Dual barrier-projection and barrier-Newton methods in linear programming

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Abstract

In this paper the stable version of dual barrier-projection method and the dual barrier-Newton method for solving linear programming problems are proposed. These methods are obtained by using space transformation techniques which are applied for releasing from nonnegativity of dual slack variables. The continuous and discrete versions of both methods are considered and their local convergence is investigated. A modification of the dual barrier-projection method is proposed in which a steepest descent is utilized.

Keywords

Linear programming, dual problem, space transformation, gradient projection method, Newton's method, steepest descent

1 INTRODUCTION

The paper deals with dual algorithms for solving linear programming problems. These algorithms are similar to primal ones which were proposed by Evtusenko and Zhadan (1978, 1994) and which are based on space transformation techniques. Using a surjective transformation we convert the original linear programming problem with nonnegative variables to a new problem where variables belong to the whole space. We can apply various numerical methods for solving this new problem, in particular, the stable version of the gradient projection method (Tanabe, 1980) and the Newton method. After an inverse transformation to the original space a family of barrier-projection and barrier-Newton methods are obtained. As a result of a space transformation additional diagonal matrices appear in the right-hand sides of formulas describing methods. These matrices prevent trajectories from leaving the positive orthant of the space. Different numerical methods are obtained by different choices of the space transformation.

This paper is devoted to dual barrier-projection and barrier-Newton methods. The space transformation technique is applied there for dual slack variables. In section 2 we outline the basic approach and propose a family of dual barrier-projection methods. These

methods are described by systems of ordinary differential equations. Numerical algorithms are obtained as discretization of dynamical systems. In the case of exponential space transformation these methods are similar to the dual affine scaling algorithm proposed by Adler, Karmarkar, Resende and Veiga (1989). We show that if the quadratic space transformation is used then we obtain an exponential rate of convergence for continuous methods and a linear rate of convergence for discrete versions.

In section 3 we consider a variant of the dual barrier-projection method with a steepest descent which converges to the solution of the dual problem in a finite number of iterations. In section 4 we propose the barrier-Newton method.

2 DUAL BARRIER-PROJECTION METHOD

We consider a linear programming problem

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax = b \\ & && x \geq 0_n, \end{aligned} \tag{1}$$

and its dual

$$\begin{aligned} & \text{maximize} && b^T u \\ & \text{subject to} && A^T u \leq c, \end{aligned} \tag{2}$$

where A is a full rank $m \times n$ matrix where $m < n$, x and c are n -dimensional vectors, and b and u are m -dimensional vectors.

We define the set of feasible solutions of dual problem (2) and its interior

$$U = \{u \in R^m : v = c - A^T u \geq 0_n\}, \quad U_0 = \{u \in R^m : v = c - A^T u > 0_n\}.$$

Throughout the paper we assume that U_0 is nonempty and that the primal and dual nondegeneracy holds.

Let $v = \varphi(w)$ be a componentwise space transformation from R^n to the nonnegative orthant R_+^n , i.e. $v^i = \varphi^i(w^i)$, $1 \leq i \leq n$. Suppose also that this transformation is differentiable and is such that the closure of $\varphi(R^n)$ coincides with R_+^n . In this case every vector in R_+^n is the image of at least one vector in R^n or it is the image of a limit point of a sequence in R^n .

Let $w^i = \psi^i(v^i)$ denote the inverse transformation of $\varphi^i(w^i)$. This transformation exists at least in a neighborhood of a point $v_0^i = \varphi^i(w_0^i)$, as long as $\dot{\varphi}^i(w_0^i) \neq 0$. If $\tilde{J}(w)$ is the Jacobian matrix of the transformation $\varphi(w)$, then the Gram matrix $G(v) = \tilde{J}(\psi(v))\tilde{J}^T(\psi(v))$ is diagonal, i.e.

$$G(v) = D(\theta(v)), \quad \theta(v) = [\theta^1(v^1), \theta^2(v^2), \dots, \theta^n(v^n)],$$

where $\theta^i(v^i) = (\gamma^i(v^i))^2$, $\gamma^i(v^i) = \dot{\varphi}^i(\psi^i(v^i))$, $1 \leq i \leq n$.

We impose the following conditions on the space transformation $\varphi(w)$:

C_1 . The functions $\theta^i(v^i)$ are defined and continuous in some neighborhood of R_+^1 and $\theta^i(v^i) = 0$ if and only if $v^i = 0$, where $1 \leq i \leq n$.

C_2 . The functions $\theta^i(v^i)$ are continuously differentiable in some neighborhood of R_+^1 and $\theta^i(0) > 0$, $1 \leq i \leq n$.

Here we consider only two simple surjective transformations:

$$v = \frac{1}{4}D(w)(w), \quad v = e^{-w}, \quad (3)$$

where the i -th component of the n -vector e^{-w} is e^{-w^i} . For these transformations we obtain, respectively

$$\theta(v) = v, \quad G(v) = D(v); \quad \theta(v) = D(v)v, \quad G(v) = D^2(v).$$

In both cases the Gram matrix is singular on the boundary of R_+^n . These transformations satisfy C_1 . Condition C_2 holds only for the first quadratic transformation (3).

By extension of the space and by converting the inequality constraints to equalities, we transform the original dual problem (2) into the following equivalent problem in the extended space

$$\begin{aligned} &\text{maximize} && b^T u \\ &\text{subject to} && \varphi(w) + A^T u - c = 0_n. \end{aligned} \quad (4)$$

The Lagrangian associated with this problem is defined by

$$\tilde{L}(u, w, x) = b^T u + x^T [\varphi(w) + A^T u - c].$$

For solving problem (4) we use the stable version of the gradient projection method which is proposed by Tanabe (1980). The method is stated as an initial-value problem involving the following system of ordinary differential equations

$$\frac{du}{dt} = \tilde{L}_u(u, w, x(u, w)), \quad \frac{dw}{dt} = \tilde{L}_w(u, w, x(u, w)). \quad (5)$$

The function $x(u, w)$ is chosen to satisfy the following condition

$$\tilde{L}_{xu}(u, w, x)\dot{u} + \tilde{L}_{xw}(u, w, x)\dot{w} = -\tau \tilde{L}_x(u, w, x). \quad (6)$$

Since $\dot{v} = \varphi_w \dot{w}$, (5) can be rewritten in terms of u and v as follows

$$\frac{du}{dt} = b - Ax(u, v), \quad \frac{dv}{dt} = -G(v)x(u, v), \quad (7)$$

where $\Phi(v)x(u, v) = A^T b + \tau(v + A^T u - c)$ and $\Phi(v) = G(v) + A^T A$.

Denote $v(u) = c - A^T u$.

Proposition 1 *Let the space transformation $\varphi(w)$ satisfy C_1 . Then the matrix $\Phi(v(u))$ is positive definite for all $u \in U_0$.*

Proposition 2 *If all extreme points of the bounded set U are nondegenerate, then $\Phi(v(u))$ is positive definite for any $u \in U$.*

Let $[u(t, z_0), v(t, z_0)]$ denote the solution of the Cauchy problem (7) with initial condition $u(0, z_0) = u_0$, $v(0, z_0) = v_0$, $z_0^T = [u_0^T, v_0^T]$. This system of differential equations has the first integral

$$v(t, z_0) + A^T u(t, z_0) - c = (v_0 + A^T u_0 - c)e^{-\tau t}.$$

Thus, if the initial point z_0 is such that $v_0 = v(u_0)$, then $v(t, z_0) \equiv c - A^T u(t, u_0)$ for all $t \geq 0$. In this case the system (7) can be simplified

$$\frac{du}{dt} = b - Ax(u), \quad (G(v) + A^T A)x(u) = A^T b, \quad (8)$$

where $u(0, u_0) = u_0 \in U$. For this system we obtain the following inequality

$$b^T \frac{du}{dt} = \|b - Ax(u)\|^2 + x^T(u)G(v(u))x(u) \geq 0. \quad (9)$$

Hence the objective function of the dual problem monotonically increases on a feasible set.

Numerical algorithms are obtained as discretization of dynamical systems (7), (8). By applying the Euler numerical integration method we obtain the following iterative algorithm

$$u_{k+1} = u_k + \alpha_k(b - Ax_k), \quad v_{k+1} = v_k - \alpha_k G(v_k)x_k, \quad (10)$$

$$(G(v_k) + A^T A)x_k = A^T b + \tau(v_k + A^T u_k - c).$$

Similarly for the system (8) we have

$$u_{k+1} = u_k + \alpha_k(b - Ax_k), \quad (G(v_k) + A^T A)x_k = A^T b, \quad (11)$$

where $v_k = v(u_k)$. Both variants solve the primal and dual problems simultaneously.

Theorem 1 *Let x_* and u_* be unique nondegenerate solutions of Problems (1) and (2), respectively, and let $v_* = c - A^T u_*$. Assume that the space transformation $\varphi(w)$ satisfies conditions C_1 , C_2 and $\tau > 0$. Then the following statements are true:*

1. *The pair $[u_*, v_*]$ is an asymptotically stable equilibrium state of system (7).*
2. *The solutions $u(t, z_0), v(t, z_0)$ of system (7) converge locally to the pair $[u_*, v_*]$. The corresponding function $x(u(t, z_0), v(t, z_0))$ converges to the optimal solution x_* of the primal problem (1).*
3. *The point u_* is an asymptotically stable equilibrium state of system (8).*
4. *The solutions $u(t, u_0)$ of system (8) converge locally to the optimal solution u_* of the dual problem (2). The corresponding function $x(u(t, u_0))$ converges to the optimal solution x_* of the primal problem (1).*
5. *There exists an $\alpha_* > 0$ such that for any fixed $0 < \alpha_k < \alpha_*$ the sequence $\{u_k, v_k\}$ generated by (10) converges locally with a linear rate to $[u_*, v_*]$ while the corresponding sequence $\{x_k\}$ converges to x_* .*

6. There exists an $\alpha_* > 0$ such that for any fixed $0 < \alpha_k < \alpha_*$ the sequence $\{u_k\}$ generated by (11) converges locally with a linear rate to u_* while the corresponding sequence $\{x_k\}$ converges to x_* .

If $u \in U_0$, then we can transform (8) to the equivalent system

$$\frac{du}{dt} = (I_m + AG^{-1}(v)A^T)^{-1}b,$$

where I_m is the $m \times m$ identity matrix. In the case where $G(v) = D^2(v)$ this method is similar to the dual affine scaling algorithm which was proposed and investigated by Adler, Karmarkar, Resende and Veiga (1989).

3 STEEPEST DESCENT

It follows from (9) that in order to maximize the value of the objective function at each iteration in the method (11) we must choose the step length α_k as large as possible provided that the point u_{k+1} belongs to the feasible set. This way of stepsize choice leads to a variant of the dual barrier-projection method (11) in which a steepest descent is utilized.

Let $u \in U$ and assume that

$$v(u) = \begin{bmatrix} v^B(u) \\ v^N(u) \end{bmatrix}, \quad v^B(u) \in R^s, \quad v^N(u) \in R^d,$$

where $v^B(u) = 0_s, v^N(u) > 0_d, d = n - s, s \leq m$. Denote the $m \times s$ matrix determined by the first s columns as B and denote the $m \times d$ matrix determined by the rest columns as N . Supposing for simplicity that $G(v) = D(v)$, we can write

$$\Phi(v) = \begin{bmatrix} B^T B & B^T N \\ N^T B & D(v^N) + N^T N \end{bmatrix}. \quad (12)$$

If B is a full rank matrix then, using the Frobenious formula, we obtain

$$\Phi^{-1} = \begin{bmatrix} Q & -QB^T W N D_N^{-1} \\ -D_N^{-1} N^T W B Q & D_N^{-1} - D_N^{-1} N^T [W - W B Q B^T W] N D_N^{-1} \end{bmatrix}, \quad (13)$$

where $Q = (B^T W B)^{-1}$, $W = (I_m + N D_N^{-1} N^T)^{-1}$, $D_N = D(v^N)$

Substituting (13) in (8) yields

$$\frac{du}{dt} = [W - W B Q B^T W] b. \quad (14)$$

Moreover, we have

$$x(u) = \begin{bmatrix} x^B(u) \\ x^N(u) \end{bmatrix}, \quad x^B(u) = Q B^T W b, \quad x^N(u) = D_N^{-1} N^T [W - W B Q B^T W] b.$$

Thus

$$a_i^T \frac{du}{dt} = 0_s, \quad i \in J^B(u), \quad (15)$$

where a_i is the i -th column of the matrix A , and $J^B(u) = \{1 \leq i \leq n : v^i(u) = 0\}$. We supposed previously that $J^B(u) = \{1, 2, \dots, s\}$.

The equality (15) shows that an adhesion property holds and that the trajectory can't leave the boundary of the constraint $a_i^T u \leq c^i$ even if $x^i(u) < 0$. In order to overcome this drawback we will perturb the righthand side of the method (14). Denote

$$J_-^B(u) = \{i \in J^B(u) : x^i(u) < 0\}, \quad J_+^B(u) = \{i \in J^B(u) : x^i(u) \geq 0\}.$$

Let $\varepsilon > 0$ and let

$$y \in S^B(u) = \{z \in R_+^s : \sum_{i \in J^B(u)} z^i = 1, z^i = 0, i \in J_+^B(u)\}.$$

We introduce a diagonal matrix Y of order s such that $Y^2 = D(y)$ and modify the matrix $\Phi(v)$ changing only the left upper submatrix

$$\Phi(v) = \begin{bmatrix} B^T B + \varepsilon Y^2 & B^T N \\ N^T B & D(v^N) + N^T N \end{bmatrix}.$$

Then instead of (14) we obtain

$$\frac{du}{dt} = \{W - WB[Q - \varepsilon P]B^T W\} b, \quad (16)$$

where $P = QY[I_s + \varepsilon YQY]^{-1}YQ$. Now equality (15) takes place only for $i \in J^B(u)$ such that $y^i = 0$. Moreover, if ε is sufficiently small and $y^i > 0$ for $i \in J_-^B(u)$, then

$$a_i^T \frac{du}{dt} > 0, \quad i \in J_-^B(u). \quad (17)$$

There are several approaches to choosing the vector y from the set $S^B(u)$. The simplest one is the approach where a single index $i_k \in J_-^B(u_k)$ is selected and y is set to be equal to the vector $e_{i_k} \in S^B(u_k)$ with the components

$$e_{i_k}^i = 0, \quad i \neq i_k, \quad e_{i_k}^{i_k} = 1.$$

In this case the inequality (17) holds for any $\varepsilon > 0$.

We denote by q_i the i -th column of the matrix Q and by q_i^j the j -th element of q_i . We also introduce a vector

$$\Delta = A^T W [I_m - BQ B^T W + (1 + q_{i_k}^{i_k})^{-1} B q_{i_k} q_{i_k}^T B^T W] b$$

and index sets

$$J^N(u) = \{1 \leq i \leq n : v^i(u) > 0\}, \quad J_+^N(u) = \{i \in J^N(u) : \Delta^i > 0\}.$$

Let, for simplicity, $\varepsilon = 1$. Then there is a variant of the method (11) with a steepest descent which can be written as

$$u_{k+1} = u_k + \alpha_k W [I_m - BQ B^T W + (1 + q_{i_k}^{i_k})^{-1} B q_{i_k} q_{i_k}^T B^T W] b, \quad (18)$$

where

$$\alpha_k = \min_{i \in J_+^N(u_k)} \frac{v_k^i}{\Delta_k^i}.$$

Theorem 2 *Let the set U be bounded and let the function $b^T u$ have different values at extreme points of U . Then the method (18) solves the problem (2) in a finite number of iterations.*

Geometrically, the adding of the matrix $D(y)$ means the shifting of active constraints in dual problem (2). In method (18) we shift only a single active constraint. This simplest variant of the dual method possesses the following property, namely, if the point u_k is an extreme point of the set U , then the further behavior of the method is the same as a dual simplex method.

4 DUAL BARRIER-NEWTON METHOD

The method (11) can be interpreted as a contraction mapping for solving the system of nonlinear equations

$$b - Ax(u) = 0_m,$$

where the function $x(u)$ is defined from the condition

$$(D(\theta(v)) + A^T A)x(u) = A^T b. \quad (19)$$

We can apply the Newton method for this purpose. The continuous version of the Newton method leads to the following system of ordinary differential equations

$$\Lambda(u) \frac{du}{dt} = Ax(u) - b, \quad \Lambda(u) = -A \frac{dx}{du}. \quad (20)$$

By differentiating the relation (19) with respect to u , we obtain

$$-D(\dot{\theta}(v))D(x)A^T + (D(\theta(v)) + A^T A) \frac{dx}{du} = 0_{nm}.$$

Therefore

$$\Lambda(u) = -A (D(\theta(v)) + A^T A)^{-1} D(\dot{\theta}(v))D(x(u))A^T.$$

Proposition 3 *Let all conditions of Theorem 1 be fulfilled. Then $\Lambda(u_*)$ is a nonsingular matrix.*

Theorem 3 *Let all conditions of Theorem 1 hold. Then u_* is an asymptotically stable equilibrium point of system (20). If the matrix $\Lambda(u)$ satisfies a Lipschitz condition in a neighborhood of u_* , then the discrete version*

$$u_{k+1} = u_k + \alpha_k \Lambda^{-1}(u_k)(Ax_k - b), \quad x_k = x(u_k), \quad (21)$$

locally converges with at least linear rate to the point u_ if the stepsize α_k is fixed and $0 < \alpha_k < 2$. If the matrix $\Lambda(u)$ satisfies a Lipschitz condition in a neighborhood of u_* and $\alpha_k = 1$, then the sequence $\{u_k\}$ converges quadratically to u_* .*

Let us suppose now that we use the quadratic space transformation (3). Then the method (20) can be rewritten as follows

$$\frac{du}{dt} = \left[A \left(D(v(u)) + A^T A \right)^{-1} D(x(u)) A^T \right]^{-1} (b - Ax(u)). \quad (22)$$

On the set U_0 the right-hand side of this equation can be simplified

$$\frac{du}{dt} = \left[AD(x(u)) D^{-1}(v(u)) A^T \right]^{-1} b, \quad x(u) = \left[I_m - (I_m + AD^{-1}(v) A^T)^{-1} \right] b. \quad (23)$$

Thus the objective function of the dual problem increases along the Newton direction.

According to Theorem 3 the trajectories of the system (22) converge locally to u_* . The trajectories of the system (23) converge locally to u_* on the set U_0 . The discrete version of the method (22) is similar to (21). It is possible to consider the discrete version of the method (23) using a steepest descent.

5 ACKNOWLEDGEMENT

Research supported by the Russian Scientific Fund, grant number 94-01-01379, and partially by the Czech Research Grant Agency, grant number 201/93/0429.

REFERENCES

- Adler, I., Karmarkar, N., Resende, M. and Veiga, G. (1989) An implementation of Karmarkar's algorithm for linear programming. *Mathematical programming*, **44**, 297-335.
- Evtushenko, Yu. and Zhadan, V. (1978) A relaxation method for solving problems of nonlinear programming. *U.S.S.R. Computational Mathematics and Mathematical Physics*, **17**, 73-87.
- Evtushenko, Yu. and Zhadan, V. (1994) Stable Barrier-projection and Barrier-Newton Methods in Linear Programming. *Computational Optimization and Applications*, **3**, 289-303.
- Tanabe K. (1980) A geometric method in nonlinear programming. *Journal of Optimization Theory and Applications*, **30**, 181-210.