text{unstable}}$, then $P_I = S_{\text{stable}}$ and $P_{-I} = S_{\text{unstable}}$.

Proof. Recall that the range of P_I and P_{-I} are the Hamiltonian invariant subspaces V_I and V_{-I} , respectively. So by Lemma 6 we know that both spectral projections map these subspaces into themselves. We shall show that $S_{\text{stable}}V_{-I} = 0$. Let Γ_n be a closed curve in the left-half plane, and let $(z, -z)^T$ be an element of V_{-I} , then we see that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_n} (\gamma I - \mathcal{H})^{-1} \begin{pmatrix} z \\ -z \end{pmatrix} d\gamma &= \frac{1}{2\pi i} \int_{\Gamma_n} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} (\gamma I - \mathcal{H}_{-I})^{-1} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} z \\ -z \end{pmatrix} d\gamma \\ &= \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \frac{1}{2\pi i} \int_{\Gamma_n} (\gamma I - \mathcal{H}_{-I})^{-1} \begin{pmatrix} z \\ 0 \end{pmatrix} d\gamma \\ &= \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \frac{1}{2\pi i} \int_{\Gamma_n} \begin{pmatrix} (\gamma I - A - Q_1)^{-1}z \\ 0 \end{pmatrix} d\gamma = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

since by Lemma 5 $A + Q_1$ has no poles inside Γ_n . From the definition of S_{stable} it follows directly that $S_{\text{stable}}V_{-I} = 0$. Similarly one can show that $S_{\text{unstable}}V_I = 0$. From these relations it follows easily that $S_{\text{stable}} = S_{\text{stable}}P_I$. Letting P_I operate on this equation gives $P_I S_{\text{stable}} = P_I S_{\text{stable}} P_I = S_{\text{stable}} P_I$, since $\text{ran } S_{\text{stable}} P_I \subset \text{ran } P_I = V_I$. Hence combining these equalities gives

$$S_{\text{stable}} = S_{\text{stable}} P_I = P_I S_{\text{stable}}. \quad (17)$$

Similarly, one can show that

$$S_{\text{unstable}} = S_{\text{unstable}} P_{-I} = P_{-I} S_{\text{unstable}}. \quad (18)$$

For $z \in Z$ we have that

$$\begin{aligned} z &= P_I z + P_{-I} z = P_I [S_{\text{stable}} z + S_{\text{unstable}} z] + P_{-I} [S_{\text{stable}} z + S_{\text{unstable}} z] \\ &= S_{\text{stable}} z + P_I S_{\text{unstable}} z + P_{-I} S_{\text{stable}} z + S_{\text{unstable}} z = z + P_I S_{\text{unstable}} z + P_{-I} S_{\text{stable}} z. \end{aligned}$$

Since $\text{ran } P_I \cap \text{ran } P_{-I} = \{0\}$, we see that $P_I S_{\text{unstable}} = P_{-I} S_{\text{stable}} = 0$. Now it follows directly that $P_I = S_{\text{stable}}$ and $P_{-I} = S_{\text{unstable}}$. $\square$

With these lemmas we can characterize all solutions of the ARE.

Theorem 8 Assume that S_{stable} and S_{unstable} are well-defined, and that $I = S_{\text{stable}} + S_{\text{unstable}}$. If X is a solution of the ARE, then X can be written as $X = I - 2\mathcal{P}$, where $\mathcal{P}$ is a projection. If X is self-adjoint, then $-I \leq X \leq I$.

Proof. From Lemma 7 we know that $P_I = S_{\text{stable}}$ and $P_{-I} = S_{\text{unstable}}$. Let X be a solution of the ARE, and thus V_X is an invariant subspace of the Hamiltonian. Now from Lemma 6 it follows directly that $P_I V_X \subset V_X$. Thus for all $z \in Z$, there exists a $y \in Z$ such that

$$P_I \begin{pmatrix} z \\ Xz \end{pmatrix} = \frac{1}{2} \begin{pmatrix} z + Xz \\ z + Xz \end{pmatrix} = \begin{pmatrix} y \\ Xy \end{pmatrix}$$

Thus for all $z \in Z$, $z + Xz = 2Xy = X(z + Xz)$, and so

$$I = X^2. \quad (19)$$

If we write $X = I - 2\mathcal{P}$, then we see that (19) implies that $\mathcal{P}^2 = \mathcal{P}$. Hence $\mathcal{P}$ is a projection.

It is easy to see that X is self-adjoint if and only if $\mathcal{P}$ is self-adjoint. Since for a self-adjoint projection one always has that $0 \leq \mathcal{P} \leq I$, the assertion follows directly. $\square$

Note that the assumptions in the theorem are always satisfied if $\mathcal{H}$ has a Riesz basis of eigenvectors. One can use this Riesz basis to characterize all solutions of the ARE, even if A is not skew-adjoint, see Kuiper and Zwart (1995). Theorem 8 is a generalization of Theorem 6 of Willems (1971). As in Willems (1971), one can make the result in this theorem if and only if. However, due to space limitations, we cannot present a proof here.

REFERENCES

- Curtain, R.F. and Zwart, H.J. (1995) *An Introduction to Infinite Dimensional Linear Systems Theory*. Text in Applied Mathematics, Vol. 21, Springer Verlag, New York.
- Kato, T. (1984) *Perturbation Theory for Linear Operators*, Grundlehren der mathematischen Wissenschaften, 132, Springer Verlag, Berlin.
- Kučera, V. (1991) Algebraic Riccati Equation: Hermitian and Definite Solutions, in *The Riccati Equation* (ed. S. Bittanti, A.J. Laub, J.C. Willems), Springer Verlag, Berlin.
- Kuiper, C.R. and Zwart, H.J. (1995) Relations between the algebraic Riccati equation and the Hamiltonian for Riesz-spectral systems, *Journal of Mathematical Systems, Estimation and Control* (to appear)
- Mårtensson, K. (1971) On the Matrix Riccati Equation, *Inform. Sci.* **3**, 17-49.
- Potter, J.E. (1966) Matrix quadratic solutions, *SIAM J. Appl. Math.* **14**, 496-501.
- Willems, J.C. (1971) Least Squares Stationary Optimal Control and The Algebraic Riccati Equation, *IEEE Trans. Autom. Control*, **16**, 621-634.
- Zwart, H.J. (1989) *Geometric Theory for Infinite Dimensional Systems*, Lecture Notes in Control and Information Sciences, **115**, Springer Verlag, Berlin.