Modelling and solving of the allocation problem of non-convex polygons with rotations

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Abstract

The work deals with the problem of optimal allocation of objects of the so-called irregular form. The objects are allocated on a strip of given width. The rotation of objects are allowed.

Keywords

Allocation, nonlinear optimization

1 INTRODUCTION

The problem is insufficiently studied, but it is of great applied importance and also of interest for the developing the theory of solving cutting and packing problems, as the problem in question belongs just that very set of problems. Problems of cutting and packing are frequently found in different spheres of human activity (cutting glass, metal, textile, leather etc., allocation of equipment in complex shape spaces, design of complex engineering systems).

A broad range of bibliography (see Sweeney and Ridenour (1989)) in certain directions of research exists. It is necessary to mention the well-known works of Dowsland (1992), Beasley (1985), Terno (1993), Li and Milenkovic (1993), Schwartz and Sharir (1983). However, very few works can practically be found in which the allocation problems are investigated taking into account the rotation possibility of allocation objects or the allocation region. Apparently it is caused by their complexity and non-triviality of mathematical models, necessary for their adequate description.

A wide class of problems connected with allocation of 2D and 3D complex shape geometric objects in domains of arbitrary shape has been studied for a number of years at the Institute for problems in Machinery (Ukraine, Kharkov).

In the work a mathematical model of the allocation problem being considered is constructed based on the original theory of Φ -functions and structures of nonlinear inequalities. The exact method searching for a problem local minimum from any feasible initial point uses the active set strategy ideas.

2 PROBLEM MATHEMATICAL MODEL CONSTRUCTION

Let there be in an arithmetic Euclidean space R^2 a connected region T_0 with a linear - piecewise frontier: $\{(x, y) \mid (x, y) \in R^2, 0 \leq y \leq A, 0 \leq x \leq z; A - \text{const}, z - \text{var}\}$, given in a fixed orthogonal coordinate system XOY . There also are objects $T_i \subset R^2$, $i = 1, 2, \dots, n$ ($n > 1$), which are canonically closed bounded point sets with an arcwise frontier and a connected interior (*int*). It is required to allocate objects T_i without overlapping in region T_0 so that value z reaches its minimum. In other words, it is necessary to determine

$$\min z \quad (1)$$

$$\text{s.t. :} \quad T_i \subset T_0 \quad (2)$$

$$\text{int}T_i \cap \text{int}T_j = \emptyset, i, j = 1, 2, \dots, n, i \neq j, \quad (3)$$

where $\emptyset$ is empty set.

In general the allocation objects are non-convex polygons T_i with allocation parameters $u_i = (v_i, \varphi_i)$, where $v_i = (x_i, y_i)$, $v_i \in \text{int}T_i$ are coordinates of the origin of eigen coordinate system OX_iY_i , φ_i is angular parameter, which equals a rotation angle of the system OX_iY_i with respect to the system OXY .

Allocation parameters (v_i, φ_i) and variable z induce space R^{3n+1} of allocation parameters, in which the problem (1-3) ought to be considered, and the conditions (2-3) describe the feasible region $D \subset R^{3n+1}$ of the problem.

The initial information about allocation objects is an ordered set of vertices $\{t_i^k(x_i^k, y_i^k)\}$, $k = 1, \dots, N_i$ of object T_i given in the system OX_iY_i .

2.1 $\Phi_{ij}(u_i, u_j)$ -functions

The constructive tool of formalizing the conditions of mutual non-overlapping of objects (condition (3)) is $\Phi_{ij}(u_i, u_j)$ -functions introduced for each pair (i, j) of objects being allocated (Stoyan (1980)). Moreover, the $\Phi_{0j}(u_i, u_j)$ -function can be also applied to describe the condition (2) of belonging of object T_i to region T_0 .

Definition 1 Any, everywhere defined and continuous function in R^6 , possessing the following characteristic property:

$$\Phi_{ij}(u_i, u_j) > 0, \text{ if } \text{cl}T_i(u_i) \cap \text{cl}T_j(u_j) = \emptyset, \quad (4)$$

$$\Phi_{ij}(u_i, u_j) = 0, \text{ if } \begin{matrix} \text{cl}T_i(u_i) \cap \text{cl}T_j(u_j) \neq \emptyset, \\ \text{int}T_i(u_i) \cap \text{int}T_j(u_j) = \emptyset, \end{matrix} \quad (5)$$

$$\Phi_{ij}(u_i, u_j) < 0, \text{ if } \text{int}T_i(u_i) \cap \text{int}T_j(u_j) \neq \emptyset, \quad (6)$$

where $\text{cl}T$ is topological closure of set T ,
is called Φ -function.

Hence, the characteristic property of the Φ_{ij} -function allows to speak about non-intersection, tangency and intersection of corresponding point sets.

Later on we shall be interested in following properties of $\Phi_{0j}(u_i, u_j)$. We assume that all the $\varphi_i = \text{const}$.

Property 1 For objects T_i and T_j the surface of the $\Phi_{ij}(v_i, v_j)$ -function defined by equation $\Phi_{ij}(v_i, v_j) = 0$, is linear-piecewise.

Property 2 The region of non-negative values of the $\Phi_{ij}(v_i, v_j)$ -function is a polyhedral closed set $D_{ij} \subset R^4$.

Property 3 The region $D_{ij} \subset R^4$ may be presented in the form $D_{ij} = \bigcup_{g=1}^{G_{ij}} \text{Tr}^g, G_{ij} \leq m_{ij} \leq O(K_i + K_j)$, where Tr^g is truncation, which is described by intersection of a finite number P_g closed half-spaces $H^l, l \in \{1, 2, \dots, m_{ij}\}$, defined by generating planes E^l , i.e. $D_{ij} = \bigcup_{g=1}^{G_{ij}} \bigcap_{p=1}^{P_g} H^l$, where $l = f(p, g)$.

The generating planes E^l are defined by equations $f^l(v_j, v_i) = 0$. Functions $f^l(v_j, v_i)$ have the form

$$f^l(v_j, v_i) = a^l \cdot (x_j - x_i) + b^l \cdot (y_j - y_i) + c^l, l = 1, 2, \dots, m_{ij}, \quad (7)$$

where coefficients a^l, b^l, c^l are functions of **certain** vertices of objects T_i and T_j .

The position of point (v_j, v_i) in half-space H^l is determined by inequality $f^l(v_j, v_i) \leq 0$.

Let us consider two-valued predicate $S(f^l(v_i, v_j))$, as Rvachev (1967), such that

$$S(f^l(v_i, v_j)) = \begin{cases} 1, & \text{if } f^l(v_i, v_j) \leq 0, \\ 0, & \text{if } f^l(v_i, v_j) > 0. \end{cases} \quad (8)$$

Then predicate $\Omega_{D_{ij}}(v_i, v_j)$, characterizing region D_{ij} , may be presented in the following form:

$$\Omega_{D_{ij}}(v_i, v_j) = \bigvee_{g=1}^{G_{ij}} \left(\bigwedge_{p=1}^{P_g} S(f_l(v_i, v_j)) \right), l \in \{1, 2, \dots, m_{ij}\}, \quad (9)$$

i.e. $\Omega_{D_{ij}}(v_i, v_j) = 1$, if $(v_i, v_j) \in D_{ij}$, and $\Omega(v_i, v_j) = 0$, if $(v_i, v_j) \notin D_{ij}$.

Structures of inequalities

To describe region D_{ij} , defined by predicate $\Omega_{D_{ij}}$, we shall use the structure of linear inequalities, following by Magas (1984). To simplify the exposition, we shall say that the l -th and r -th inequalities of the form (7) are connected by the conjunction operation, if $S(f^l)$ and $S(f^r)$ enter together at least into the one conjunct in expression (9). Otherwise, we shall say that the l -th and the r -th inequalities are connected by the disjunction operation.

Definition 2 *The structure of linear inequalities $S(F(x), \Delta, m)$ is an ordered set $F(x)$ of linear inequalities of the form $f_t(x) \geq 0$ ($x \in R^k, t = 1, 2, \dots, m$) with defined operations of conjunction or disjunction for each pair of inequalities; the operations are determined by the relation matrix $\Delta = \|\delta_{lr}\|_{m \times m}$ such that $\delta_{lr} = 1$ corresponds to the conjunction operation between the l -th and r -th inequalities ($l, r \in \{1, 2, \dots, m\}$) and $\delta_{lr} = 0$ corresponds to the disjunction operation.*

As is obvious from the definition, the notion of a structure of inequalities generalizes the notion of a system of inequalities. That is, if all the elements of matrix Δ of the inequalities structure $S(F(x), \Delta, m)$ are equal to unit, then the inequalities structure shall be identical to a system of inequalities of set $F(x)$. We denote the matrix of operations with unit elements as Δ^1 . Otherwise, if none pair of inequalities of structure $S(F(x), \Delta, m)$ is connected by conjunction operation, e.g. elements $\delta_{ll} = 1$ only, then we denote such an operation matrix as Δ^0 .

Therefore to describe region D_{ij} we shall use the structure $S_g(F_{ij}(v_i, v_j), \Delta, m_{ij})$,

Then under assumption that all the $\varphi_i = \text{const}$ the conditions of mutual pairwise non-overlapping of n objects being allocated are described by the structure

$$\bigcap_{g=1}^{C_2^n} S_{g=y(i,j)}(F_{ij}(v_i, v_j), \Delta, m_{ij}), \quad (10)$$

Using such a methodology one can define the analytical description of the allocation conditions of n objects T_i into region T_0 as the system

$$\bigcap_{i=1}^n S_{i0}(F_{i0}(v_i, z), \Delta^1, m_{i0}), \quad (11)$$

where each system S_{i0} is a system of a four linear inequalities of the form:

$$\begin{aligned} x_i - z &\leq -p_i^l, p_i^l = \max_k x_i^k; & -x_i &\leq p_i^r, p_i^r = \min_k x_i^k \\ y_i &\leq -p_i^d, p_i^d = \min_k y_i^k; & y_i + A &\leq -p_i^u, p_i^u = \max_k x_i^k, \\ & & k &= 1, \dots, N_i, \end{aligned} \quad (12)$$

In the case if φ_i are not *const* an ordered set $F(x)$ of non-linear inequalities with defined operations by analogy with above operations for each pair of inequalities is a structure of **non-linear** inequalities.

The feasible region D of the problem (1-3) is determined by the structure of non-linear inequalities.

2.2 Analytical model of the basic problem

Let set $C_i = \text{conv } T_i$ be a convex hull of object T_i and m_i^c is the number of vertices of C_i . Then the structure Σ_0 of inequalities describing the object T_i position in T_0 must include $4 \times m_i^c$ systems consisting of both support function $f_c(v_i, \varphi_i)$ of kind

$$x_i - z \leq -x_i^c(\varphi_i), \quad y_i \leq -y_i^c(\varphi_i), \quad y_i + A \leq -y_i^c(\varphi_i), \quad -x_i \leq x_i^c(\varphi_i), \quad (13)$$

where $x_i^c(\varphi_i) = x_i^c \cos \varphi_i - y_i^c \sin \varphi_i$, $x_i^c(\varphi_i) = y_i^c \cos \varphi_i + x_i^c \sin \varphi_i$, and pair of inequalities which determine the range of parameter φ_i . Then structure Σ_0 has a form:

$$S_1(F_1(x_i, 0, \varphi_i), \Delta^0, m_i^c) \cap S_2(F_2(x_i, z, \varphi_i), \Delta^0, m_i^c) \cap S_3(F_3(y_i, 0, \varphi_i), \Delta^0, m_i^c) \cap S_4(F_4(y_i, A, \varphi_i), \Delta^0, m_i^c) \quad (14)$$

It is necessary to note that after introducing into consideration the angular parameters the analytical defining the condition (3) is getting more complex one.

1. The preimage of each generating plane E^l in space R^6 is the surface γ_l which has the analytical description in the form of system $\{f_1, f_2, f_3\}$ depending on the geometric characteristics of the surface γ_l like

$$\begin{cases} f_1 : & (a^l \cos \varphi_i - b^l \sin \varphi_i)(x_j - x_i) + (b^l \cos \varphi_i + a^l \sin \varphi_i)(y_j - y_i) + \\ & (a^l x_j^k + b^l y_j^k) \cos(\varphi_j - \varphi_i) + (b^l x_j^k - a^l y_j^k) \sin(\varphi_j - \varphi_i) - \\ & (a^l x_i^h + b^l y_i^h) \leq 0 \\ f_2 : & 0 \leq \varphi_i \leq 2\pi \\ f_3 : & \varphi_i - \varphi_{jk}^+ \leq \varphi_j \leq \varphi_i + \varphi_{jk}^+ \end{cases}, \quad (15)$$

where $t_j^k = \{x_j^k, y_j^k\}$, $k \in \{1, \dots, N_j\}$ and $t_i^h = \{x_i^h, y_i^h\}$, $h \in \{1, \dots, N_j\}$ are vertices of objects T_i and T_j respectively.

2. Using the representation of each of objects T_i in the form $T_i = \bigcup_{g_i=1}^{G_i} T_{g_i}$, where T_{g_i}

is convex and $\bigcap_{g_i=1}^{G_i} T_{g_i} = \emptyset$ and taking into account that under consideration of the condition (3) for each pair of objects T_{g_i} and T_{g_j} coefficients a^l , b^l , c^l are functions of arbitrary vertices of objects T_{g_i} and T_{g_j} , we can write general structure S_{ij} describing condition (3) for objects T_i and T_j in the following way:

$$S_{ij} = \bigcap_{g_i} \bigcap_{g_j} S(F(v_{g_i}, v_{g_j}, \varphi_i, \varphi_j), m(\varphi_i, \varphi_j), \Delta^0). \quad (16)$$

Remark 1 Number $m(\varphi_i, \varphi_j)$ defining the amount of inequalities in set $F(u_{g_i}, u_{g_j})$ is a variable value and it depends on values φ_i and φ_j .

It is necessary to note that:

1. Support functions $f_c(v_i, \varphi_i)$ are concave at each point of region D .
2. Functions like f_1 from (11) in general (depending on the geometric characteristics of objects) have indefinite Hessian.
3. In general feasible region D is a non-connected set with a finite number η of connected components D_r , each of which in turn admits a representation in the form of a union of a finite number ω of such a non-convex subsets D_{rk} , that

$$D = \bigcup_r D_r = \bigcup_r \left(\bigcup_k D_{rk} \right) = \bigcup_{w=1}^W D_w, \quad (17)$$

4. It follows from above properties the problem (1-3) is multiextremal.

3 METHOD OF SOLVING BASIC PROBLEM

The following strategy for solving the local optimization problem is proposed:

1. Having fixed all the parameters φ_i (e.g. $\varphi_i = 0$) so that all the problem constraints became linear ones we solve the problem of local minimization objective function z on a some non-convex connected component D_δ of region $D_{linear} \subset R^{2n+1}$ ($D_\delta = \bigcup_{q=1}^Q D_{\delta q}$, $D_{\delta q}$ is convex subregion). In other words we solve the problem searching for $z(v^*) = \min_{D_\delta \subset R^{2n+1}} z$ on a structure $S_\delta(F(v, z), \Delta, m)$ of linear inequalities (see Stoyan, Yu.G., Novozhilova, M.V., and Kartashov, A.V., (1994)) and structure S_δ may be represented by formula:

$$S_\delta(F(v, z), \Delta, m) = \bigcup_{q=1}^Q S_q(F^q(v, z), \Delta^1, (m^q = 4n + n(n-1)/2)), \quad (18)$$

where each $S_q(F^q(v, z), \Delta^1, m^q)$ is a system of linear inequalities.

The point v^* in general is described by several systems of equations $Fx = B$, which may belong to different systems S_q .

2. We choose one of the systems $S_q^*(F^q(v, z), \Delta^1, m^q)$ and construct the mapping of the set of linear systems $\{S_q(F^q(v, z), \Delta^1, m^q)\}$ onto a set of nonlinear systems $\{S_q^n(F^q(v, z), \Delta^1, m^q)\}$ by the following rule: each constraint of type (7) is substituted by constraint of type (15) and each constraint of type (12) is substituted by constraint of type (13). Then the system $S_q^{*n}(F^q(v, z), \Delta^1, m^q)$ will define some region $D \subset R^{3n+1}$.

3. Considering the problem of local optimization on the system $S_q^{*n}(F^q(v, z), \Delta^1, m^q)$ as a minimax problem in the space R^{3n} with the objective function $\mathfrak{R} = \max_i (x_i + x_i^*(\varphi_i))$, where function $\mathfrak{R}$ is convex, we solve the problem

$$\begin{aligned} & \min \mathfrak{R} \\ \text{s.t.} \quad & S_q^*(F^q(v, z), \Delta^1, m^q) \end{aligned} \quad (19)$$

The problem belongs to a well-known problems of non-linear programming, the method-

ology of projected Lagrangian was selected by the authors for solving the problem. More exactly, the method of sequential quadratic programming is utilized, where the approximation of Lagrange function of problem (19) is present in the objective function of auxiliary problem. The initial active set is chosen from set of constraints, which corresponds to linear ones active at the point v^* .

It is essential that

1. Hessian of problem (19) Lagrange function may be easily calculated at each point of the region D by means of obvious transformations, however it is undetermined at each point of the region D that is why in finding a descent direction of a objective function the modified LDL^T decomposition of Hessian is performed.
2. The step value is chosen not only from method convergence condition but with regard for geometric characteristics of objects being allocated, namely range of parameters φ_i , as well. It is necessary to note, that in defining descent direction the constraints determining the range of φ_i are not taken into consideration.

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