Dual Bregman proximal methods for large-scale 0–1 problems

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Abstract

We describe an extension of the Bregman proximal method for convex programming, employing B -functions as generalizations of Bregman functions that cover more applications. We allow inexact subproblem solutions, increasing their accuracy successively to retain global convergence. Our framework is applied to Lagrangian relaxations of large-scale set covering problems that arise in airline crew scheduling.

Keywords

Proximal point methods, Bregman functions, set covering problems.

1 INTRODUCTION

We describe an extension of the Bregman proximal method for convex programming (Censor and Zenios, 1992; Chen and Teboulle, 1993; Eckstein, 1993). First, we consider B -functions of (Kiwiel, 1994) as generalizations of Bregman functions (Censor and Lent, 1981) that cover more applications. Second, we allow inexact subproblem solutions, increasing their accuracy successively to retain global convergence. Our framework is applied to Lagrangian relaxations of large-scale set covering problems that arise in airline crew scheduling (Wedelin, 1993). We report encouraging preliminary computational experience.

Our notation is fairly standard. $\langle \cdot, \cdot \rangle$ and $|\cdot|$ are the Euclidean inner product and norm respectively. $\mathbb{R}_+$ and $\mathbb{R}_>$ are the nonnegative and positive reals respectively. $\text{cl } C$ and $\text{ri } C$ denote the closure and relative interior of $C \subset \mathbb{R}^n$. We let $\iota_C(x) = 0$ if $x \in C$, ∞ otherwise, and $\iota_C^* = \sup_{x \in C} \langle \cdot, x \rangle$. For a convex function f , $\mathcal{D}_f = \{x : f(x) < \infty\}$, $\partial f(\cdot) = \{g : f(x) \geq f(\cdot) + \langle g, \cdot - x \rangle \ \forall x\}$ and $\mathcal{D}_{\partial f} = \{x : \partial f(x) \neq \emptyset\}$.

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2 B-FUNCTIONS

For any convex function h on $\mathbb{R}^n$, we define its *difference functions*

$$\begin{aligned} D_h^b(x, y) &= h(x) - h(y) - v_{\partial h(y)}^*(x - y) & \forall x, y \in \mathcal{D}_h, \\ D_h^\sharp(x, y) &= h(x) - h(y) + v_{\partial h(y)}^*(y - x) & \forall x, y \in \mathcal{D}_h. \end{aligned}$$

By convexity, $h(x) \geq h(y) + v_{\partial h(y)}^*(x - y)$ and

$$0 \leq D_h^b(x, y) \leq h(x) - h(y) - \langle g, x - y \rangle \leq D_h^\sharp(x, y) \quad \forall x, y \in \mathcal{D}_h, g \in \partial h(y). \quad (1)$$

D_h^b and $D_h^\sharp$ generalize the usual *D-function* of h (Censor and Lent, 1981), defined by

$$D_h(x, y) = h(x) - h(y) - \langle \nabla h(y), x - y \rangle \quad \forall x \in \mathcal{D}_h, y \in \mathcal{D}_{\nabla h}, \quad (2)$$

since

$$D_h(x, y) = D_h^b(x, y) = D_h^\sharp(x, y) \quad \forall x \in \mathcal{D}_h, y \in \mathcal{D}_{\nabla h}. \quad (3)$$

(Readers only interested in our applications may assume h is s.t. $\partial h(y) = \{\nabla h(y)\}$ for all points y of interest.)

Definition 1 (Kiwiel, 1994) A closed proper (possibly nondifferentiable) convex function h is called a *B-function* (*generalized Bregman function*) if

- (a) h is strictly convex on $\mathcal{D}_h$.
- (b) h is continuous on $\mathcal{D}_h$.
- (c) For every $\alpha \in \mathbb{R}$ and $x \in \mathcal{D}_h$, the set $\mathcal{L}_h^b(x, \alpha) = \{y \in \mathcal{D}_{\partial h} : D_h^b(x, y) \leq \alpha\}$ is bounded.
- (d) For every $\alpha \in \mathbb{R}$ and $x \in \mathcal{D}_h$, if $\{y^k\} \subset \mathcal{L}_h^b(x, \alpha)$ is a convergent sequence with limit $y^* \in \mathcal{D}_h \setminus \{x\}$, then $D_h^\sharp(y^*, y^k) \rightarrow 0$.

General properties of *B-functions* are discussed by (Kiwiel, 1994). From lack of space, we only quote the following simple consequence of Lemmas 2.8–2.10 of (Kiwiel, 1994).

Lemma 2 Let ψ be a proper convex function on $\mathbb{R}$. Then $h(x) = \sum_{i=1}^n \psi(x_i)$ is a *B-function* iff ψ is closed, essentially strictly convex and $\mathcal{D}_{\psi^*} = \text{ri } \mathcal{D}_{\psi^*}$, where $\psi^*(\cdot) = \sup_t \langle \cdot, t \rangle - \psi(t)$ is the conjugate of ψ . In particular $\mathcal{D}_{\psi^*} = \text{ri } \mathcal{D}_{\psi^*} = \mathbb{R}$ if $\mathcal{D}_{\psi}$ is bounded.

Examples 3 Let $\psi : \mathbb{R} \rightarrow (-\infty, \infty]$ and $h(x) = \sum_{i=1}^n \psi(x_i)$; then $h^*(y) = \sum_i \psi^*(y_i)$. In each example, h is an essentially smooth *B-function* with $\partial h = \{\nabla h\}$ and $\mathcal{D}_h = \mathbb{R}^n$.

1 (Fermi/Dirac). $\psi(t) = t \ln t + (1 - t) \ln(1 - t)$ on $\mathcal{D}_\psi = [0, 1]$ ($0 \ln 0 = 0$). Then $\psi^*(t) = \ln(1 + \exp t)$ on $\mathcal{D}_{\psi^*} = \mathbb{R}$.

2 (Burg's entropy). $\psi(t) = -\ln t - \ln(1 - t)$ on $\mathcal{D}_\psi = [0, 1]$. Then $\psi^*(t) = \frac{1}{2}[(t^2 + 4)^{1/2} + t - 2] + \ln[(t^2 + 4)^{1/2} - 2] - \ln t^2$ for $t \neq 0$, $\psi^*(0) = -\ln 4$.

3 (Hellinger). $\psi(t) = -\sqrt{t(1 - t)}$ on $\mathcal{D}_\psi = [0, 1]$. Then $\psi^*(t) = \frac{1}{2}(t + \sqrt{1 + t^2})$.

3 AN INEXACT B -PROXIMAL METHOD

Consider the convex minimization problem $f_* = \inf_X f$, where $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is closed proper convex and X is a nonempty closed convex set in $\mathbb{R}^n$. Suppose h is a B -function s.t. $\mathcal{D}_{f_X} \cap \mathcal{D}_h \neq \emptyset$, where $f_X = f + \iota_X$ is the essential objective. Let $\{t_k\} \subset \mathbb{R}_+$ and $\{\epsilon_k\} \subset \mathbb{R}_+$ satisfy $\sum_{k=1}^\infty t_k = \infty$ and $\lim_{l \rightarrow \infty} \sum_{k=1}^l t_k \epsilon_k / \sum_{k=1}^l t_k = 0$.

The *inexact Bregman-prox method* of (Kiwiel, 1995) runs as follows. At iteration $k \geq 1$, having $x^k \in \mathcal{D}_{f_X} \cap \mathcal{D}_{\partial h}$, $\gamma^k \in \partial h(x^k)$ and $D_h^k(\cdot, x^k) = h(\cdot) - h(x^k) - \langle \gamma^k, \cdot - x^k \rangle$, minimize $f_X + D_h^k(\cdot, x^k)/t_k$ approximately to find x^{k+1} , γ^{k+1} and p^{k+1} satisfying

$$\gamma^{k+1} \in \partial h(x^{k+1}), \quad (4)$$

$$t_k p^{k+1} + \gamma^{k+1} - \gamma^k = 0, \quad (5)$$

$$p^{k+1} \in \partial_{\epsilon_k} f_X(x^{k+1}), \quad (6)$$

$$f_X(x^{k+1}) \leq f_X(x^k). \quad (7)$$

We note that (4)–(6) imply

$$x^{k+1} \approx \arg \min \{ f(x) + D_h^k(x, x^k)/t_k : x \in X \} \quad (8)$$

with $0 \leq D_h^k(\cdot, x^k) \leq D_h^k(\cdot, x^k) \leq D_h^h(\cdot, x^k)$; in fact x^{k+1} is an ϵ_k -minimizer of $f_X + D_h^k(\cdot, x^k)/t_k$ (Kiwiel, 1995). The *exact* Bregman-prox method (Censor and Zenios, 1992; Chen and Teboulle, 1993; Eckstein, 1993) corresponds to assuming that $\epsilon_k \equiv 0$ and $\mathcal{D}_{\partial h} = \mathcal{D}_{\nabla h}$, so that $p^{k+1} \in \partial f_X(x^{k+1})$, $\gamma^k = \nabla h(x^k)$ and $D_h^k = D_h$ for all k (cf. (3)). A particular implementation of our inexact method will be described in the next section.

The following global convergence results are established in (Kiwiel, 1995).

- Theorem 4** (a) $f_X(x^k) \downarrow \inf_{\mathcal{D}_h} f_X = \inf_{\text{cl}(\mathcal{D}_h \cap \mathcal{D}_{f_X})} f$. Hence $f_X(x^k) \downarrow \inf_X f$ if $\mathcal{D}_{f_X} \subset \mathcal{D}_h$. If $\text{ri } \mathcal{D}_h \cap \text{ri } \mathcal{D}_{f_X} \neq \emptyset$ (e.g., $\text{int } \mathcal{D}_h \cap \mathcal{D}_{f_X} \neq \emptyset$) then $\inf_{\mathcal{D}_h} f_X = \inf_{\text{cl } \mathcal{D}_h} f_X = \inf_{(\text{cl } \mathcal{D}_h) \cap (\text{cl } \mathcal{D}_{f_X})} f$. If $\text{ri } \mathcal{D}_{f_X} \subset \text{cl } \mathcal{D}_h$ (e.g., $\mathcal{D}_{\partial f_X} \subset \text{cl } \mathcal{D}_h$) then $\text{cl } \mathcal{D}_h \supset \text{cl } \mathcal{D}_{f_X}$ and $\text{Arg min}_X f \subset \text{cl } \mathcal{D}_h$.
- (b) If $\sum_{k=1}^\infty t_k \epsilon_k < \infty$ and $X_* = \text{Arg min}_{\mathcal{D}_h} f_X$ is nonempty then $\{x^k\}$ converges to some $x^\infty \in X_*$, and $x^\infty \in \text{Arg min}_X f$ if $\mathcal{D}_{f_X} \subset \mathcal{D}_h$.
- (c) If $\mathcal{D}_{f_X} \subset \mathcal{D}_h$ and $X_* = \emptyset$ then $|x^k| \rightarrow \infty$.

In certain applications it may be difficult to satisfy the descent requirement (7). Fortunately, it may be replaced by another condition on the tolerances $\{\epsilon_k\}$ (Kiwiel, 1995).

Theorem 5 Let $s_k = \sum_{j=1}^k t_j$ for all k . The assertions of Theorem 4 hold if the descent condition (7) is replaced by the accuracy condition $\sum_{k=1}^l s_k \epsilon_k / s_l \rightarrow 0$.

For choosing $\{\epsilon_k\}$, one may use the following result of (Kiwiel, 1995).

Lemma 6 (i) If $\epsilon_k \rightarrow 0$ then $\sum_{k=1}^l t_k \epsilon_k / s_l \rightarrow 0$ as $l \rightarrow \infty$.

- (ii) If $\sum_{k=1}^{\infty} \epsilon_k < \infty$ and $\{t_k\} \subset (0, t_{\max}]$ for some $t_{\max} < \infty$ then $\sum_{k=1}^{\infty} t_k \epsilon_k < \infty$.
 (iii) If $\sum_{l=1}^{\infty} t_l/s_l = \infty$ and $\lim_{k \rightarrow \infty} \epsilon_k s_k/t_k = 0$ then $\lim_{l \rightarrow \infty} \sum_{k=1}^l s_k \epsilon_k/s_l = 0$.
 (iv) If $\{t_k\} \subset [t_{\min}, t_{\max}]$ for some $0 < t_{\min} \leq t_{\max}$ and $k \epsilon_k \rightarrow 0$ then $\sum_{k=1}^l s_k \epsilon_k/s_l \rightarrow 0$.

4 LAGRANGIAN RELAXATIONS OF SET COVERING PROBLEMS

Let $A \in \{0, 1\}^{m \times n}$, $e = (1, \dots, 1)^T \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Consider the *set covering problem*

$$\min \{c^T x : Ax \geq e, x \in \{0, 1\}^n\}. \quad (9)$$

To solve its LP *relaxation*

$$\min \{c^T x : Ax \geq e, x \in [0, 1]^n\} \quad (10)$$

via the Bregman-prox method, let $f(x) = c^T x$, $X = \{x : Ax \geq e\} \cap [0, 1]^n$ and choose a B-function h as in Ex. 3, so that $\text{cl } \mathcal{D}_h = [0, 1]^n$ and $\partial h = \{\nabla h\}$ on $\mathcal{D}_{\partial h} = (0, 1)^n$. Then $\gamma^k = \nabla h(x^k)$ and (8) reads

$$x^{k+1} \approx \arg \min \{c^T x + [h(x) - \nabla h(x^k)^T x]/t_k : Ax \geq e\}. \quad (11)$$

The *dual problem*, obtained by assigning a multiplier λ to the constraints $Ax \geq e$, is

$$\max \{q(\lambda) : \lambda \in \mathbb{R}_+^m\}, \quad (12)$$

where q is the concave *continuously differentiable* dual functional (Kiwiel, 1994)

$$\begin{aligned} q(\lambda) &= \min_x \{c^T x + [h(x) - \nabla h(x^k)^T x]/t_k + \lambda^T (e - Ax)\} \\ &= -h^*(t_k(A^T \lambda - c) + \nabla h(x^k))/t_k + e^T \lambda \end{aligned} \quad (13)$$

with $\nabla q(\lambda) = e - Ax(\lambda)$ for all $\lambda \in \mathbb{R}^m$, where

$$\begin{aligned} x(\lambda) &= \arg \min_x \{c^T x + [h(x) - \nabla h(x^k)^T x]/t_k + \lambda^T (e - Ax)\} \\ &= \nabla h^*(t_k(A^T \lambda - c) + \nabla h(x^k)). \end{aligned} \quad (14)$$

Note that $q(\lambda)$ and $x(\lambda)$ are easily computed for h as in Ex. 3 even in the large-scale case.

In effect, (12) may be solved approximately in the large-scale case by coordinate or block coordinate ascent (Kiwiel, 1994). However, having obtained an approximate solution λ^k of (12), we cannot simply set x^{k+1} to $x(\lambda^k)$, because $x(\lambda^k)$ will not, in general, be feasible in (11). The following result helps in constructing x^{k+1} from $x(\lambda^k)$ to satisfy (4)–(6).

Theorem 7 *Let λ^k be an approximate solution of (12). Suppose $x^{k+1} \in \mathcal{D}_{\nabla h}$ satisfies $Ax^{k+1} \geq e$ and $(Ax^{k+1} - e)^T \lambda^k + e^T \delta_+^k - \langle \delta^k, x^{k+1} \rangle \leq \epsilon_k$, where $\delta^k = [\nabla h(x(\lambda^k)) - \nabla h(x^{k+1})]/t_k$ and $[\delta_+^k]_i = \max\{\delta_i^k, 0\}$, $i = 1:n$. Then (4)–(6) hold with $\gamma^{k+1} = \nabla h(x^{k+1})$.*

The algorithm of (Kiwiel, 1994) applied to (12) generates a sequence $\{\lambda^{(l)}\} \in \mathbb{R}_+^m$ s.t. $(Ax(\lambda^{(l)}) - e)^T \lambda^{(l)} \rightarrow 0$, $x(\lambda^{(l)}) \rightarrow \hat{x}^{k+1} \in (0, 1)^n$ and $\nabla h(x(\lambda^{(l)})) \rightarrow \nabla h(\hat{x}^{k+1})$ as $l \rightarrow \infty$, where $\hat{x}^{k+1}$ solves (11), so that $A\hat{x}^{k+1} \geq e$. Hence, for l sufficiently large, one may take $\lambda^k = \lambda^{(l)}$ and obtain x^{k+1} by increasing slightly certain components of $x(\lambda^k)$ (using $A \in \{0, 1\}^{m \times n}$) to satisfy the conditions of Theorem 7 for any $\epsilon_k > 0$. In effect, Theorem 5 applies, i.e., $\{c^T x^k\}$ converges to the optimal value of the LP relaxation (10), and $\{x^k\}$ converges to a solution of (10) if $\sum_k t_k \epsilon_k < \infty$.

Integer programming (IP) problems of the form (9) are usually solved by branch and bound, and their LP relaxations are often highly degenerate, which makes them difficult for simplex-based methods (Lindberg and Nöu, 1994). Our Bregman-prox method may be used not only for solving LP relaxations, but also for finding approximate solutions to the IP problem (9). Specifically, while solving the current dual subproblem (12), we may interpret $x_j(\lambda^{(l)}) \in (0, 1)$ as the probability of having $x_j = 1$ in a solution of (9). Hence we may employ the randomized heuristic of (Lindberg and Nöu, 1994) for generating primal feasible solutions (PFS's).

Broadly speaking, our randomized PFS heuristic runs as follows.

- Start with $\tilde{x} = 0$, i.e., all rows uncovered.
- While the set of uncovered rows $\mathcal{V} = \{i : (A\tilde{x} - e)_i < 0\}$ is nonempty:
 - Choose an uncovered row $i \in \mathcal{V}$.
 - Choose $\hat{j} \in \{j : A_{ij} = 1\}$ at random, using the probabilities $x_j(\lambda^{(l)})$. Set $\tilde{x}_{\hat{j}} = 1$.
- If possible, set $\tilde{x}_j = 0$ for one or more j s s.t. the final $\tilde{x}$ satisfies $A\tilde{x} \geq e$.

Of course, this heuristic may be implemented in various ways. The next section shows that it may produce points $\tilde{x}$ close to the primal integer optimum.

5 NUMERICAL RESULTS

We now report on preliminary numerical experience obtained with an implementation of our method programmed in Matlab. To save space, we only mention some of its features.

We chose h as the Fermi/Dirac ‘ $x \log x$ ’ entropy of Example 3.1, leaving other possibilities for future work.

The dual subproblems (12) were solved to relatively high accuracy by the coordinate ascent method of (Kiwiel, 1994). Each coordinate ascent employed a safeguarded Newton’s scheme to reduce the directional derivative of q by at least a quarter, i.e., the relaxation factor stayed in $[0.75, 1]$, unless a bound became active. For simplicity, the coordinates (rows) were chosen cyclically. The randomized PFS heuristic was called once every three cycles (minor iterations), i.e., complete sweeps over the constraint set. We used constant proximal stepsizes $t_k \equiv 100/\max_i c_i$ (with $c > 0$ for our problems) and the initial accuracy tolerance $\epsilon_1 = v_{LP}/10$, where v_{LP} was the (known) optimal value of the LP relaxation (10). The accuracy tolerances were decreased relatively slowly by setting $\epsilon_{k+1} = \kappa_\epsilon \epsilon_k$, where $\kappa_\epsilon = 0.8$.

We present numerical results for a set of real-world set covering problems, emanating from airline crew scheduling problems (Wedelin, 1993). Table 1 highlights main features

Table 1 Airline scheduling test problems

Name	Rows	Columns	Density (%)	NNZ		IP-gap (%)
				col	row	
B727	29	157	8.2	2.4	12.8	0.372
ALITALIA	116	1163	3.1	3.6	35.8	0
A320	195	6927	2.4	4.6	163.1	0
LH_A320	234	18752	1.9	4.4	351.4	0.003
SASjump	725	10353	0.57	4.1	58.9	0.025
SASD9.2	1360	25026	0.33	4.4	81.5	≤ 0.018

Table 2 Convergence to the LP relaxation value

Name	Relative accuracy (%) at minor iteration			
	10	25	50	100
B727	7.13	3.09	0.74	0.11
ALITALIA	1.12	0.54	0.13	0.03
A320	9.91	0.98	0.37	0.23
LH_A320	10.36	0.83	0.40	0.28
SASjump	7.85	2.15	0.66	0.41
SASD9.2	6.77	1.72	0.70	0.16

of these problems (NNZ stands for the mean number of nonzeros, and IP-gap for the difference in the optimal values of (9) and (10)). We may add that preprocessing has considerably reduced the size of these problems, and that CPLEX (version 2.1beta) could not solve the largest problem (Wedelin, 1993).

Table 2 shows the accuracy in the objective value relative to the (known) LP-value v_{LP} obtained after a given number of minor iterations (cycles). The numbers of major prox-iterations were fairly small, e.g., 15 for the ALITALIA problem. Anyway, most effort goes into minor iterations.

Table 3 shows the accuracy of the best primal solution obtained by the PFS heuristic relative to the (known) IP-value v_{IP} after a given number of sweeps (v_{IP} is not known for SASD9.2). We note that the PFS heuristic was not called every cycle.

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Table 3 Convergence to the IP value

Name	Relative accuracy (%) at minor iteration			
	10	25	50	100
B727	10.54	6.67	3.50	0.64
ALITALIA	1.39	1.39	1.39	1.39
A320	1.35	0.86	0.86	0.75
LH_A320	2.13	1.64	1.64	1.64
SASjump	22.86	8.71	7.51	6.41
SASD9.2	15.78	8.19	6.98	5.69

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