Algebraic methods in control, theory and applications

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Abstract

Fractional representations are a useful algebraic tool for control system design. They result in a simple parametrization of all controllers that stabilize a given plant. One could then, in principle, choose the best controller for various applications.

Keywords

Algebraic system theory; control system synthesis; feedback control; linear systems; transfer functions.

1 INTRODUCTION

This paper is a tutorial whose aim is to explain a useful algebraic tool for control system design, the so-called *fractional representation* approach. This approach is based on the input-output properties of linear systems. The central idea is that of representing the transfer function of a (not necessarily stable) system as the ratio of two *stable* transfer functions. This is a natural step for the linear systems whose transfer functions are rational, i.e., for the lumped-parameter systems. Under certain conditions, however, this approach is productive also for the distributed-parameter systems.

The starting point of the design is to obtain a simple parametrization of *all* controllers that stabilize a given plant. One could then, in principle, choose the best controller for various applications. The key point is that the parameter appears in the closed-loop system transfer function in a *linear* manner, thus making it easier to meet additional design specifications.

The actual design of control systems is an engineering task that cannot be reduced to algebra. Design contains many additional aspects which have to be taken into account: sensor placement, computational constraints, actuator constraints, redundancy, performance robustness, among many others. There is a need for understanding of the control process, a feeling for what kinds of performance objectives are unrealistic, or even dangerous, to

ask for. The algebraic approach to be presented, nevertheless, is an elegant and useful tool for the mathematical part of the controller design (Kučera, 1993).

2 SYSTEMS AND SIGNALS

The fundamentals of the fractional representation approach will be explained for linear systems with rational transfer functions whose input u and output y are scalar quantities. We suppose that u and y live in a space of functions mapping a time set into a value set. The time set is the set of non-negative reals, R_+ , and the value set is taken to be the set of real numbers R .

Let the input and output spaces of a system be the spaces of locally (Lebesgue) integrable functions f from R_+ into R and define a p -norm

$$\|f\|_{L_p} = \left[\int_0^\infty |f(t)|^p dt \right]^{1/p} \quad \text{if } 1 \leq p < \infty$$

$$\|f\|_{L_\infty} = \operatorname{ess\,sup}_{t \geq 0} |f(t)| \quad \text{if } p = \infty.$$

The corresponding normed space is denoted by L_p .

Systems having the desirable property of preserving the functional space are called *stable*. More precisely, a system is said to be L_p -stable if any input $u \in L_p$ gives rise to an output $y \in L_p$. The systems that are L_∞ -stable are also termed to be *bounded-input bounded-output* (BIBO) *stable*.

The transfer function of a system is the Laplace transform of its impulse response $g(t)$,

$$G(s) = \int_0^\infty g(t)e^{-st} dt.$$

It is well known that a system with a rational transfer function $G(s)$ is BIBO stable if and only if $G(s)$ is proper and stable, i.e., bounded at infinity with all poles having negative real parts (Desoer and Vidyasagar, 1975).

3 FRACTIONAL REPRESENTATIONS

Consider a rational function $G(s)$. By definition, it can be expressed as the ratio

$$G(s) = \frac{B(s)}{A(s)}$$

of two qualified rational functions A and B .

A well known example is the polynomial description, in which case A and B are coprime polynomials, i.e., polynomials having no roots in common.

Another example is to take for A and B two coprime, *proper and stable* rational functions. When $G(s)$ is, say,

$$G(s) = \frac{s+1}{s^2+1}$$

then one can take

$$A(s) = \frac{s^2 + 1}{(s + \lambda)^2}, \quad B(s) = \frac{s + 1}{(s + \lambda)^2}$$

where λ is a positive real number. We recall that two proper and stable rational functions are coprime if they have no infinite nor unstable zeros in common. Therefore, in the example above, the denominator of A and B can be any strictly Hurwitz polynomial of degree exactly two; if the degree is lower, then A would not be proper and if it is higher, then A and B would have a common zero at infinity. The set of proper and stable rational functions will be denoted by $R_{ps}(s)$. It has the algebraic structure of a ring, in particular a euclidean domain (Vidyasagar, 1985).

4 FEEDBACK SYSTEMS

To control a system means altering its dynamics so that a desired behaviour is obtained. This can be done by feedback. A typical feedback system consists of two subsystems, S_1 and S_2 , connected as shown in Fig. 1

Figure 1 Feedback system

In most applications, it is desirable that the feedback system be BIBO stable in the sense that whenever the inputs u_1 and u_2 are bounded in magnitude so too are the outputs y_1 and y_2 .

In order to study this property, we express the transfer functions of S_1 and S_2 as ratios of proper stable rational functions and seek for conditions under which the transfer function of the feedback system is proper and stable.

Thus we write

$$S_1(s) = \frac{B(s)}{A(s)}, \quad S_2(s) = -\frac{Y(s)}{X(s)}$$

where A, B and X, Y are two couples of coprime rational functions from $R_{ps}(s)$. The transfer matrix of the feedback system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{S_1}{1-S_1S_2} & \frac{S_1S_2}{1-S_1S_2} \\ \frac{S_1S_2}{1-S_1S_2} & \frac{S_2}{1-S_1S_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is then given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{AX + BY} \begin{bmatrix} BX & -BY \\ -BY & -AY \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

We observe that the numerator matrix has all its entries in $R_{ps}(s)$ and that no infinite nor unstable zeros of the denominator can be absorbed in all these entries. We therefore conclude that the four transfer functions belong to $R_{ps}(s)$ if and only if $AX + BY$ is a unit of $R_{ps}(s)$, i.e., the inverse of $AX + BY$ belongs to $R_{ps}(s)$.

We illustrate with the example where S_1 is a differentiator and S_2 is an inverter such that

$$S_1(s) = s, \quad S_2(s) = -1.$$

We take

$$A(s) = \frac{1}{s + \lambda}, \quad B(s) = \frac{s}{s + \lambda}$$

for any real $\lambda > 0$ and

$$X(s) = 1, \quad Y(s) = 1.$$

Then

$$(AX + BY)^{-1}(s) = \frac{s + \lambda}{s + 1}$$

resides in $R_{ps}(s)$ and hence the closed-loop system is BIBO stable.

To summarize, the fractional representation used should be matched with the goal of the analysis. The analysis is then more transparent and leads to a simple algebraic condition: $AX + BY$ is a unit of the underlying ring (Desoer et al., 1980).

5 PARAMETRIZATION OF STABILIZING CONTROLLERS

The design of feedback control systems consists of the following: given one subsystem, say S_1 , we seek to determine the other subsystem, S_2 , so that the resulting feedback system shown in Fig. 1 meets the design specifications. We call S_1 the *plant* and S_2 the *controller*.

The primary design specification is some sort of stability. We shall focus on achieving BIBO stability. Any controller that BIBO stabilizes the plant will be called a *stabilizing* controller for that plant.

Suppose the plant gives rise to the transfer function

$$S_1(s) = \frac{B(s)}{A(s)}$$

for some coprime elements A and B of $R_{ps}(s)$. It follows from the foregoing analysis (Kučera, 1979) that a stabilizing controller exists and that all controllers that stabilize the given plant are generated by all solution pairs X, Y with $X \neq 0$ of the Bézout equation

$$AX + BY = 1$$

over $R_{ps}(s)$. There is no loss of generality in setting $AX + BY$ to the identity rather than to an arbitrary unit of $R_{ps}(s)$: this unit is absorbed by X and Y and therefore cancels in forming

$$S_2(s) = -\frac{Y(s)}{X(s)}.$$

The solution set of the Bézout equation can be parametrized (Kučera, 1979) as

$$X = X' + BW, \quad Y = Y' - AW$$

where X', Y' is a particular solution of the equation and W is a free parameter, which is an arbitrary function in $R_{ps}(s)$.

The parametrization of the family of all stabilizing controllers for the plant now falls out (Kučera, 1974; Youla et al., 1976) as

$$S_2(s) = -\frac{Y'(s) - A(s)W(s)}{X'(s) + B(s)W(s)}$$

where the parameter W varies over $R_{ps}(s)$ while satisfying $X' + BW \neq 0$. This condition is not very restrictive, as $X' + BW$ can identically vanish for at most one choice of W .

As an example, we shall stabilize an integrator plant S_1 . Its transfer function can be expressed as

$$S_1(s) = \frac{1}{\frac{s+1}{s}}$$

where $s + 1$ is an arbitrarily chosen Hurwitz polynomial of degree one. One stabilizing controller for S_1 is easy to find, namely

$$S_2(s) = -1.$$

It corresponds to a particular solution $X' = 1$, $Y' = 1$ of the Bézout equation

$$\frac{s}{s+1}X + \frac{1}{s+1}Y = 1.$$

The solution set in $R_{ps}(s)$ of this equation is

$$X(s) = 1 + \frac{1}{s+1}W(s), \quad Y(s) = 1 - \frac{1}{s+1}W(s).$$

Hence all controllers S_2 that BIBO stabilize S_1 have the transfer function

$$S_2(s) = -\frac{1 - \frac{s}{s+1}W(s)}{1 + \frac{1}{s+1}W(s)}$$

where W is any function in $R_{ps}(s)$.

It is clear that the result is independent of the particular fraction taken to represent S_1 . Indeed, if $s + 1$ is replaced by another Hurwitz polynomial $s + \lambda$ in the above example, one obtains

$$S_2(s) = -\frac{\lambda - \frac{s}{s+\lambda}W'(s)}{1 + \frac{1}{s+\lambda}W'(s)},$$

which is the same set when

$$W'(s) = \left(\frac{s+\lambda}{s+1}\right)^2 W(s) + \frac{s+\lambda}{s+1}(\lambda - 1).$$

6 PARAMETRIZATION OF CLOSED-LOOP TRANSFER FUNCTIONS

The utility of the fractional approach derives not merely from the fact that it provides a parametrization for all controllers that stabilize a given plant in terms of a free parameter

W , but also from the simple manner in which this parameter enters the resulting (proper and stable) closed-loop transfer functions.

In fact,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} B(X' + BW) & -B(Y' - AW) \\ -B(Y' - AW) & -A(Y' - AW) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and we observe that all four transfer functions are *linear* in the free parameter W .

This result serves to parameterize the performance specifications and it is the starting point for the selection of the best controller for the application at hand. The search for S_2 is thus replaced by a search for W . The crucial point is that the resulting selection/optimization problem is linear in W while it is non-linear in S_2 .

7 OPTIMAL PERFORMANCE

The performance specifications often involve a norm minimization.

Let us consider the problem of *disturbance attenuation*. We are given a plant S_1 having two inputs: the control input u and an unmeasurable disturbance d , see Fig. 2.

Figure 2 Disturbance attenuation

The objective is to determine a BIBO stabilizing controller S_2 for the plant S_1 such that the effect of d on the plant output y is minimized in some sense.

We describe the plant by two transfer functions

$$S_{1u}(s) = \frac{B(s)}{A(s)}, \quad S_{1d}(s) = \frac{C(s)}{A(s)}$$

where A, B and C is a triple of coprime functions from $R_{ps}(s)$. The set of stabilizing controllers for S_1 is given by the transfer function

$$S_2(s) = -\frac{Y'(s) - A'(s)W(s)}{X'(s) + B'(s)W(s)}$$

where A', B' is a *coprime* fraction over $R_{ps}(s)$ for S_{1u} ,

$$\frac{B(s)}{A(s)} = \frac{B'(s)}{A'(s)}$$

and X', Y' is a particular solution over $R_{ps}(s)$ of the Bézout equation

$$A'X + B'Y = 1$$

such that $X' + B'W \neq 0$.

The transfer function, $G(s)$, between d and y in a stable feedback system is

$$G = \frac{S_{1d}}{1 - S_{1u}S_2} = C(X' + B'W)$$

and it is indeed linear in the proper stable rational parameter W .

Now suppose that the disturbance d is any function from L_∞ , i.e., any essentially bounded real function on R_+ . Then (Doyle et al., 1992)

$$\|y\|_{L_\infty} \leq \|G\|_1 \|d\|_{L_\infty},$$

where

$$\|G\|_1 = \int_0^\infty |g(t)| dt$$

and $g(t)$ is the impulse response corresponding to $G(s)$. The parameter W can be used to minimize the norm $\|G\|_1$ and hence the maximum output amplitude.

If d is stationary white noise, the steady-state output variance equals (Kučera, 1979)

$$Ey^2 = \|G\|_2^2 Ed^2,$$

where

$$\|G\|_2^2 = \int_0^\infty |g(t)|^2 dt = \frac{1}{2\pi j} \oint G(-s)G(s) ds.$$

The last integral is a contour integral up the imaginary axis and then around an infinite semicircle in the left half-plane. Again, W can be selected so as to minimize the norm $\|G\|_2$, thus minimizing the steady-state output variance.

Finally suppose that d is any function from L_2 , i.e., any finite-energy real function on R_+ . Then one obtains (Doyle et al., 1992)

$$\|y\|_{L_2} \leq \|G\|_\infty \|d\|_{L_2},$$

where

$$\|G\|_\infty = \sup_{\operatorname{Re} s > 0} |G(s)|.$$

Therefore choosing W so as to make the norm $\|G\|_\infty$ minimal, one minimizes the maximum output energy.

Here is an illustrative example. The plant is given by

$$S_{1u}(s) = \frac{1}{s}, \quad S_{1d}(s) = 1 + \frac{1}{s}$$

and we seek to find a stabilizing controller S_2 such that $G(s)$ has minimum ∞ -norm.

We write

$$A(s) = \frac{s}{s+1}, \quad B(s) = \frac{1}{s+1}, \quad C(s) = 1$$

and recall that the set of stabilizing controllers is given by

$$S_2(s) = -\frac{1 - \frac{s}{s+1} W(s)}{1 + \frac{1}{s+1} W(s)}$$

where W is a free parameter in $R_{ps}(s)$. Then

$$G(s) = \frac{S_{1d}(s)}{1 - S_{1u}(s)S_2(s)} = 1 + \frac{1}{s+1} W(s)$$

so that the least norm $\|G\|_\infty = 0$ is achieved by $W(s) = -(s+1)$. This parameter, however, does not belong to $R_{ps}(s)$. So we approximate it by

$$W_\varepsilon(s) = -\frac{s+1}{\varepsilon s+1}$$

for any real $\varepsilon > 0$. Then

$$G_\varepsilon(s) = \frac{\varepsilon s}{\varepsilon s+1}$$

and the least ∞ -norm attainable by a proper and stable rational parameter W_ε is $\|G_\varepsilon\|_\infty = 1$. The optimal stabilizing controller guarantees $\|y\|_{L_2} \leq \|d\|_{L_2}$ for any $d \in L_2$.

8 ROBUST STABILIZATION

The actual plant can differ from its nominal model. We suppose that a nominal plant description is available together with a description of the plant uncertainty. The objective is to design a controller that stabilizes the nominal plant as well as all plants lying within the specified domain of uncertainty. Such a controller is said to *robustly* stabilize the plant.

The plant uncertainty can be modelled conveniently in terms of the fractional representation over $R_{ps}(s)$. We endow $R_{ps}(s)$ with the ∞ -norm: for any function $F(s)$ from $R_{ps}(s)$,

$$\|F\|_\infty = \sup_{\operatorname{Re} s > 0} |F(s)|.$$

For any two such function, $F_1(s)$ and $F_2(s)$, we define

$$\| \begin{bmatrix} F_1 & F_2 \end{bmatrix} \|_\infty = \left\| \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \right\|_\infty = \sup_{\operatorname{Re} s > 0} (|F_1(s)|^2 + |F_2(s)|^2)^{1/2}.$$

Let S_{10} be a nominal plant giving rise to a transfer function

$$S_{10}(s) = \frac{B(s)}{A(s)}$$

where A and B are coprime functions from $R_{ps}(s)$. We denote $S_1(A, B, \mu)$ the family of plants having transfer functions

$$S_1(s) = \frac{B(s) + \Delta B(s)}{A(s) + \Delta A(s)}$$

where ΔA and ΔB are functions from $R_{ps}(s)$ such that

$$\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \|_\infty < \mu$$

for some non-negative real number μ .

Now, let S_2 be a BIBO stabilizing controller for S_{10} . Therefore

$$S_2(s) = -\frac{Y'(s) - A(s) W(s)}{X'(s) + B(s) W(s)}$$

where $AX' + BY' = 1$ and W is an element of $R_{ps}(s)$. Then S_2 will BIBO stabilize all plants from $S_1(A, B, \mu)$ if and only if

$$(A + \Delta A)(X' + BW) + (B + \Delta B)(Y' - AW) = 1 + [\Delta A \quad \Delta B] \begin{bmatrix} X' + BW \\ Y' - AW \end{bmatrix}$$

is a unit of $R_{ps}(s)$. This is the case whenever

$$\| [\Delta A \quad \Delta B] \begin{bmatrix} X' + BW \\ Y' - AW \end{bmatrix} \|_{\infty} < 1$$

so we have the following condition of robust stability (Vidyasagar, 1985):

$$\mu \left\| \begin{bmatrix} X' + BW \\ Y' - AW \end{bmatrix} \right\|_{\infty} \leq 1.$$

The best controller that robustly stabilizes the plant corresponds to the parameter W that minimizes the ∞ -norm above.

To illustrate, consider the family of plants

$$S_1 = \left(\frac{s}{s+1}, \frac{1}{s+1}, \mu \right)$$

around the nominal model

$$S_{10}(s) = \frac{1}{s}.$$

This family accommodates, among other things, gain perturbations and stable second-order dynamics of the form

$$S_1(s) = \frac{1 + \mu}{s(1 + \varepsilon s)}.$$

All controllers that BIBO stabilize the nominal plant S_{10} are given by

$$S_2(s) = -\frac{1 - \frac{s}{s+1} W(s)}{1 + \frac{1}{s+1} W(s)}$$

where W is a free parameter in $R_{ps}(s)$.

Suppose we wish to obtain a robust *proportional* controller, $S_2(s) = -\lambda$. Then

$$W(s) = (1 - \lambda) \frac{s+1}{s+\lambda}$$

and $\lambda > 0$. The norm

$$\left\| \begin{bmatrix} 1 + \frac{1}{s+1} W \\ 1 - \frac{s}{s+1} W \end{bmatrix} \right\|_{\infty} = \sqrt{1 + \lambda^2} \left\| \frac{s+1}{s+\lambda} \right\|_{\infty}$$

is calculated to be

$$N(\lambda) = \begin{cases} \frac{\sqrt{1+\lambda^2}}{\lambda}, & \text{if } 0 < \lambda < 1 \\ \sqrt{1+\lambda^2}, & \text{if } \lambda \geq 1. \end{cases}$$

Thus our proportional controller is robust if and only if

$$N(\lambda) \leq \frac{1}{\mu}.$$

Which proportional controller will maximize the stability margin μ ? The one which minimizes $N(\lambda)$. Since

$$\min_{\lambda > 0} N(\lambda) = \sqrt{2}$$

is attained by $\lambda = 1$, which corresponds to $W(s) = 0$, the consequent controller $S_2(s) = -1$ achieves the stability margin as large as

$$\mu = \frac{1}{\sqrt{2}}.$$

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