Risk-sensitive control and dynamic games: The discrete-time case

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Abstract

We show that a suitable small parameter limit of a family of (possibly partially observed) risk-sensitive stochastic control problems is a deterministic dynamic game. We identify the corresponding cost functional through large deviations techniques.

Keywords

Risk-sensitive control, dynamic games, large deviations.

1 INTRODUCTION

The aim of this paper is to investigate connections between risk-sensitive stochastic optimal control and dynamic games. The existence of a link between these two families of optimization problems was first noticed by Jacobson (1973, 1977) in the case of linear systems with quadratic cost, where the explicit solutions are available. In the nonlinear setting the relation is more subtle: a deterministic game is recovered from a risk-sensitive stochastic control problem through a limiting procedure, consisting in letting the noise variance and the risk parameter going to zero and infinity respectively, at the same rate. As pointed out by Whittle (1990, 1991), large deviation theory is a natural probabilistic tool to study small-noise high-risk limit of stochastic control problems, providing rather convincing heuristic arguments but, so far, no rigorous proof.

In the case of continuous time diffusion processes, a different approach to the limit of risk-sensitive control problems was proposed by Fleming and McEneaney (1992, 1995). Their method relies on the theory of viscosity solutions for Hamilton-Jacobi (Dynamic Programming) equations, and has been applied to both finite time horizon and infinite time horizon problems.

When mimicking Fleming's method for discrete time models, one has to deal with a Dynamic Programming equation, for the risk-sensitive control problem, having the form of an integral difference equation, in which the noise distribution enters explicitly. The structure of this difference equation naturally suggests, in the analysis of its small-noise high-risk limit, the use of large deviation techniques, exploiting thus some of Whittle's ideas. Such large deviation analysis of the Dynamic Programming equation has been

completed by James, Baras and Elliott (1994) for nonlinear discrete time systems driven by a Gaussian additive noise.

The aim of this paper is to extend the results of James, Baras and Elliott by making a systematic use of large deviation methods. We give here a rather wide class of systems whose small noise-high risk limit can be identified as a dynamic game. The models in James, Baras and Elliott (1994) are covered by our approach, that in turn does not rely on any specific assumption on the distribution of the noise. In particular, we can treat nontrivial models with non-Gaussian and/or non-additive noise.

We finally mention that different, but close in spirit, limit theorems for risk-sensitive control problems have been obtained by Barron and Jensen (1989), and Runolfsson (1994).

The paper is organized as follows. In Section 2 we introduce the (new) concept of Uniform Large Deviation Family, that is used in Section 3 to analyze the limit of totally observed risk-sensitive control problems. Section 4 is devoted to the partially observed case, and to the discussion of a relevant example.

2 UNIFORM LARGE DEVIATION FAMILIES

We introduce here the main technical tool of this paper. In what follows we let X be a Polish (metric, separable and complete) space, and Θ be a set.

Definition 1 We say that a family $\{P^\epsilon(dx; \theta) : \epsilon > 0, \theta \in \Theta\}$ of probability measures on X is a Uniform Large Deviation Family (ULDF) if there exists a function $H : X \times \Theta \rightarrow [0, +\infty]$ such that

- i) $x \rightarrow H(x; \theta)$ is lower semicontinuous (l.s.c.), for every fixed $\theta \in \Theta$;
- ii) for $C \subset X$ closed

$$\limsup_{\epsilon \rightarrow 0} \sup_{\theta \in \Theta} \left[\epsilon \log P^\epsilon(C; \theta) + \inf_{x \in C} H(x; \theta) \right] \leq 0; \quad (1)$$

- iii) for $A \in X$ open

$$\liminf_{\epsilon \rightarrow 0} \inf_{\theta \in \Theta} \left[\epsilon \log P^\epsilon(A; \theta) + \inf_{x \in A} H(x; \theta) \right] \geq 0. \quad (2)$$

The function H is called the *rate function*. The following result is a version of Varadhan's Lemma (Varadhan (1984)); its proof requires a modification of the classical argument in Varadhan (1984), and can be found in Dai Pra and Rudari (1995a).

Lemma 2 Let $F_\epsilon, F : X \rightarrow \mathbb{R}$ be bounded continuous functions such that $F_\epsilon \rightarrow F$ uniformly as $\epsilon \rightarrow 0$. Then

$$\lim_{\epsilon \rightarrow 0} \epsilon \log \int e^{\epsilon^{-1} F_\epsilon(x)} P^\epsilon(dx; \theta) = \sup_{x \in X} [F(x) - H(x; \theta)] \quad (3)$$

uniformly in $\theta \in \Theta$.

3 SMALL-NOISE HIGH-RISK LIMIT: THE CASE OF COMPLETE OBSERVATION

Let $x^{\epsilon, \mathbf{u}} = (x_n^{\epsilon, \mathbf{u}})_{n=0}^N$ be a stochastic process, defined on a probability space $(\Omega, \mathcal{F}, P)$, taking values in a Polish space X and adapted to a filtration $(\mathcal{F}_n)_{n=0}^N$. The process is indexed by a parameter $\epsilon > 0$ and by a $(\mathcal{F}_n)$ -adapted process $\mathbf{u} = (u_0, \dots, u_{N-1})$, taking values in a metric space U . The law of the process $x^{\epsilon, \mathbf{u}}$ is determined by a $\mathcal{F}_0$ -measurable initial condition x_0 , and by the following transition probabilities:

$$P(x_{n+1}^{\epsilon, \mathbf{u}} \in \cdot | \mathcal{F}_n) = P_n^{\epsilon}(\cdot; x_n^{\epsilon, \mathbf{u}}, u_n); \quad n = 0, \dots, N-1 \quad (4)$$

where the probability kernels P_n^{ϵ} satisfy the following

Assumption A.

1. For fixed $\epsilon > 0$, $n = 0, \dots, N-1$, the map $(x, u) \rightarrow P_n^{\epsilon}(\cdot; x, u)$, from $X \times U$ to $\mathcal{M}_1(X)$ (space of probability measures on X) is continuous in x , uniformly for $u \in U$, with respect to the weak topology in $\mathcal{M}_1(X)$;

2. For n fixed, $\{P_n^{\epsilon}(\cdot; x, u) : \epsilon > 0, (x, u) \in X \times U\}$ is a ULDF with rate function $H_n(z; x, u)$.

We denote by $\mathcal{U}$ the class of *admissible controls*, i.e. the $(\mathcal{F}_n)$ -adapted, U -valued stochastic processes. The aim of the controller is to minimize over $\mathcal{U}$ the cost functional

$$J^{\epsilon}(\mathbf{u}) = E \left\{ \exp \left[\epsilon^{-1} \left(\sum_{n=0}^{N-1} g_n(x_n^{\epsilon, \mathbf{u}}, u_n) + g_N(x_N^{\epsilon, \mathbf{u}}) \right) \right] \right\} \quad (5)$$

where the g_n 's are bounded functions, continuous in x , uniformly for $u \in U$.

The *value* $J_*^{\epsilon} = \inf_{\mathbf{u}} J^{\epsilon}(\mathbf{u})$ can be obtained through the Dynamic Programming equation (Bertsekas (1976))

$$\begin{cases} V_n^{\epsilon}(x) &= \inf_{u \in U} \left[e^{\epsilon^{-1} g_n(x, u)} \int V_{n+1}^{\epsilon}(x') P_n^{\epsilon}(dx'; x, u) \right] \\ V_N^{\epsilon}(x) &= e^{\epsilon^{-1} g_N(x)} \end{cases} \quad (6)$$

which yields $J_*^{\epsilon} = E(V_0^{\epsilon}(x_0))$. The function $(n, x) \rightarrow V_n^{\epsilon}(x)$ is called the *value function* for the risk-sensitive stochastic control problem (4)(5), where ϵ^{-1} plays the role of a risk parameter. Under Assumption A 1., it is easily proved that V_n^{ϵ} is a bounded continuous function of x .

To the above risk-sensitive stochastic control problem we will associate a deterministic dynamic game of the following type. The dynamics are given by the difference equations

$$\begin{aligned} x_{n+1} &= f_n(x_n, u_n, w_n) \quad n = 0, \dots, N-1 \\ x_0 &\text{ given} \end{aligned} \quad (7)$$

with $x_n \in X$, $u_n \in U$, $w_n \in W$, W being an arbitrary set. The controls $\mathbf{u} = (u_0, \dots, u_{N-1})$ are such that u_n is a function of $(x_0, \dots, x_n)$. Letting $\tilde{\mathcal{U}}$ denote this class of controls, and $\mathbf{w} = (w_0, \dots, w_{N-1})$, the aim of the controller is to minimize over $\tilde{\mathcal{U}}$ the cost functional

$$K(\mathbf{u}) = \sup_{\mathbf{w}} \left\{ \sum_{n=0}^{N-1} G_n(x_n, u_n, w_n) + G_N(x_N) \right\} \quad (8)$$

where the G_n 's are given functions. The quantity $K_* = \inf_u K(u)$ is the (upper) value for the game. The dynamic programming equation for this game is

$$\begin{aligned} S_n(x) &= \inf_{u \in U} \sup_{w \in W} \left\{ G_n(x, u, w) + S_{n+1}(f_n(x, u, w)) \right\} \\ S_N(x) &= G_N(x). \end{aligned} \quad (9)$$

Also in this context the map $(n, x) \rightarrow S_n(x)$ is called the (upper) *value function* for the game, and $K_* = S_0(x_0)$. The main result of this Section is the following Theorem, which identifies the $\epsilon \rightarrow 0$ limit of the stochastic control problem (4)(5).

Theorem 3 *Suppose Assumption A is satisfied and, for $n = 0, \dots, N-1$, let $f_n : X \times U \times W \rightarrow X$ be functions such that for each $(x, u) \in X \times U$ the map $w \rightarrow f_n(x, u, w)$ is onto. Then the value function $V_n^\epsilon(x)$ defined in (6) is such that*

$$\lim_{\epsilon \rightarrow 0} \epsilon \log V_n^\epsilon = S_n \quad (10)$$

uniformly on X , where S_n is the value function of the dynamic game (7)(8) with

$$\begin{aligned} G_N(x) &= g_N(x) \\ G_n(x, u, w) &= g_n(x, u) - H_n(f_n(x, u, w); x, u) \quad \text{for } n = 0, \dots, N-1. \end{aligned} \quad (11)$$

Proof. The proof consists in a simple use of Lemma 2. Indeed, we define $S_n^\epsilon = \epsilon \log V_n^\epsilon$, and write (6) in the equivalent form

$$S_n^\epsilon(x) = \inf_{u \in U} \left\{ g_n(x, u) + \epsilon \log \int e^{\epsilon^{-1} S_{n+1}^\epsilon(x')} P_n^\epsilon(dx'; x, u) \right\}. \quad (12)$$

Since $S_N^\epsilon \equiv g_N = S_N$, we can proceed by induction, and assume $S_{n+1}^\epsilon \rightarrow S_{n+1}$ uniformly, as $\epsilon \rightarrow 0$. Then, by Lemma 2,

$$\epsilon \log \int e^{\epsilon^{-1} S_{n+1}^\epsilon(x')} P_n^\epsilon(dx'; x, u) \rightarrow \sup_{x'} [S_{n+1}(x') - H_n(x'; x, u)] \quad (13)$$

uniformly. Thus

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} S_n^\epsilon(x) &= \inf_{u \in U} \sup_{x' \in X} [g_n(x, u) - H_n(x'; x, u) + S_{n+1}(x')] \\ &= \inf_{u \in U} \sup_{w \in W} [g_n(x, u) - H_n(f(x, u, w); x, u) + S_{n+1}(x')] \\ &= \inf_{u \in U} \sup_{w \in W} [G_n(x, u, w) + S_{n+1}(x')] \\ &= S_n(x) \end{aligned}$$

where the limit is uniform. $\square$

4 THE CASE OF PARTIAL OBSERVATION

Consider a risk-sensitive stochastic control problem of type (4)(5). When the state process (4) is only *partially observed* one changes the class of admissible controls to the set of those

control processes $\mathbf{u}$ that are adapted to the filtration $(\mathcal{F}_n^y)_{n=0}^N$ generated by an *observation process* $y^{\epsilon, \mathbf{u}}$. This observation process, that takes value in a Polish space Y , is assumed to obey to the following requirements:

$$P\{y_n^{\epsilon, \mathbf{u}} \in \cdot | \mathcal{F}_n \vee \mathcal{F}_{n-1}^y\} = Q_n^\epsilon(\cdot; x_n^{\epsilon, \mathbf{u}}) \quad n = 0, \dots, N-1, \quad (14)$$

$$P\{x_{n+1}^{\epsilon, \mathbf{u}} \in \cdot | \mathcal{F}_n \vee \mathcal{F}_n^y\} = P_n^\epsilon(\cdot; x_n^{\epsilon, \mathbf{u}}, u_n) \quad (15)$$

where the Q_n^ϵ are given probability kernels. Clearly, (4), (14) and (15) together determine the joint law of the process $(x^{\epsilon, \mathbf{u}}, y^{\epsilon, \mathbf{u}})$. Denoting by $\mathcal{U}^y$ the class of control processes that are adapted to the filtration $(\mathcal{F}_n^y)$, the aim of the controller is to minimize over $\mathcal{U}^y$ the cost functional $J^\epsilon(u)$ given by (5).

The following (rather long!) list of requirements make the problem tractable with our method.

Assumption B.

1. For fixed $\epsilon > 0$, $n = 0, \dots, N-1$ the map $(x, u) \rightarrow P_n^\epsilon(dx'; x, u)$ is continuous in x uniformly for $u \in U$, and the map $x \rightarrow Q_n^\epsilon(dy; x)$ is continuous.
2. For any n , $\{P_n^\epsilon(dx'; x, u) : \epsilon > 0, (x, u) \in X \times U\}$ and $\{Q_n^\epsilon(dy; x); \epsilon > 0, x \in X\}$ are ULDF's with rate functions $H_n^P(x'; x, u)$, $H_n^Q(y; x)$ respectively.
3. The spaces X, Y are locally compact Polish spaces, and U is a compact metric space.
4. The functions g_n , $n = 0, \dots, N$, in the cost function (5) are uniformly continuous.
5. There are sequences $(C_m)_{m \geq 0}$, $(K_m)_{m \geq 0}$ of compact sets in X and Y respectively, such that, for $n = 0, \dots, N-1$

$$\limsup_{m \rightarrow \infty} \limsup_{\epsilon \rightarrow 0} \sup_{(x, u) \in X \times U} \epsilon \log P_n^\epsilon(C_m^c; x, u) = -\infty$$

$$\limsup_{m \rightarrow \infty} \limsup_{\epsilon \rightarrow 0} \sup_{x \in X} \epsilon \log Q_n^\epsilon(K_m^c; x) = -\infty.$$

6. Let C_1, C_2 be arbitrary compact subsets of X , and $A_n(C_1, C_2) = \{(x', x, u) : H_n^P(x'; x, u) < \infty\}$. Then H_n^P is uniformly continuous in $A_n(C_1, C_2)$, for any choice of C_1, C_2 .
7. The rate function $H_n^Q(y; x)$ is uniformly continuous in $K \times X$, for any $K \subset Y$ compact.

For simplicity, we also assume from now on that the initial condition x_0 for the state dynamics is deterministic; this requirement can be considerably weakened.

The $\epsilon \rightarrow 0$ limit of the partially observed stochastic control problems defined above will be described as a dynamic game of the following type. The state-observation process is determined by the difference equations

$$\begin{cases} x_{n+1} &= f_n(x_n, u_n, w_n) \\ x_0 &\text{given} \\ y_n &= h_n(x_n, v_n) \end{cases} \quad (16)$$

with $x_n \in X, y_n \in Y, w_n \in W, v_n \in V$, W and V being arbitrary sets. The class $\hat{\mathcal{U}}^y$ of admissible controls is given by those controls $\mathbf{u}$ such that u_n is a function of $(y_0, \dots, y_n)$. The aim of the controller is to minimize over $\hat{\mathcal{U}}^y$ a cost functional of the form

$$K(\mathbf{u}) = \sup_{\mathbf{w}, \mathbf{v}} \left\{ \sum_{n=0}^{N-1} G_n(x_n, u_n, w_n, v_n) + G_N(x_N) \right\}. \quad (17)$$

In order to use the machinery introduced in Section 3, we transform the problem (14)(5) into an equivalent (i.e. having the same cost functional) problem with complete observation.

The transformation consists in defining the process $z^{\epsilon, \mathbf{u}} = (z_0^{\epsilon, \mathbf{u}}, \dots, z_N^{\epsilon, \mathbf{u}})$, with $z_n^{\epsilon, \mathbf{u}} = (y_0^{\epsilon, \mathbf{u}}, \dots, y_n^{\epsilon, \mathbf{u}}, u_0, \dots, u_{n-1})$. The process $z^{\epsilon, \mathbf{u}}$ is adapted to the filtration $(\mathcal{F}_n^y)$, and it can be shown that there are probability kernels $P_n^{z, \epsilon}(dz_{n+1}; z_n, u_n)$ such that

$$P\{z_{n+1}^{\epsilon, \mathbf{u}} \in \cdot | \mathcal{F}_n^y\} = P_n^{z, \epsilon}(\cdot; z_n^{\epsilon, \mathbf{u}}, u_n) \quad (18)$$

for any admissible $\mathbf{u} \in \hat{\mathcal{U}}^y$. Moreover, the cost functional $J^\epsilon(\mathbf{u})$ can be rewritten in terms of the process $z^{\epsilon, \mathbf{u}}$ as

$$J^\epsilon(\mathbf{u}) = E\{e^{\epsilon^{-1}\gamma^\epsilon(z_N^{\epsilon, \mathbf{u}})}\} \quad (19)$$

with

$$\gamma^\epsilon(z_N^{\epsilon, \mathbf{u}}) = \epsilon \log E\left\{ \exp \left[\epsilon^{-1} \left(\sum_{n=0}^{N-1} g_n(x_n^{\epsilon, \mathbf{u}}, u_n) + g_N(x_N) \right) \right] | z_N^{\epsilon, \mathbf{u}} \right\}.$$

We have therefore transformed the partially observed problem (14)(5) into the equivalent totally observed (18)(19). The value function $V_n^\epsilon(z_n)$ for this last problem will be referred to as the value function for (14)(5).

It turns out that Theorem 3 cannot immediately applied to this transformed problem, since the probability kernels $P_n^{z, \epsilon}$ do not necessarily satisfy Assumption A. However, using Assumption B and a refinement of Lemma 2, the proof can be repeated in its essential lines, yielding the Theorem that follows. The details are, however, rather long and technical, and are omitted. The complete analysis of the partially observed case is the subject of a forthcoming paper (Dai Pra and Rudari (1995b)).

Theorem 4 *Suppose Assumption B is satisfied and, for $n = 0, \dots, N-1$, let $f_n : X \times U \times W \rightarrow X$, $h_n : X \times V \rightarrow Y$ be functions such that for each $(x, u) \in X \times U$ the maps $w \rightarrow f_n(x, u, w)$ and $v \rightarrow h_n(x, v)$ are onto. Then the value function $V_n^\epsilon(z_n)$ for the problem (14)(5) is such that*

$$\lim_{\epsilon \rightarrow 0} \epsilon \log V_n^\epsilon = S_n \quad (20)$$

exists, uniformly on $Y^n \times U^{n-1}$. Moreover, $\sup_{z_0} S_0(z_0)$ is the value function of the dynamic game (16)(17) with

$$\begin{aligned} G_N(x) &= g_N(x) \\ G_n(x, u, w, v) &= g_n(x, u) - H_n^P(f_n(x, u, w); x, u) - H_n^Q(h_n(x, v); x) \\ &\quad \text{for } n = 1, \dots, N-1 \\ G_0(u, w) &= g_0(x_0, u) - H_0^P(f_0(x_0, u, w); x_0, u). \end{aligned} \quad (21)$$

Example 5 Consider the risk-sensitive stochastic control problems with cost functional given by (6), and dynamical equations

$$\begin{aligned} x_{n+1}^{\epsilon, u} &= F\left(a_n(x_n^{\epsilon, u}, u_n) + c_n(x_n^{\epsilon, u}, u_n)w_n^\epsilon\right) \\ x_0^{\epsilon, u} &= x_0 \\ y_n^{\epsilon, u} &= G\left(b_n(x_n^{\epsilon, u}) + d_n(x_n^{\epsilon, u})v_n^\epsilon\right) \end{aligned} \quad (22)$$

where $x_n^{\epsilon, u}, w_n^\epsilon \in \mathbb{R}^d$, $y_n^{\epsilon, u}, v_n^\epsilon \in \mathbb{R}^k$, $u_n \in U \subset \mathbb{R}^q$ compact, a_n, b_n are bounded uniformly continuous functions, c_n, d_n are matrix valued functions, uniformly continuous, bounded, and such that $c_n c_n^T, d_n d_n^T$ have a spectrum which is uniformly bounded away from zero. Moreover F, G are injective, uniformly continuous functions, and $w_0^\epsilon, \dots, w_{N-1}^\epsilon, v_0^\epsilon, \dots, v_{N-1}^\epsilon$ are two sequences of i.i.d. random variables with the following distribution:

$$w_n^\epsilon \sim C_1 \sum_{m \in M} \alpha_m \exp\left[\frac{1}{2\epsilon} \|w - m\|^2\right]$$

$$v_n^\epsilon \sim C_2 \sum_{m \in N} \beta_m \exp\left[\frac{1}{2\epsilon} \|v - m\|^2\right]$$

where $\alpha_m, \beta_m > 0$, M, N are finite subsets of $\mathbb{R}^d$ and $\mathbb{R}^k$ respectively, and C_1, C_2 are normalization factors. The model studied in James, Baras and Elliott (1994) corresponds to $F = id$, $G = id$, c_n, d_n scalar matrices, M, N consisting of a single element. By using the results in Dai Pra and Rudari (1995a), we have that the transition probabilities corresponding to (22) form ULDF's with rate functions

$$H_n^P(x'; x, u) = \frac{1}{2} \min_{m \in M} \|F_n^{-1}(x') - m - a_n(x, u)\|^2$$

$$H_n^Q(y; x) = \frac{1}{2} \min_{m \in N} \|G_n^{-1}(y) - m - b_n(x)\|^2.$$

If we assume the functions g_n in (6) to be uniformly continuous and bounded, then it can be shown that Assumption B is satisfied (see Dai Pra and Rudari (1995b)). By using Theorem 4 one obtain, among all possible choices, the following dynamic game:

$$\begin{aligned} x_{n+1} &= F\left(a_n(x_n, u_n) + c_n(x_n, u_n)w_n\right) \\ y_n &= G\left(b_n(x_n) + d_n(x_n)v_n\right) \end{aligned}$$

$$K(u) = \sup_{w,v} \left[g_0(x_0, u_0) - \frac{1}{2} \text{dist}^2(w_0, M) + \sum_{n=1}^{N-1} \left(g_n(x_n, u_n) - \frac{1}{2} \text{dist}^2(w_n, M) - \frac{1}{2} \text{dist}^2(v_n, N) \right) + g_N(x_N) \right].$$

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