On abstract boundary control problems for vibrations

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Abstract

We consider abstract control problems which are governed by a homogeneous linear wave equation for Hilbert space valued functions under nonhomogeneous boundary conditions which contain the controls that have values in an abstract Hilbert space. These problems are transformed into distributed control problems under homogeneous boundary conditions which have the controls on the right-hand side of a nonhomogeneous wave equation.

After this transformation the problem of null-controllability is studied.

Keywords

Abstract boundary control, distributed control, problem of null-controllability

0 INTRODUCTION

In (Fattorini, 1968) boundary control problems are considered for abstract differential equations of first and second order with respect to the time which in the case of second order are slightly more general than the abstract wave equation (1.1) and under boundary conditions of which (1.2) is a special case. Among others an assumption is made there which in the case of the abstract wave equation (1.1) and the boundary conditions (1.2) reads as follows: There exists a bounded linear operator $G : H_B \rightarrow D(A)$ such that

$$B(G(\psi)) = \psi \quad \text{for all } \psi \in H_B$$

(which, of course, requires that $D(A) \subseteq D(B)$).

As control space $C^\infty([0, T], H_B)$ is considered. If then, for some $\varphi \in C^\infty([0, T], H_B)$, $y \in C^2([0, T], D(A)) \cap C^1([0, T], H)$ is a solution (1.1), (1.2), (1.3), we put $\tilde{y} = y - G(\varphi)$ and conclude that

$$B(\tilde{y}(t)) = \Theta_H \quad \text{for all } t \in [0, T], \quad (0.1)$$

$$\ddot{\tilde{y}}(t) + \tilde{A}\tilde{y}(t) = -G(\ddot{\varphi})(t) - A(G(\varphi))(t) \quad (0.2)$$

for all $t \in (0, T)$ and

$$\tilde{y}(0) = y_0 - G(\varphi)(0), \quad \dot{\tilde{y}}(0) = \dot{y}_0 - G(\dot{\varphi})(0). \quad (0.3)$$

Conversely, if $\tilde{y} \in C^2([0, T], D(\tilde{A})) \cap C^1([0, T], H)$ is a solution of (0.1), (0.2), (0.3) for some $\varphi \in C^\infty([0, T], H_B)$, then $y = \tilde{y} + G(\varphi)$ is in $C^2([0, T], D(A)) \cap C^1([0, T], H)$ and solves (1.1), (1.2), and (1.3).

This approach is taken in (Fattorini, 1968) in order to reduce a boundary control problem to a problem of distributed control. Based on this reduction then problems of approximate controllability are investigated.

In this paper we assume in addition that the bounded linear operator $G : H_B \rightarrow D(A)$ is such that $A(G(\psi)) = \Theta_H$ for all $\psi \in H_B$. Further we replace the control space $C^\infty([0, T], H_B)$ by $H_0^1([0, T], H_B) = \{\varphi \in H^1([0, T], H_B) \mid \varphi(0) = \varphi(T) = \Theta_{H_B}\}$ and define a weak solution of (1.1), (1.2), (1.3) in such a way that it can be explicitly represented in terms of an infinite series (see (1.7)).

Then we formulate the problem of (exact) null-controllability and investigate under which conditions it can be solved. Finally, we illustrate the abstract results by an example.

1 REDUCTION OF BOUNDARY CONTROL TO DISTRIBUTED CONTROL

We consider a vibrating system whose states are Hilbert space valued functions $y : [0, T] \rightarrow H$ for some $T > 0$ (H is a real Hilbert space) which satisfy an abstract wave equation

$$\ddot{y}(t) + Ay(t) = \Theta_H, \quad t \in (0, T), \quad (1.1)$$

and boundary conditions

$$By(t) = \varphi(t), \quad t \in [0, T], \quad (1.2)$$

where A and B are linear operators on dense linear subspaces $D(A)$ and $D(B)$ of H , respectively, mapping $D(A)$ and $D(B)$ into H and H_B , respectively, with H_B being a second Hilbert space. We assume that $D(A) \subseteq D(B)$. The control function $\varphi : [0, T] \rightarrow H_B$ on the right-hand side of (1.2) is chosen in the control space

$$H_0^2([0, T], H_B) = \{\varphi \in H^2([0, T], H_B) \mid \varphi(0) = \varphi(T) = \dot{\varphi}(0) = \dot{\varphi}(T) = \Theta_{H_B}\}.$$

Let

$$D_0 = \{z \in D(A) \mid Bz = \Theta_{H_B}\} \quad \text{and} \quad \tilde{A} = A|_{D_0}.$$

We assume $\tilde{A} : D_0 \rightarrow H$ to be self adjoint and positive definite and we also assume that $E = D(\tilde{A}^{\frac{1}{2}}) \subseteq D(B)$. For $y_0 \in E$ and $\dot{y}_0 \in H$ being given we further consider initial

conditions

$$y(0) = y_0 \quad \text{and} \quad \dot{y}(0) = \dot{y}_0 . \quad (1.3)$$

Assumption 1: For every $\psi \in H_B$ there exists exactly one element $w_\psi \in D(A)$ such that

$$A w_\psi = \Theta_H \quad \text{and} \quad B w_\psi = \psi$$

and the linear operator $G : H_B \rightarrow H$ given by $G(\psi) = w_\psi$ is continuous.

Let $\varphi \in H_0^2([0, T], H_B)$ be given. Then we define

$$r(t) = G(\varphi)(t) \quad \text{for every} \quad t \in [0, T]$$

and conclude that

$$\begin{aligned} r &\in H_0^2([0, T], H) = \{w \in H^2([0, T], H) \mid w(0) \\ &= w(T) = \dot{w}(0) = \dot{w}(T) = \Theta_H\} . \end{aligned}$$

Let, for some $\varphi \in H_0^2([0, T], H_B)$, $y \in C([0, T], D(A)) \cap C^1([0, T], H)$ be a solution of (1.1), (1.2), (1.3). Then we put $r = G(\varphi)$ and define $\tilde{y} = y - r$. Then we conclude

$$B\tilde{y}(t) = By(t) - Br(t) = \varphi(t) - \varphi(t) = \Theta_H ,$$

hence $\tilde{y}(t) \in D_0$ for every $t \in [0, T]$.

From $r \in C([0, T], D(A)) \cap C^1([0, T], H)$ we infer that $\tilde{y} \in C([0, T], D_0) \cap C^1([0, T], H)$. Finally, we obtain

$$\begin{aligned} \ddot{\tilde{y}}(t) &= \ddot{y}(t) - \ddot{r}(t) = -Ay(t) - \ddot{r}(t) \\ &= -A(y(t) - r(t)) - \ddot{r}(t) = -\tilde{A}\tilde{y}(t) - \ddot{r}(t) , \end{aligned}$$

hence

$$\ddot{\tilde{y}}(t) + \tilde{A}\tilde{y}(t) = -\ddot{r}(t) \quad \text{for all} \quad t \in (0, T) . \quad (1.4)$$

It also follows that

$$\tilde{y}(0) = y_0 \quad \text{and} \quad \dot{\tilde{y}}(0) = \dot{y}_0 . \quad (\widetilde{1.3})$$

Conversely, if $\tilde{y} \in C([0, T], D_0) \cap C^1([0, T], H)$ is a solution of (1.4), $(\widetilde{1.3})$ where $r = G(\varphi)$ for some $\varphi \in H_0^2([0, T], H_B)$, then $y = \tilde{y} + r \in C([0, T], D(A)) \cap C^1([0, T], H)$ is a solution of (1.1), (1.2), (1.3).

If we replace (1.4) by the weak equation

$$\begin{aligned} -\int_0^T \langle \dot{\tilde{y}}(t), \dot{v}(t) \rangle_H dt + \int_0^T \langle \tilde{A}^{\frac{1}{2}} \tilde{y}(t), \tilde{A}^{\frac{1}{2}} v(t) \rangle_H dt \\ = -\int_0^T \langle \ddot{r}(t), v(t) \rangle_H dt \end{aligned} \quad (1.5)$$

for all $v \in H_0^1([0, T], E) = \{w \in H^1([0, T], E) \mid w(0) = w(T) = \Theta_E\}$, then there is exactly one solution $\tilde{y} \in C([0, T], E) \cap C^1([0, T], H)$ of (1.5), (1.3) which is given by

$$\begin{aligned} \tilde{y}(t) &= \sum_{j=1}^{\infty} (\langle y_0, \varphi_j \rangle_H \cos \sqrt{\lambda_j} t + \frac{1}{\sqrt{\lambda_j}} \langle \dot{y}_0, \varphi_j \rangle_H \sin \sqrt{\lambda_j} t) \varphi_j \\ &- \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \int_0^t \langle \ddot{r}(s), \varphi_j \rangle_H \sin \sqrt{\lambda_j} (t-s) ds \varphi_j \end{aligned} \quad (1.6)$$

for $t \in [0, T]$, if we assume that $\tilde{A} : D_0 \rightarrow H$ has a complete orthonormal system of eigenelements $\lambda_j \in D_0$, $j \in \mathbb{N}$, and corresponding eigenvalues $\lambda_j \in \mathbb{R}$, $j \in \mathbb{N}$, of finite multiplicity such that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{and} \quad \lambda_j \rightarrow \infty \quad \text{as} \quad j \rightarrow \infty.$$

The time derivative of $\tilde{y}$ is given by

$$\begin{aligned} \dot{\tilde{y}}(t) &= \sum_{j=1}^{\infty} (-\sqrt{\lambda_j} \langle y_0, \varphi_j \rangle_H \sin \sqrt{\lambda_j} t + \langle \dot{y}_0, \varphi_j \rangle_H \cos \sqrt{\lambda_j} t) \varphi_j \\ &- \sum_{j=1}^{\infty} \int_0^t \langle \ddot{r}(s), \varphi_j \rangle_H \cos \sqrt{\lambda_j} (t-s) ds \varphi_j \end{aligned}$$

for $t \in [0, T]$.

We therefore define a weak solution $y \in C([0, T], D(B)) \cap C^1([0, T], H)$ of (1.1), (1.2) for some $\varphi \in H_0^2([0, T], H_B)$ and (1.3) by $y = \tilde{y} + r$ with $r = G(\varphi)$ and $\tilde{y}$ given by (1.6). This can then be represented in the form

$$\begin{aligned} y(t) &= \sum_{j=1}^{\infty} (\langle y_0, \varphi_j \rangle_H \cos \sqrt{\lambda_j} t + \frac{1}{\sqrt{\lambda_j}} \langle \dot{y}_0, \varphi_j \rangle_H \sin \sqrt{\lambda_j} t) \varphi_j \\ &+ \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^t \langle r(s), \varphi_j \rangle_H \sin \sqrt{\lambda_j} (t-s) ds \varphi_j \end{aligned} \quad (1.7)$$

for $t \in [0, T]$ and its time derivative reads

$$\begin{aligned} \dot{y}(t) &= \sum_{j=1}^{\infty} (-\sqrt{\lambda_j} \langle y_0, \varphi_j \rangle_H \sin \sqrt{\lambda_j} t + \langle \dot{y}_0, \varphi_j \rangle_H \cos \sqrt{\lambda_j} t) \varphi_j \\ &+ \sum_{j=1}^{\infty} \lambda_j \int_0^t \langle r(s), \varphi_j \rangle_H \cos \sqrt{\lambda_j} (t-s) ds \varphi_j \end{aligned} \quad (1.8)$$

for $t \in [0, T]$.

If the control space $H_0^2([0, T], H_B)$ is replaced by

$$H_0^1([0, T], H_B) = \{\varphi \in H^1([0, T], H_B) \mid \varphi(0) = \varphi(T) = \Theta_{H_B}\},$$

then $y = y(t)$, $t \in [0, T]$, defined by (1.7) with $r = G(\varphi)$ for some $\varphi \in H_0^1([0, T], H_B)$ is taken as weak solution of (1.1), (1.2) for this φ , and (1.3) und $\dot{y} = \dot{y}(t)$, $t \in [0, T]$, defined by (1.8) is taken as its time derivative.

2 THE PROBLEM OF NULL-CONTROLLABILITY

Given $T > 0$ and $y_0 \in E$, $\dot{y}_0 \in H$, find $\varphi \in H_0^1([0, T], H_B)$ such that the weak solution $y = y(t)$, $t \in [0, T]$, of (1.1), (1.2), (1.3) which is defined by (1.7) with $r = G(\varphi)$ satisfies the end conditions

$$y(T) = \Theta_H \quad \text{and} \quad \dot{y}(T) = \Theta_H \quad (2.1)$$

where $\dot{y} = \dot{y}(t)$, $t \in [0, T]$, is given by (1.8).

From (1.7) and (1.8) it follows that the end conditions (2.1) are equivalent to

$$\begin{aligned} \lambda_j \int_0^T \langle r(t), \varphi_j \rangle_H \sin \sqrt{\lambda_j} t \, dt &= \sqrt{\lambda_j} \langle y_0, \varphi_j \rangle_H, \\ \lambda_j \int_0^T \langle r(t), \varphi_j \rangle_H \cos \sqrt{\lambda_j} t \, dt &= -\langle \dot{y}_0, \varphi_j \rangle_H \end{aligned} \quad (2.2)$$

for $j \in IN$.

Assumption 2: There exists a linear operator $\tilde{B} : D_0 \rightarrow H_B$ such that

$$\langle r, \tilde{A} \varphi_j \rangle_H = \langle Br, \tilde{B} \varphi_j \rangle_{H_B} \quad \text{for all } j \in IN$$

and all $r \in D(A)$.

Then (2.2) can be rewritten as

$$\begin{aligned} \int_0^T \langle \varphi(t), \tilde{B} \varphi_j \rangle_H \sin \sqrt{\lambda_j} t \, dt &= \sqrt{\lambda_j} \langle y_0, \varphi_j \rangle_H, \\ \int_0^T \langle \varphi(t), \tilde{B} \varphi_j \rangle_H \cos \sqrt{\lambda_j} t \, dt &= -\langle \dot{y}_0, \varphi_j \rangle_H \end{aligned} \quad (2.3)$$

for $j \in IN$. Integration by parts shows that this is equivalent to

$$\begin{aligned} \frac{1}{\sqrt{\lambda_j}} \int_0^T \langle \dot{\varphi}(t), \tilde{B} \varphi_j \rangle_H \cos \sqrt{\lambda_j} t \, dt &= \sqrt{\lambda_j} \langle y_0, \varphi_j \rangle_H, \\ \frac{1}{\sqrt{\lambda_j}} \int_0^T \langle \dot{\varphi}(t), \tilde{B} \varphi_j \rangle_H \sin \sqrt{\lambda_j} t \, dt &= \langle \dot{y}_0, \varphi_j \rangle_H \end{aligned} \quad (2.4)$$

for $j \in IN$.

For every $u \in L^2([0, T], H_B)$ let us define

$$\varphi(t) = \int_0^t u(s) \, ds, \quad t \in [0, T]. \quad (2.5)$$

Then it follows that $\dot{\varphi}(t) = u(t)$ for almost all $t \in [0, T]$, i.e., $\varphi \in H^1([0, T], H_B)$ and $\varphi(0) = \Theta_{H_B} \cdot \varphi(T) = \Theta_{H_B}$ is equivalent to

$$\int_0^T u(t) dt = \Theta_{H_B} . \quad (2.6)$$

Result: If $u \in L^2([0, T], H_B)$ satisfies (2.4) with u instead of $\dot{\varphi}$ and (2.6) and if $\varphi \in H^1([0, T], H_B)$ is defined by (2.5), then $\varphi \in H_0^1([0, T], H_B)$ and $y = y(t)$, $t \in [0, T]$, defined by (1.7) with $r = G(\varphi)$ satisfies the end conditions (2.1).

Thus, in order to solve the problem of null-controllability one has to determine some $u \in L^2([0, T], H_B)$ which satisfies (2.4) for all $j \in \mathcal{N}$ with u instead of $\dot{\varphi}$ and (2.6).

3 ON THE SOLVABILITY OF THE PROBLEM OF NULL — CONTROLLABILITY

Let us define a linear operator $S : L^2([0, T], H_B) \rightarrow (\mathbb{R}^2)^{\mathcal{N}}$ by $S(u) = (S_j^1(u), S_j^2(u))_{j \in \mathcal{N}}$ where

$$\begin{aligned} S_j^1(u) &= \frac{1}{\lambda_j} \int_0^T \langle u(t), \tilde{B} \varphi_j \rangle_H \cos \sqrt{\lambda_j} t dt , \\ S_j^2(u) &= \frac{1}{\lambda_j} \int_0^T \langle u(t), \tilde{B} \varphi_j \rangle_H \sin \sqrt{\lambda_j} t dt \end{aligned} \quad (3.1)$$

for all $j \in \mathcal{N}$ and $u \in L^2([0, T], H_B)$.

Further let us define a sequence $c = (c_j^1, c_j^2)_{j \in \mathcal{N}} \in \ell^2$ by

$$c_j^1 = \langle y_0, \varphi_j \rangle_H \quad \text{and} \quad c_j^2 = \frac{1}{\sqrt{\lambda_j}} \langle \dot{y}_0, \varphi_j \rangle_H \quad \text{for all } j \in \mathcal{N} . \quad (3.2)$$

Then (2.4) for all $j \in \mathcal{N}$ with u instead of $\dot{\varphi}$ can be rewritten in the form

$$S(u) = c . \quad (3.3)$$

By Assumption 1, for every $u \in L^2([0, T], H_B)$ there is exactly one $w_u \in L^2([0, T], H)$ such that $w_u(t) \in D(A)$ for all $t \in [0, T]$ and

$$\begin{aligned} A w_u(t) &= \Theta_H , \quad B w_u(t) = u(t) \\ &\text{for almost all } t \in [0, T] . \end{aligned}$$

By Assumption 2 we obtain

$$\frac{1}{\lambda_j} \langle u(t), \tilde{B} \varphi_j \rangle_{H_B} = \langle w_u(t), \varphi_j \rangle_H \quad \text{for all } j \in \mathcal{N} ,$$

hence

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} \langle u(t), \tilde{B} \varphi_j \rangle_H^2 = \sum_{j=1}^{\infty} \langle w_u(t), \varphi_j \rangle_H^2 = \|w_u(t)\|_H^2$$

for almost all $t \in [0, T]$. This implies

$$\sum_{j=1}^{\infty} [S_j^1(u)^2 + S_j^2(u)^2] \leq T \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} \int_0^T \langle u(t), \tilde{B} \varphi_j \rangle_H dt = T \int_0^T \|w_u(t)\|_H^2 dt$$

and shows that $S(L^2[0, T], H_B) \subseteq \ell^2$. In addition, S is closed. Therefore, by the closed graph theorem, S is continuous. The adjoint operator $S^* : \ell^2 \rightarrow L^2([0, T], H_B)$ of S is given by

$$S^*(y)(t) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} (y_j^1 \cos \sqrt{\lambda_j} t + y_j^2 \sin \sqrt{\lambda_j} t) \tilde{B} \varphi_j$$

for almost all $t \in [0, T]$ and $y = (y_j^1, y_j^2)_{j \in \mathbf{N}} \in \ell^2$.

Let us define a continuous linear operator $G : L^2([0, T], H_B) \rightarrow L^2([0, T], H_B)$ by

$$G(u) = u - \frac{1}{T} \int_0^T u(t) dt \quad \text{for } u \in L^2([0, T], H_B).$$

Obviously it follows that

$$\int_0^T G(u)(t) dt = \Theta_{H_B} \quad \text{for all } u \in L^2([0, T], H_B).$$

Therefore, finding some $u \in L^2([0, T], H_B)$ which satisfies (3.3) and (2.6) is equivalent to finding some $v \in L^2([0, T], H_B)$ which satisfies

$$S(G(v)) = c. \tag{3.4}$$

If $u \in L^2([0, T], H_B)$ solves (3.3) and (2.6), then $G(u) = u$ solves (3.4). Conversely, if $v \in L^2([0, T], H_B)$ solves (3.4) then $G(v) \in L^2([0, T], H_B)$ is a solution of (3.3) and (2.6).

Let $u, v \in L^2([0, T], H_B)$ be given arbitrarily. Then one can show that

$$\langle v, G(u) \rangle_{L^2([0, T], H_B)} = \langle G(v), u \rangle_{L^2([0, T], H_B)}$$

which implies that the adjoint operator $G^* : L^2([0, T], H_B) \rightarrow L^2([0, T], H_B)$ of G is equal to G and therefore

$$(S \circ G)^*(y) = G(S^*(y)) = S^*(y) - \frac{1}{T} \int_0^T S^*(y)(t) dt$$

for all $y \in \ell^2$.

By (Krabs, 1982), Theorem 3.6, the operator equation (3.4) has a solution $v \in L^2([0, T], H_B)$, if and only if there is a constant $\lambda > 0$ such that

$$\begin{aligned} \langle c, y \rangle_{\ell^2} &\leq \lambda \|(S \circ G)^*(y)\|_{L^2([0, T], H_B)} \\ &= \lambda \|S^*(y) - \frac{1}{T} \int_0^T S^*(y)(t) dt\|_{L^2([0, T], H_B)} \end{aligned}$$

for all $y \in \ell^2$.

This leads to the following

Problem: Minimize

$$\frac{1}{2} \|S^*(y) - \frac{1}{T} \int_0^T S^*(y)(t) dt\|_{L^2([0, T], H_B)}^2$$

for $y \in \ell^2$ subject to

$$\langle c, y \rangle_{\ell^2} = 1.$$

Let $\hat{y} \in \ell^2$ be a solution of this problem. Then we have to distinguish between two cases:

a) $\frac{1}{2} \|S^*(\hat{y}) - \frac{1}{T} \int_0^T S^*(\hat{y})(t) dt\|_{L^2([0, T], H_B)}^2 = 0$, hence

$$S^*(\hat{y}) - \frac{1}{T} \int_0^T S^*(\hat{y})(t) dt = \Theta_{H_B} \quad \text{a.e.}$$

Let $u \in L^2([0, T], H_B)$ be any solution of $S(G(u)) = c$. Then it follows that

$$\langle c, \hat{y} \rangle_{\ell^2} = \langle S(G(u)), \hat{y} \rangle_{\ell} = \langle u, \underbrace{G(S^*(\hat{y}))}_{=\Theta_{H_B} \text{ a.e.}} \rangle_{L^2([0, T], H_B)} = 0$$

which contradicts $\langle c, \hat{y} \rangle_{\ell^2} = 1$.

Therefore $S(G(u)) = c$ has no solution $u \in L^2([0, T], H_B)$.

b) $\frac{1}{2} \|S^*(\hat{y}) - \frac{1}{T} \int_0^T S^*(\hat{y})(t) dt\|_{L^2([0, T], H_B)}^2 > 0$.

Then, by the Lagrangian multiplier rule, there is some $\lambda \in \mathbb{R}$ such that

$$\langle (G \circ S^*)(\hat{y}), (G \circ S^*)(y) \rangle_{L^2([0, T], H_B)} = \lambda \langle c, y \rangle_{\ell^2}$$

for all $y \in \ell^2$ which implies

$$\lambda = \|(G \circ S^*)(\hat{y})\|_{L^2([0, T], H_B)}^2 > 0$$

and

$$\langle (S \circ G \circ G \circ S^*)(\hat{y}), y \rangle_{\ell^2} = \langle c, y \rangle_{\ell^2} \quad \text{for all } y \in \ell^2 ,$$

hence

$$S(G(\hat{u})) = c ,$$

if we put

$$\hat{u} = \frac{1}{\lambda} (G \circ S^*)(\hat{y}) = \frac{1}{\|(G \circ S^*)(\hat{y})\|_{L^2([0,T], H_B)}} (G \circ S^*)(\hat{y}) .$$

If we define $y^* = \frac{1}{\lambda} \hat{y}$, then $y^* \in \ell^2$ and solves the operator equation

$$(S \circ G \circ S^*)(y^*) = c \tag{3.5}$$

(because of $G \circ G = G$) which can be rewritten in the form

$$S(S^*(y^*)) - \frac{1}{T} \int_0^T S^*(y^*)(t) dt = c$$

where

$$S^*(y^*)(t) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} (y_j^{1*} \cos \sqrt{\lambda_j} t + y_j^{2*} \sin \sqrt{\lambda_j} t) \tilde{B} \varphi_j .$$

for almost all $t \in [0, T]$.

If $y^* \in \ell^2$ is a solution of (3.5) and if we define $\hat{u} \in L^2([0, T], H_B)$ by

$$\hat{u} = (G \circ S^*)(y^*) = S^*(y^*) - \frac{1}{T} \int_0^T S^*(y^*)(t) dt ,$$

then $S(G(\hat{u})) = c$ and $G(\hat{u}) \in L^2([0, T], H_B)$ solves (3.3) and (2.6).

If $S \circ G : L^2([0, T], H_B) \rightarrow \ell^2$ is surjective, then one can show that the operator $S \circ G \circ S^* : \ell^2 \rightarrow \ell^2$ is positive definite and therefore invertible. As a result we conclude that (3.5) has a unique solution which is given as

$$y^* = (S \circ G \circ S^*)^{-1} c .$$

4 AN EXAMPLE

As an example of (1.1) we consider the wave equation

$$y_{tt}(x, t) - \Delta y(x, t) = 0 \quad \text{for } x \in \Omega , \quad t \in (0, T)$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded open set with a sufficiently smooth boundary $\partial\Omega = \bar{\Omega} \setminus \Omega$ and $T > 0$ is given. By Δ we denote the Laplacian operator $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. As boundary condition we consider

$$y(x, t) = \varphi(x, t) \quad \text{for } x \in \partial\Omega, \quad t \in [0, T].$$

We choose $H = L^2(\Omega)$, $A = -\Delta$, $D(A) = H^2(\Omega)$, $B = \cdot|_{\partial\Omega}$, $D(B) = H^1(\Omega)$, $H_B = L^2(\partial\Omega)$.

This implies $D_0 = H^2(\Omega) \cap H_0^1(\Omega)$ and $E = D(\tilde{A}^{\frac{1}{2}}) = H_0^1(\Omega)$. For every $\psi \in L^2(\partial\Omega)$ there exists exactly one $w_\psi \in H^2(\Omega)$ such that

$$\Delta w_\psi = 0 \quad \text{and} \quad w_\psi|_{\partial\Omega} = \psi$$

and $\psi \rightarrow w_\psi$ is continuous. Therefore Assumption 1 holds true. Assumption 2 is a consequence of Green's formula

$$\int_{\Omega} (v \Delta u - u \Delta v) dx = \int_{\partial\Omega} (v u_\nu - u v_\nu) ds$$

for all $u, v \in H^2(\Omega)$ where $\frac{\partial}{\partial \nu}$ denotes the normal derivative. If we choose $v = r \in D(A)$ and $u = \varphi_j$, $j \in \mathbb{N}$, then it follows

$$\int_{\Omega} r \Delta \varphi_j dx = \int_{\partial\Omega} (Br)(\varphi_j)_\nu ds$$

which is equivalent to Assumption 2, if $\tilde{B} : D_0 \rightarrow H_B$ is defined by $\tilde{B}v = v_\nu$, $v \in D_0$.

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