Optimal control problem governed by a semilinear parabolic equation

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Abstract

This paper deals with optimal control problems governed by semilinear parabolic equations. We obtain necessary optimality conditions in the form of an exact and an approximate Pontryagin's minimum principle for distributed controls and boundary controls.

Keywords

Nonlinear boundary controls, Semilinear parabolic equations, Pontryagin's minimum principle.

1 INTRODUCTION

Let Ω be a smooth bounded open subset in R^N ($N \geq 2$), and Γ its boundary. Let A be a second order symmetric uniformly elliptic operator with regular coefficients ($Ay = -\sum_{i,j} D_j(a_{ij}(x)D_i y)$) and let T be a positive number. We consider the following control problem.

Minimize (P)

$$J(y, u, v) = \int_Q F(y(x, t), u(x, t)) dx dt + \int_\Sigma G(y(s, t), v(s, t)) ds dt + \int_\Omega \ell(y(x, T)) dx, \quad (1)$$

where y is the solution of the initial-boundary value problem

$$\begin{aligned} \frac{\partial y}{\partial t}(x, t) + Ay(x, t) + f(y(x, t), u(x, t)) &= 0 & \text{in } Q = \Omega \times]0, T[, \\ \frac{\partial y}{\partial n_A}(s, t) + g(y(s, t), v(s, t)) &= 0 & \text{on } \Sigma = \Gamma \times]0, T[, \\ y(x, 0) &= y_0(x) & \text{in } \Omega, \end{aligned} \quad (2)$$

and the control variables u and v satisfy the constraints:

$$u \in U_{ad} = \{u \in L^q(Q) \mid u(x, t) \in K_U \text{ for a.e. } (x, t) \in Q\},$$

$$v \in V_{ad} = \{v \in L^r(\Sigma) \mid v(s, t) \in K_V \text{ for a.e. } (s, t) \in \Sigma\},$$

K_U and K_V are closed subsets of R , $y_0 \in C(\bar{\Omega})$ and

$$q > \frac{N}{2} + 1, \quad r > N + 1.$$

Throughout the paper, the following assumptions are in force.

(A1) - f and F are continuous on R^2 , for every $u \in R$, $f(., u)$ and $F(., u)$ are of class C^1 on R and we have the following estimates

$$|f(y, u)| + |F(y, u)| + |F'_y(y, u)| \leq M_1(1 + |u|)\eta(|y|),$$

$$C_0 \leq f'_y(y, u) \leq M_1(1 + |u|)\eta(|y|),$$

where η is a nondecreasing function from R^+ into R^+ , $M_1 \in R^+$ and $C_0 \in R$. (We have denoted by F'_y and f'_y the partial derivatives of F and f with respect to y , in all the sequel we adopt the same kind of notation for other functions).

(A2) - g and G are continuous on R^2 , for every $v \in R$, $g(., v)$ and $G(., v)$ are of class C^1 on R and we have the following estimates

$$|g(y, v)| + |G(y, v)| + |G'_y(y, v)| \leq M_1(1 + |v|)\eta(|y|),$$

$$C_0 \leq g'_y(y, v) \leq M_1(1 + |v|)\eta(|y|),$$

where η , M_1 and C_0 are as in (A1).

(A3) - ℓ is of class C^1 on R .

(A4) - The infimum of (P) is finite.

We first prove that the state equation (2) admits a unique weak solution in the space $W(0, T; H^1(\Omega), (H^1(\Omega))' \cap C(\bar{Q}))$, this new regularity result (continuity of the state on $\bar{Q}$) is particularly interesting to next consider control problems with pointwise state constraints.

We obtain optimality conditions for (P) in the form of pointwise Pontryagin's principles. For this we follow the method developed in (Bonnans and Casas, 1991) for elliptic problems. This method requires C^0 -regularity results for linearized equations. In our knowledge such results are not known in our case (indeed some coefficients of the equation are not bounded and can be negative, see Proposition 1) and we state them in Section 2.

2 EXISTENCE RESULT FOR THE STATE EQUATION

2.1 Linear equation

We first give a new regularity result for some linear equations. This result will be used to prove the existence of a bounded weak solution of (2) and will be the main tool to establish convergence results in the proof of Pontryagin's principles.

Proposition 1.

Let a be in $L^q(Q)$, let b be in $L^r(\Sigma)$ verifying $a(x, t) \geq C_0$ in Q and $b(s, t) \geq C_0$ on Σ . There exists a positive constant $C_1 = C_1(N, q, r, \Omega, T, C_0)$ (independent of a and b), such that for every $\phi \in L^q(Q)$, every $\psi \in L^r(\Sigma)$ and every $y_0 \in C(\bar{\Omega})$, the weak solution $y \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ of the equation

$$\begin{aligned} \frac{\partial y}{\partial t}(x, t) + Ay(x, t) + a(x, t)y(x, t) &= \phi(x, t) && \text{in } Q, \\ \frac{\partial y}{\partial n_A}(s, t) + b(s, t)y(s, t) &= \psi(s, t) && \text{on } \Sigma, \\ y(x, 0) &= y_0(x) && \text{in } \Omega, \end{aligned} \tag{3}$$

belongs to $C(\bar{Q})$ and satisfies the estimate:

$$\|y\|_{L^2(0, T; H^1(\Omega))} + \|y\|_{L^\infty(Q)} \leq C_1(\|\phi\|_{L^q(Q)} + \|\psi\|_{L^r(\Sigma)} + \|y_0\|_{L^\infty(\Omega)}). \tag{4}$$

Moreover, this solution belongs to $W(0, T; H^1(\Omega), (H^1(\Omega))')$. $\square$

The proof is given in (Raymond and Zidani, 1995) and is based on duality arguments (to estimate y thanks to estimate on the solution of a Cauchy problem), comparison principle and estimates on analytic semigroups.

In the particular case when $a \in C^\infty(\bar{\Omega})$, $b(s, t) = \alpha \geq 0$, $\phi \in L^\infty(Q)$ and $\psi \in L^\infty(\Sigma)$ such a result is given in (Mackenroth, 1982).

2.2 State equation

Theorem 1.

If f and g verify the above assumptions, if $y_0 \in C(\bar{\Omega})$, $u \in L^q(Q)$ and $v \in L^r(\Sigma)$, then the equation (2) admits a unique weak solution y in $W(0, T; H^1(\Omega), (H^1(\Omega))') \cap C(\bar{Q})$, this solution satisfies the estimate

$$\|y\|_{L^\infty(Q)} \leq C_2(\|u\|_{L^q(Q)} + \|v\|_{L^r(\Sigma)} + \|y_0\|_{L^\infty(\Omega)} + 1),$$

where $C_2 = C_2(T, \Omega, N, q, r, C_0)$ is a positive constant. $\square$

Idea on the proof.

Since we cannot directly apply a Faedo Galerkin method to equation (2), we first replace $f(., u)$ and $g(., v)$ by truncated functions $f_k(., u)$ and $g_k(., v)$ (such that $|f'_{ky}(y, u)|$ and $|g'_{ky}(y, v)|$ are bounded with respect to y).

We next regularize $f_k(y, u(., .))$ and $g_k(y, v(., .))$ in order that the derivatives of the regularized functions $(f_k^\varepsilon)'_y(y, u(., .))$ and $(g_k^\varepsilon)'_y(y, v(., .))$ be bounded respectively in Q and Σ .

We prove the existence of a bounded weak solution y_k^ε for the equation corresponding to f_k^ε and g_k^ε .

Thanks to Proposition 1, we get estimates on $(y_k^\varepsilon)_\varepsilon$ in $L^2(0, T; H^1(\Omega)) \cap C(\bar{Q})$. We pass to the limit when ε tends to zero and we prove that $(y_k^\varepsilon)_\varepsilon$ converges in $L^2(0, T; H^1(\Omega)) \cap C(\bar{Q})$ to the solution y_k of the equation corresponding to f_k and g_k . Next we prove that if k is big enough, then y_k is also a solution of (2). The uniqueness is proved thanks to a comparison principle. Moreover, if we denote by y the solution of (2), we can see that y is also a solution of

$$\begin{aligned} \frac{\partial z}{\partial t}(x, t) + Az(x, t) + a(x, t)z(x, t) &= f(0, u(x, t)) && \text{in } Q, \\ \frac{\partial z}{\partial n_A}(s, t) + b(s, t)z(s, t) &= g(0, v(s, t)) && \text{on } \Sigma, \\ z(x, 0) &= y_0(x) && \text{in } \Omega, \end{aligned} \tag{5}$$

with $a(x, t) = \int_0^1 f'_y(\theta y(x, t), u(x, t)) d\theta \geq C_0$ in Q and $b(s, t) = \int_0^1 g'_y(\theta y(s, t), v(s, t)) d\theta \geq C_0$ on Σ . From (A1), (A2), we deduce that $a \in L^q(Q)$ and $b \in L^r(\Sigma)$. Thanks to Proposition 1, it yields $y \in C(\bar{Q})$ and

$$\begin{aligned} \|y\|_{C(\bar{Q})} &\leq C_1(\|f(0, u)\|_{L^q(Q)} + \|g(0, v)\|_{L^r(\Sigma)} + \|y_0\|_{L^\infty(\Omega)}), \\ \|y\|_{C(\bar{Q})} &\leq C_2(\|u\|_{L^q(Q)} + \|v\|_{L^r(\Sigma)} + \|y_0\|_{L^\infty(\Omega)} + 1). \end{aligned}$$

2.3 Pontryagin principles

We define a distributed Hamiltonian function and a boundary Hamiltonian function by:

$$H_Q(y, u, p) = F(y, u) - pf(y, u)$$

for every $(y, u, p) \in R \times R \times R$,

$$H_\Sigma(y, v, p) = G(y, v) - pg(y, v)$$

for every $(y, v, p) \in R \times R \times R$.

With the regularity results and the estimates stated in Theorem 1, we prove optimality conditions for (P) in the form of two decoupled Pontryagin principles.

Theorem 2.

If $(\bar{y}, \bar{u}, \bar{v})$ is a solution of (P), there then exists $\bar{p} \in W(0, T; H^1(\Omega), (H^1(\Omega))')$ satisfying the equation

$$-\frac{\partial p}{\partial t}(x, t) + Ap(x, t) + f'_y(\bar{y}(x, t), \bar{u}(x, t))p(x, t) = F'_y(\bar{y}(x, t), \bar{u}(x, t)) \quad \text{in } Q,$$

$$\frac{\partial p}{\partial n_A}(s, t) + g'_y(\bar{y}(s, t), \bar{v}(s, t))p(s, t) = G'_y(\bar{y}(s, t), \bar{v}(s, t)) \quad \text{on } \Sigma,$$

$$p(x, T) = \ell'_y(\bar{y}(x, T)) \quad \text{in } \Omega,$$

and such that:

$$H_Q(\bar{y}(x, t), \bar{u}(x, t), \bar{p}(x, t)) = \min_{u \in K_U} H_Q(\bar{y}(x, t), u, \bar{p}(x, t)) \quad \text{for a.e. } (x, t) \in Q,$$

$$H_\Sigma(\bar{y}(s, t), \bar{v}(s, t), \bar{p}(s, t)) = \min_{v \in K_V} H_\Sigma(\bar{y}(s, t), v, \bar{p}(s, t)) \quad \text{for a.e. } (s, t) \in \Sigma. \quad \square$$

This kind of results has been already proved by Wu and Teo (1983) for problems with a linear state equation and a convex cost criterion and by Bonnans and Casas (1991) for problems governed by semilinear elliptic equations with homogeneous Dirichlet boundary conditions.

Very recently, Fattorini and Murphy (1994) have obtained a Pontryagin principle for problems with a boundary control in a Robin boundary condition and a terminal state constraint. Here our assumptions and our proof are completely different.

Sketch of the proof of Theorem 2.

We only give the proof of the boundary Pontryagin's principle, the distributed Pontryagin's principle can be obtained in similar way.

We first write the variation of the cost functional in the form of an Hamiltonian variation. (For this, we introduce an intermediate adjoint state as in (Bonnans and Casas, 1991)). We next use spike perturbations.

i) Thanks to Taylor formula we define $\tilde{G}_y(y_1, y_2, v)$ by:

$$\begin{aligned} G(y_1, v) - G(y_2, v) &= \left(\int_0^1 G'_y(y_2 + \theta(y_1 - y_2), v) d\theta \right) (y_1 - y_2) \\ &= \tilde{G}_y(y_1, y_2, v)(y_1 - y_2), \end{aligned}$$

for every $v \in R$, $y_1 \in R$ and $y_2 \in R$. We define $\tilde{F}_y(y_1, y_2, u)$, $\tilde{f}_y(y_1, y_2, u)$, $\tilde{g}_y(y_1, y_2, v)$ and $\tilde{\ell}_y(y_1, y_2)$ in a similar manner.

Let v_1 be in V_{ad} , we denote by y_1 the weak solution of (2) corresponding to $(\bar{u}, v_1)$ and we define the intermediate adjoint state p_1 as the weak solution of the equation:

$$\begin{aligned} -\frac{\partial p}{\partial t} + Ap + \tilde{f}_y(y_1, \bar{y}, \bar{u})p &= \tilde{F}_y(y_1, \bar{y}, \bar{u}) \quad \text{in } Q, \\ \frac{\partial p}{\partial n_A} + \tilde{g}_y(y_1, \bar{y}, v_1)p &= \tilde{G}_y(y_1, \bar{y}, v_1) \quad \text{on } \Sigma, \\ p(T) &= \tilde{\ell}_y(y_1(T), \bar{y}(T)) \quad \text{in } \Omega. \end{aligned} \tag{6}$$

Let us notice that if $v_1 = \bar{v}$ then $y_1 = \bar{y}$ and the function $p_1 = \bar{p}$ is the so-called adjoint state associated with $(\bar{u}, \bar{v})$.

By a straightforward calculation, we get

$$J(y_1, \bar{u}, v_1) - J(\bar{y}, \bar{u}, \bar{v}) = \int_{\Sigma} \{H_{\Sigma}(\bar{y}, v_1, p_1) - H_{\Sigma}(\bar{y}, \bar{v}, p_1)\} ds dt.$$

ii) We define

$$S_k(s_0, t_0) = \{(s, t) \in \Sigma \mid |(s_0, t_0) - (s, t)|_{R^{N+1}} \leq 1/k\}.$$

Let v be in K_V , we consider a sequence $(v_k)_k$ of spike perturbations in V_{ad} , given by:

$$v_k(s, t) = \begin{cases} \bar{v}(s, t) & \text{on } \Sigma \setminus S_k(s_0, t_0), \\ v & \text{on } S_k(s_0, t_0), \end{cases} \quad (7)$$

and we denote by y_k (respectively p_k) the state associated with $(\bar{u}, v_k)$ (respectively the intermediate adjoint state associated with $(\bar{u}, v_k)$).

We first prove that $(y_k)_k$ converges to $\bar{y}$ in $C(\bar{Q})$. For this, we remark that $y_k - \bar{y}$ is the weak solution of the equation:

$$\frac{\partial z}{\partial t}(x, t) + Az(x, t) + a_k(x, t)z(x, t) = 0 \quad \text{in } Q,$$

$$\frac{\partial z}{\partial n_A}(s, t) + b_k(s, t)z(s, t) = g(\bar{y}(s, t), \bar{v}(s, t)) - g(\bar{y}(s, t), v_k(s, t)) \quad \text{on } \Sigma,$$

$$z(x, 0) = 0 \quad \text{in } \Omega,$$

where

$$a_k(x, t) = \int_0^1 f'_y((\bar{y} + \theta(y_k - \bar{y}))(x, t), \bar{u}(x, t)) d\theta \geq C_0,$$

$$b_k(s, t) = \int_0^1 g'_y((\bar{y} + \theta(y_k - \bar{y}))(s, t), v_k(s, t)) d\theta \geq C_0.$$

Thus, from Proposition 1 it yields

$$\|y_k - \bar{y}\|_{L^\infty(Q)} \leq C_1 \|g(\bar{y}, \bar{v}) - g(\bar{y}, v_k)\|_{L^r(\Sigma)}.$$

We get convergence result from assumption (A2). In a similar manner we prove that $(p_k)_k$ converges to $\bar{p}$ in $C(\bar{Q})$.

iii) If (s_0, t_0) is a Lebesgue point for the functions:

$$(s, t) \longmapsto g(\bar{y}(s, t), v),$$

$$(s, t) \longmapsto g(\bar{y}(s, t), \bar{v}(s, t)),$$

$$(s, t) \longmapsto H_{\Sigma}(\bar{y}(s, t), v, \bar{p}(s, t)),$$

$$(s, t) \longmapsto H_{\Sigma}(\bar{y}(s, t), \bar{v}(s, t), \bar{p}(s, t)),$$

it yields :

$$0 \geq \lim_k \frac{1}{\text{meas}(S_k(s_0, t_0))} [J(y_k, \bar{u}, v_k) - J(\bar{y}, \bar{u}, \bar{v})] = \\ H_\Sigma(\bar{y}(s_0, t_0), v, \bar{p}(s_0, t_0)) - H_\Sigma(\bar{y}(s_0, t_0), \bar{v}(s_0, t_0), \bar{p}(s_0, t_0)).$$

Let us notice that the set of Lebesgue points of the four above functions is of full measure.

iv) Thus we have proved that, for every $v \in K_V$, there exists a set $\Sigma(v, \bar{v})$ of full measure for the N - dimensional Lebesgue measure on Σ , such that

$$H_\Sigma(\bar{y}(s, t), \bar{v}(s, t), \bar{p}(s, t)) \leq H_\Sigma(\bar{y}(s, t), v, \bar{p}(s, t))$$

for all $(s, t) \in \Sigma(v, \bar{v})$. Now we conclude as in (Bonnans and Casas, 1991).

Assumptions (A1)-(A3) are not in general sufficient to prove existence of an optimal control for (P). However, since the infimum of (P) is finite (see (A4)), then, for every $\varepsilon > 0$, (P) admit ε^2 -solutions that we can characterize when $q = r = \infty$. Moreover this kind of result will be useful to consider problems with state constraints.

Theorem 3.

For every $\epsilon > 0$ and every $\alpha > 0$, there exists an ϵ^2 -solution for (P), denoted by $(y_{\epsilon, \alpha}, u_{\epsilon, \alpha}, v_{\epsilon, \alpha})$ such that:

$$H_Q(y_{\epsilon, \alpha}(x, t), u_{\epsilon, \alpha}(x, t), p_{\epsilon, \alpha}(x, t)) \leq H_Q(y_{\epsilon, \alpha}(x, t), u, p_{\epsilon, \alpha}(x, t)) + \epsilon$$

a.e. $(x, t) \in Q$, for every $u \in K_U$ verifying $|u| \leq \alpha$ and

$$H_\Sigma(y_{\epsilon, \alpha}(s, t), v_{\epsilon, \alpha}(s, t), p_{\epsilon, \alpha}(s, t)) \leq H_\Sigma(y_{\epsilon, \alpha}(s, t), v, p_{\epsilon, \alpha}(s, t)) + \epsilon$$

a.e. $(s, t) \in \Sigma$, for every $v \in K_V$ verifying $|v| \leq \alpha$. $\square$

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