78

Learning mathematics with CAS

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ABSTRACT

To say that the use of a Computer Algebra System (CAS) will revolutionize the learning of Mathematics has become a common place remark. However, main research into the implications of CAS on teaching and learning is either theoretical, with no classroom examples, or practical without any general views. To overcome this problem we have started a study for the French Ministry of Education which is based on practical classroom examples, but which also aims at development of general views and identification of possible changes in curriculum and teaching practice in higher education. Our first experiments show that the use of CAS is only helpful to students who already have a certain knowledge of Mathematics. But the experiments also show that CAS may be used for detecting difficulties in learning of those who do not have this knowledge. In this paper some principles are presented which should underpin the design of new curricula and new practices of teaching.

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INTRODUCTION

In 1992, the French Ministry of Education asked us to study the implications of using CAS on teaching and learning mathematics at the further education level. The central question in the study was: will its use help the students? Although thinking the desired answer was 'yes', we tackled the question using a pragmatic approach in the first experiment described below. Its results were unexpected. We used these results to answer other questions, such as: what are the implications for the teaching practice, for the curriculum and for the way we do mathematics? Curriculum design is directed by the problems we want students to be able to solve. For the time being the problems are supposed to be the same. The only change is that CAS may be used as a tool: a new way of doing mathematics. A new curriculum can therefore not be designed before insight is gained into the central question: is use of CAS as a tool helpful?

The examples in this paper are in MAPLE V (Release 2) and DERIVE (Version 2.57).

FIRST EXPERIMENT

In our first experiment which involved undergraduate students of a rather good level, students were asked to solve the same kind of mathematical problems as usual, but with the possibility of using CAS. We did a pretest before the experiment without use of CAS. The same students did another test after we had showed them how to use CAS for doing mathematics. The tests are reported in [1] so we shall not describe them in this paper. In any case, other experiments have shown that the exact choice of exercises does not matter. What matters, of course, is to choose exercises where CAS can be helpful. The results were amazing. Although two thirds of our students improved their results, there was a third without any improvement at all. To be more precise: being not successful without CAS, they were also not successful with CAS.

Starting an investigation we discovered that students who were not successful with CAS, were not familiar with the calculation methods. During the tests, we tried to help them by telling them what CAS can do and how it works, but they did obviously not realize what its possibilities were for the solving of the problems. They failed in translating a mathematical problem into a computational one. Therefore the first result of our experiment was improvement of our ability to find the exact difficulties of students and to remove these. As a matter of fact, it is not always easy to discern between those who understand methods, but fail in their use because of technical difficulties in the computations involved, and those who do not understand the methods. To use the words of Bernhard Kutzler [2], in the first case 'scaffolding didactics' works, in the second case it does not. Thus, the answer to the first question: "does CAS help students to learn Mathematics?" is 'yes' and 'no'. We have discovered the following principle: there is a certain level of knowledge under which CAS is helpless and above which it is helpful. Obviously, CAS is of little help in reaching this level, but it is of great help to check whether the level has been reached or not. Then, teachers can tackle the misconceptions of their students.

QUESTIONS

We now can go on to some of the other questions which concern doing mathematics and teaching practice. As we have chosen to use CAS, the first question is how to use it effectively. In fact, this question has several aspects. The first one is technical. What does one have to know to understand the failures and to correct the errors of CAS, to lead the computations in the right way? The second aspect concerns the teaching of Mathematics. Which methods will we have to teach to understand the results given by CAS? At the same time, we shall have to be careful with respect to the effects of this new tool on the conceptions of students and correct possible misconceptions. The third aspect is more fundamental. It concerns Mathematics itself. What Mathematics is useful for solving mathematical problems? It is clear that at this moment these questions cannot be answered for all time. But we can seek some principles to lead us to an answer.

TECHNICAL ASPECT

The technical aspect can be divided into three questions: how to detect errors, how to help CAS when it does not answer a question, and how to deal with unusable results?

Detecting errors

A way of answering this question is to say: "tomorrow, CAS will be perfect so it is useless to tackle it", another way is to say: "if CAS can answer wrongly, do not use it". We think that these two ways are too idealistic. On the one hand, even if perfect CAS could eventually be build, in the meantime we have to use existing ones. On the other hand, we have the same kind of problem with computations 'by hand'. The real problem is that with CAS, we do not know how computations are done. To teach that knowledge would be so difficult that it would be better not to use CAS at all. However, our experiments show that a simple check on the form of results is sufficient to correct most of the wrong results of CAS.

For example, MAPLE gives the result:

$$\int_a^b \sqrt{(x-a)(b-x)}\,\mathrm{d}x = \frac{\pi}{8}(b-a)^2$$

It is easy to see that this is incorrect, because the integral is positive only if *b>a*. This is easy to correct by exchanging the parameters *a* and *b*, thus changing the sign of the integral. In the same way, MAPLE and DERIVE give the result:

$$\int_0^\infty \frac{\mathrm{d}x}{x^2-a^2} = \frac{i\pi}{2a}$$

In this case, it is easy to see the absurdity of a non real result. We can give multiples of this kind of examples. Sometimes, errors may need more knowledge to be detected. For $a = e^{\frac{2i\pi}{5}}$,MAPLE gives the amazing result:

$$(x-a)(x-a^2)(x-a^3)(x-a^4) = x^4 - 4x^3 + 6x^2 - 4x + 1$$

if we expand and simplify the left hand side of the equation given above. In such a case, average students will only detect the error by trying a number of values. But, doing so, they loose interest in using CAS.

In fact, we can see that results are reliable if they do not contain parameters and if calculations do not need the use of multimorphic functions. It follows that, at least, the notion of logarithm branches should be taught.

Helping CAS

Sometimes, CAS does not achieve the computation we asked it to do. Using MAPLE, this is the case for:

$$\int_0^\infty \frac{e^{-x}}{x^2+1}\mathrm{d}x \text{ and } \int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1+x^2}+\sqrt{1-x^2}}$$

Knowledge of usual methods allows us to doubt the possibility of computing the first integral. It also allows us to think that we still might be able to calculate the second one. In fact, this integral can easily be transformed into:

$$\int_{-1}^1 \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{2x^2}\mathrm{d}x$$

In its transformed form MAPLE is able to compute the integral. From these two examples, we can see that the user must know computational methods to be able to decide what to do when CAS fails.

Unusable results

We find the same kind of problems when trying to use a result which seems not usable at first glance. In general these problems stem from insufficient knowledge of the functions used by CAS. In some cases, the result can be false. For example, MAPLE gives the result:

$$\int_0^{\pi} \frac{dx}{1+\cos a \cos x} = -\frac{i\,\mathrm{signum}\left(\frac{\sqrt{-1+\cos a}}{\sqrt{1+\cos a}}\right)\pi}{\sqrt{1+\cos a}\sqrt{-1+\cos a}}$$

PEDAGOGICAL ASPECT

This aspect falls into three parts. The first concerns the teaching of computing 'by hand'. The second concerns the effects of using CAS on mathematical concepts. And the third concerns the problem of using CAS to help intuition.

Teaching calculation rules

If you are very enthusiastic, you may think that it is not necessary to teach how to do calculations which can be done using CAS. Our paragraph called 'Technical Aspect' shows how dangerous it could be to think so. A student must, at least, remain able to lead CAS and for that needs a minimum knowledge about these methods. On the other hand, one can think we must keep on teaching everything. This also seems too much. In fact, some methods teach nothing about what we want students be able to do. In order to distinguish between these two kinds of methods, we have to find some criteria of choice. In fact, what matters is being able to understand results given by CAS, to find their conditions of validity, to predict their forms, the influence of parameters and, sometimes, to lead CAS to a result. On the other hand, students must be able to do all the qualitative studies out of reach of CAS.

Let us apply those principles to a simple example of algebraic calculation on numbers. If we ask DERIVE to exactly compute:

$$\left(\frac{1-\sqrt{5}}{2}\right)^{40}$$

The result is:

$$\frac{228826127-102334155\sqrt{5}}{2}$$

Then, if we ask DERIVE to give an approximation in six digits of the integral and the result, it gives 11 and 4.37020 10^{-9}. The error is easy to understand with a minimum knowledge of approximations. Thus, for this question of algebraic calculation on numbers, our principles require students to have some knowledge about calculation in quadratic fields and numerical calculations on real numbers. They also require students to know about the way of computing of CAS and to study the consequences of rounding or truncating.

Let us also check the consequences of our principles on the teaching of integration rules. It is clear that all the general rules like linearity, change of variables or integration by parts should still be taught. Students should remain able to perform them 'by hand', even in rather difficult cases. Qualitative studies in many cases require their use. The more particular rules are less useful. Nevertheless, it seems to be difficult to predict the form of certain results if we do not know integration rules of rational fractions and the rules based upon them. But, in this case, we can forget about any further skill in performing them 'by hand'. The theory allows one to know the form of the results without performing calculations. Thus, even if our principles are quite different from the black box / white box one (see [3]), their consequences are so far apart. But we think that in some other fields like Algebra, the consequences could be rather different. For example, the teaching of some particular methods for solving of algebraic equations can be avoided without any consequences according to our principles.

Balance between mathematical conceptions

One of the indirect consequences of using CAS is to transform students' mathematical conceptions as shown by the following exercise:

'Is there a derivable function f on the closed interval [0,6] which has values on the interval [0,4] such that f(0)=4 and $f(x) \leq 0.4$ if $x \geq 0.4$?' The best way to deal with this kind of problem is to have a geometrical representation of the function f. But half of our students tried formulas, most of them succeeding. The others preferred to draw a line in the domain: $([0,0.4]\times[0,4])\cup([0,6]\times[0,0.4])$ before proving the result using interpolation. This used to be their way of thinking before using CAS. If we consider giving a proof, the use of a formula has some advantages. But where intuition is concerned, this way has the drawback of emphasizing the formal notion of function too much. This will strengthen the usual bad effect of the beginning of Calculus teaching. For example, the derivative is too often seen through calculation rules. The geometrical aspect which is at the heart of the notion is often forgotten.

To counterbalance this predominance of the formal aspect, we think we have to insist more than previously on the link between a function and its graph. From this point of view, tools to draw graphs are of great help. In the old days, students did not draw a graph to just get some idea about the function because long and hazardous computations were seen as an obstacle. Now, their attitude has changed (see [4]).

Helping intuition

For example, the study of the sequence (f_n) defined by $f_n(x)=\dfrac{\sin(nx)}{n\sqrt{x}}$ was considered by our students as a very difficult exercise. Generally, they did not see the result. Now, most of them see the convergence to 0 is uniform after drawing graphs of a number of f_n. Moreover most of them can also prove this. Nevertheless, we have to be careful with the meaning of 'to see'. Obviously, this is not a passive sight, but one of interpretation. Evidence of that fact came with the study of the sequence (u_n) defined by:

$u_0=a$ and $u_{n+1}=\dfrac{3}{2u_n^2+1}$ for all $n\geq 0$ where a is a real number. A third of our students 'saw' wrongly that $\lim_{n\to\infty} u_n=1$. Some of them could even 'prove' this. Others 'saw' that the sequence had two limit points, and proved this after having 'seen' the properties to prove. Thus we notice that 'seeing' takes a minimum knowledge of mathematics.

MATHEMATICAL ASPECT

Now, we shall have a look at cases where using CAS can transform the way of solving mathematical problems. We mean giving proofs and not 'helping intuition'. However, we do not intend to go into the problem of proof validity (see [5]). The fact that a proof is valid or not is a matter of consensus between mathematicians. Let us assume that this will not change in the near future. We divide our study into two parts. First we shall look at cases where CAS just takes over calculations previously done 'by hand'. Secondly, we shall look at cases where the choice of calculations is influenced by CAS.

Taking charge of calculations

This way of using CAS is the first one which comes to mind. Being used to doing such calculations in a given situation, we now use CAS to do these. We find a number of such examples in geometry. For example, the following exercise:

> 'We consider the parabola $y^2=2x$. Generally, the osculatory circle at the point M of the parabola meets the parabola at another point N. Determine the envelope of the lines MN.'

The crucial point of that exercise is to factorize

$$\left[\frac{y^2}{2}-(1+6t^2)\right]^2+(y+8t^3)^2-(1+4t^2)^3$$

Of course, this is easy with the help of CAS. Here is another example:

'Let a, b and c be three complex numbers. On which condition can the matrix:

$$M = \begin{pmatrix} a-b-c & 2a & 2a \\ 2b & b-a-c & 2b \\ 2c & 2c & c-a-b \end{pmatrix}$$

be brought into diagonal form?'

We noticed that more students than previously chose to prove the result through computing $M^2-(a+b+c)^2 I$ This shows already a new way of dealing with the problem.

New ways of dealing with problems

Here is another example:

'Let C be the curve parametrized by $M(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ Show that the osculatory planes to the curve at three different points in their plane have a common point.'

The most obvious way of dealing with this exercise is to write the equations of those four planes and to solve the linear system they form. In the past, this method did not work for the average student because of the calculations involved. Now it works when using CAS.

Thus, we see that for some mathematical problems we can emphasize the theoretical aspect against the computing one. This will change a little bit the practice of teaching, but will not affect the curricula much.

CONCLUSIONS

We see that CAS does liberate the learner of mathematics and helps 'to see' some results. But this works only for those having already a certain knowledge. Moreover, using CAS could have some undesirable effects which must be counterbalanced by emphasizing geometrical aspects.

On the other hand, using CAS requires knowledge of calculations methods. What matters is being able to understand results given by CAS, to find their conditions of validity, to predict their forms, the influence of parameters and, sometimes, to lead CAS to a result. Moreover, students must be able to do all the qualitative studies which are out of reach of CAS. We must also either teach functions used by CAS, such as the complex logarithm, or use a purpose built version of CAS in which these are not used.

As we have shown, those few principles can lead to the design of new curricula. They can also lead to a practice of teaching.

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