Homomorphisms of Secret Sharing Schemes: A Tool for Verifiable Signature Sharing

Mike Burmester

Information Security Group, Department of Mathematics
Royal Holloway – University of London, Egham, Surrey TW20 OEX, U.K.
m.burmester@vms.rhbc.ac.uk.

Abstract. Franklin and Reiter introduced at Eurocrypt '95 verifiable signature sharing, a primitive for a fault tolerant distribution of signature verification. They proposed various practical protocols. For RSA signatures with exponent $e = 3$ and n processors their protocol allows for up to $(n - 1)/5$ faulty processors (in general $(n - 1)/(2 + e)$).

We consider a new unifying approach which uses homomorphisms of secret sharing schemes, and present a verifiable signature sharing scheme for which as many as $(n - 1)/3$ processors can be faulty (for any value of e), and for which the number of interactions is reduced.

1 Introduction

Verifiable signature sharing schemes enable the holder of a signed document (who may not be the original signer) to distribute the signature among a set of proxies in such a way that each proxy can verify whether a valid signature can be reconstructed later, even if some of the proxies (but not too many) are malicious or if the original signer is malicious. The signature itself is not revealed until later when it is reconstructed. Verifiable signature sharing is related to the distributed verification of undeniable signatures [18] and to threshold signatures schemes [8]. However there are fundamental differences. With the distributed verification of undeniable signatures, the signed document is public and secret information for the signature is shared. With threshold signatures the power to sign (not to verify) is shared. Such schemes enable any sufficiently large set of shareholders to construct the signature of any document, without any further interaction with the signer.

Verifiable signature sharing has many applications. It can be used to escrow digital cash, by verifiably sharing the signature of a bank for a banknote. Another application is the secure distribution of financial services such as auction bidding [11].

At Eurocrypt '95 Franklin and Reiter presented practical verifiable signature sharing protocols for several signature schemes, including RSA (exponentiation) based signatures with small exponents¹ e and El Gamal based signatures [10]. In this paper we introduce a new approach based on homomorphisms of secret

¹ At the Rump session of Crypto '95 the authors pointed out that their scheme for RSA with $e = 3$ was flawed, and that various repairs were considered [12].

sharing schemes to describe verifiable signature sharing. We use this to extend the Franklin Reiter protocol for El Gamal signatures to a more general setting, and obtain an optimal² verifiable signature sharing scheme. Our construction is natural and simplifies significantly the protocol for RSA based signatures in [10].

2 Definitions and notation

Let $A = \{1, \dots, n\}$ be a set of proxies, K a set of secrets, $S_1, \dots, S_n$ sets of shares, and $V = S_1 \times \dots \times S_n$ the set of n -tuples of elements from S_i .

Definition 1. [3, 19] A (t, n) -threshold scheme consists of a pair of algorithms, a sharing algorithm $\mathcal{S}$ and a reconstruction algorithm $\mathcal{R}$. $\mathcal{S}$ is probabilistic and takes as input a secret $k \in K$. It generates an element $c = (s_1, \dots, s_n) \in V$ according to a distribution depending on k , and distributes the share s_i to proxy $i \in A$. $\mathcal{R}$ is deterministic. It takes as input an element $c \in V$ with up to $n - t$ erasures and will output the secret k , if c is an output of $\mathcal{S}$ when k is input. A (t, n) -threshold scheme is *perfect* if $\text{Prob}(k \mid s_{i_1}, \dots, s_{i_{t-1}}) = \text{Prob}(k)$, for all distributions on $k \in K$, all $\{i_1, \dots, i_{t-1}\} \subset A$, and all subsets of shares $(s_{i_1}, \dots, s_{i_{t-1}})$ output by $\mathcal{S}$ on input k .

Notation. We write $\mathcal{S}(k) = (s_1, \dots, s_n)$ if (the probabilistic algorithm) $\mathcal{S}$ outputs $(s_1, \dots, s_n)$ on input k , and $\mathcal{R}(s_1, \dots, s_n) = k$ if $\mathcal{R}$ outputs k on input $(s_1, \dots, s_n)$. If $B = \{i_1, \dots, i_r\} \subset A$ and $c = (s_1, \dots, s_n) \in V$, then $c_B = (s_{i_1}, \dots, s_{i_r})$. We write $\mathcal{S}_B(k) = (s_{i_1}, \dots, s_{i_r})$. If $r \geq t$ then $\mathcal{R}_B(s_{i_1}, \dots, s_{i_r}) = k$. Finally we denote by $\{\mathcal{S}(k)\}$ the set of all $c \in V$ output by $\mathcal{S}$ on input k .

We shall assume that the scheme $(\mathcal{S}, \mathcal{R})$ belongs to a family $\{(\mathcal{S}_x, \mathcal{R}_x)\}$ indexed by a parameter $x \in J$, $J \subset \{0, 1\}^*$, but for simplicity of notation, when there is no ambiguity, we do not refer explicitly to x and just consider single schemes rather than families. The proxies are polynomially bounded in $|x|$, the binary length of x . A threshold scheme is *polynomial-time* if for any set B of $r \geq t$ proxies, $\mathcal{R}_B$ runs in polynomial-time in r and $|x|$. Algorithm $\mathcal{S}$ is not necessarily polynomial-time (in $|x|$ and n).

Definition 2. A threshold scheme is *perfectly zero-knowledge* if for any set B' of $t - 1$ or less proxies, the view of B' (i.e. their shares and public information) can be simulated perfectly in expected polynomial-time bounded by $t|x|^c$, c constant. A threshold scheme is *perfectly minimal-knowledge* if given any $k \in K$ by an oracle, the view of any $r \geq t$ proxies can be simulated perfectly in expected polynomial-time bounded by $r|x|^c$, c constant. For a formal definition the reader is referred to [15, 13, 8].

Zero-knowledge guarantees that no knowledge at all leaks. Minimal knowledge guarantees that no more than what follows from the joint knowledge of the proxies leaks.

² The optimality is with respect to the number of faulty processors T . In the information theoretic model the bound is $T < n/3$ [2, 4]; in the computational model it is $T < n/2$ [14].

Remark. A threshold scheme which is perfectly zero-knowledge is also perfect.

Remark. If a (t, n) -threshold scheme is perfect and if $(s_{i_1}, \dots, s_{i_{t-1}})$ are shares of a set B' of $t - 1$ proxies, then for any $k \in K$: $(s_{i_1}, \dots, s_{i_{t-1}}) \in \{S_{B'}(k)\}$.

Definition 3. A (t, n) -threshold scheme is an *erasure* scheme if for any $k \in K$, the shares $(s_{i_1}, \dots, s_{i_{t-1}}) \in \{S_{B'}(k)\}$ of any $t - 1$ proxies B' have a *unique* extension $(s_1, \dots, s_n) \in \{S(k)\}$. An erasure scheme is *polynomial-time* if this extension can be computed in $t \text{ poly}(|x|)$ time.

Remark. With an erasure scheme, a share-assignment $(s_1, \dots, s_n) \in \{S(k)\}$ can be reconstructed if $n - t + 1$ shares are erased, given k .

Definition 4. [5], [8, p. 676] A (t, n) -threshold scheme for which the set of secrets is a group $K(*)$ is *multiplicative* if for any set of t proxies $B = \{i_1, \dots, i_t\}$ there are selection functions $f_{j,B} : S_j \rightarrow K$, $j \in B$, and a public ordering of $\{i_1, \dots, i_t\}$ such that,

$$k = f_{i_1,B}(s_{i_1}) * \dots * f_{i_t,B}(s_{i_t}), \quad (1)$$

for all $k \in K$ and shares $(s_{i_1}, \dots, s_{i_t})$ output by S on input k . A multiplicative scheme is *polynomial-time* if the operation “ $*$ ” and its inverse are polynomial-time in $|x|$, and if all selection functions are polynomial-time in $|x|$.

Definition 5. Let $\sigma = (S, \mathcal{R})$ and $\sigma' = (S', \mathcal{R}')$ be (t, n) -threshold schemes with secret sets K, K' and share sets S_i, S'_i , $i = 1, \dots, n$, respectively, and let $h_0 : K \rightarrow K'$, $h_i : S_i \rightarrow S'_i$, $i = 1, \dots, n$, be mappings. The mapping $h = (h_0, h_1, \dots, h_n)$ is a *homomorphism* of σ to σ' if the following conditions hold. For any $k \in K$, if $S(k) = (s_1, \dots, s_n)$,

- $(h_1(s_1), \dots, h_n(s_n)) \in \{S'(h_0(k))\}$
 - $h_0(\mathcal{R}(s_1, \dots, s_n)) = \mathcal{R}'(h_1(s_1), \dots, h_n(s_n)).$
- (2)

That is, the mapping h commutes with the sharing and reconstruction operators.

Fig. 1. The commutative diagram of a homomorphism $h : \sigma \rightarrow \sigma'$.

A homomorphism h is *polynomial-time* if the mappings $h_i, i = 0, 1, \dots, n$, are polynomial-time in $|x|$.

Notation. If $B = \{i_1, \dots, i_r\}$ then $h_B(s_{i_1}, \dots, s_{i_r}) = (h_{i_1}(s_{i_1}), \dots, h_{i_r}(s_{i_r}))$.

A homomorphism $h : \sigma \rightarrow \sigma'$ induces in a natural way a threshold scheme $h(\sigma) = (S^*, \mathcal{R}^*)$ with secret set $K^* = h_0(K) \subset K'$, share sets $S_i^* = h_i(S_i) \subset S'$, and $V^* = S_1^* \times \dots \times S_n^* \subset V'$. S^* on input $k^* \in K^*$ outputs $(s_1^*, \dots, s_n^*) \in V^*$ where $s_1^* = h_1(s_1), \dots, s_n^* = h_n(s_n)$, with the distribution induced by $(s_1, \dots, s_n) \in \{S(k)\}$ where $h_0(k) = k^*$. $\mathcal{R}^*$ on input $(s_1^*, \dots, s_n^*) \in V^*$ with $s_1^* = h_1(s_1), \dots, s_n^* = h_n(s_n)$ and $k = \mathcal{R}(s_1, \dots, s_n)$, outputs $k^* = h_0(k)$. We call this, the scheme *induced* by the homomorphism h .

Remark. One should distinguish between homomorphisms of threshold schemes, as introduced here, and homomorphic threshold schemes [1]. With the former no algebraic structure is imposed on the set of secrets or on the shares. However, if “ $*$ ” is a binary operation on the set of secrets K and “ $\cdot$ ” a binary operation on the sets of shares S_i , with $\mathcal{R}(s_1, \dots, s_n) * \mathcal{R}(s'_1, \dots, s'_n) = \mathcal{R}((s_1 \cdot s'_1), \dots, (s_n \cdot s'_n))$ for all $(s_1, \dots, s_n), (s'_1, \dots, s'_n)$ output by $\mathcal{S}$, as in [1], then a homomorphism $h : \sigma \times \sigma \rightarrow \sigma$ is defined in a natural way, with $h_0(k, k') = k * k'$ and $h_i(s_i, s'_i) = s_i \cdot s'_i$. So the notion of homomorphisms of sharing schemes is an extension of homomorphic sharing schemes.

Remark. Homomorphisms are an important structure preserving mechanism in algebra. As we shall see they also play an important role in the analysis of cryptographic primitives, by linking verifiable signature sharing to secret sharing.

3 Verifiable signature sharing based on homomorphisms of secret sharing schemes

The scheme we describe is based on a homomorphism $h = (h_0, h_1, \dots, h_n)$ of the (t, n) -threshold schemes σ, σ' , and uses the induced threshold scheme $h(\sigma)$. The secrets in K are the signatures $\text{sgn}(m)$ and the elements in $K^* = h_0(K)$ are the (hashed/processed) messages m . We take $\sigma = (\mathcal{S}, \mathcal{R})$, $h(\sigma) = (S^*, \mathcal{R}^*)$ and $T = t - 1$. Our scheme consists of a pair of protocols: a sharing protocol and a reconstruction protocol. In the sharing protocol a dealer D runs $\mathcal{S}$ on input $\text{sgn}(m)$. The dealer gives privately to each of n proxies a share s_i of $\text{sgn}(m)$, and sends by reliable broadcast to all proxies the message M and the shares $h_j(s_j)$, $j = 1, \dots, T$, in $h(\sigma)$. At the end of this protocol the proxies either accept or reject. In the reconstruction protocol a reconstructor R computes $\text{sgn}(m)$ by majority vote. The scheme is described schematically in Fig 2. In the upper part we use the zero-knowledge scheme $\sigma = (\mathcal{S}, \mathcal{R})$ and the shares s_i are kept secret. In the lower part we use the erasure scheme $h(\sigma) = (\mathcal{R}^*, S^*)$ and the shares $h_i(s_i)$ are in the clear. The homomorphism h links the parts. The path $\mathcal{S}(\text{sgn}(m)) \rightarrow S^*(m)$ is used for verification: proxy i checks that the map $h_i(s_i)$ of his share s_i is the appropriate component of $(h_1(s_1), \dots, h_n(s_n))$, which he

gets by extending $(h_1(s_1), \dots, h_T(s_T))$ using the message m . The commutativity of the diagram guarantees correctness.

Fig. 2. A diagram for verifiable signature sharing.

We now describe our scheme in detail.

4 A Verifiable Signature Sharing scheme

Sharing Protocol

- Step 1** The dealer D uses algorithm S of σ to get shares $s_1, \dots, s_n$ of the signature $sgn(m)$.
- a. $D \rightarrow P_i$ privately: the share s_i , $i = 1, \dots, n$.
 - b. $D \rightarrow P_1, \dots, P_n$ by reliable broadcast: the message M , and the shares $h_1(s_1), \dots, h_T(s_T)$ in $h(\sigma)$, or something equivalent³.
- Step 2** Let the broadcast values in Step 1b be $M, u_1, \dots, u_T$. Each proxy P_i computes $m = hash(M)$ and makes the following reliable broadcasts to all proxies:
- a. COMPLAIN if the share $s_i \notin S_i$, or if the share $h_i(s_i)$ is not the i -th component of the extension $(u_1, \dots, u_n) \in \{S^*(m)\}$ of $(u_1, \dots, u_T)$.
 - b. ALLOW otherwise.
- Step 3** Without further communication, each proxy ACCEPTS if (REJECTS otherwise):
- a. At most T proxies COMPLAINED (Nonsensical responses are counted as COMPLAINS).
 - b. $(u_1, \dots, u_T) \in \{S_{\{1, \dots, T\}}^*(m)\}$.

Reconstruction Protocol

- Step 1** Each proxy P_i sends to the reconstructor R the following information:
- a. The shares s_i sent in Step 1a of the Sharing Protocol.
 - b. The values $M, u_1, \dots, u_T$ broadcast in Step 1b of the Sharing Protocol.

³ For example a polynomial-time invertible bijection of these, as in [10].

Step 2 Without further communication:

- a. The reconstructor R finds $\bar{M}, \bar{u}_1, \dots, \bar{u}_T$ by majority vote on the lists received in Step 1b (of this protocol).
- b. R discards every share s_i received in Step 1a (of this protocol) that is inconsistent with $\bar{M}, \bar{u}_1, \dots, \bar{u}_T$, in $h(\sigma)$, i.e., for which the share $s_i \notin S_i$, or the share $h_i(s_i)$ is not the i -th component of the extension $(\bar{u}_1, \dots, \bar{u}_n) \in \{\mathcal{S}^*(\bar{m})\}$ of $(\bar{u}_1, \dots, \bar{u}_T)$.
- c. Let $B = \{i_1, \dots, i_r\}$ be the set of proxies with the remaining shares. R computes $\mathcal{R}_B(s_{i_1}, \dots, s_{i_r})$ and outputs this as its signature.

We shall prove that this protocol achieves a slightly stronger goal than in [10].

Definition 6. A verifiable signature sharing scheme is T -resilient if the following conditions hold.

- (i) (*Completeness*) If the dealer is honest and at most T proxies are faulty then honest proxies accept, and the reconstruction is successful.
- (ii) (*Soundness*) If at most T proxies are faulty and an honest proxy accepts, then every proxy accepts and the reconstruction is successful.
- (iii) (*Secrecy*) Anything that can be computed by an adversary who, (a) controls up to T faulty proxies who have participated in earlier sharing protocols with an honest dealer and (b) has access to the view of the reconstructor in earlier reconstructions, can also be computed by the adversary without participating in any of these protocols.

Remark. Secrecy is based on the simulation of the view of the adversary. To simulate the shares of faulty (dishonest) proxies in the sharing protocols, σ must be zero-knowledge. To simulate the (uncorrupted) shares of the honest proxies revealed in the reconstruction protocols, $h(\sigma)$ must be minimal knowledge.

Remark. Our definition of secrecy is stronger than that in [10], since we allow the adversary to access earlier reconstruction protocols. We feel that this is an essential security requirement for verifiable signature sharing. Although the schemes proposed in [10] satisfy the stronger secrecy requirement, their model overlooked this aspect.

Remark. If h_0 is a one-way function then this scheme can also be used as a verifiable secret sharing scheme with heuristic security. The security is only heuristic because with one-way functions there is no guarantee that the input will not leak.

Theorem 7. Suppose that σ, σ' are polynomial-time (t, n) -threshold schemes with $t = T + 1$ and $T \leq (n - 1)/3$. The protocol above is a T -resilient verifiable signature sharing scheme if $h : \sigma \rightarrow \sigma'$ is a polynomial-time homomorphism and if the following conditions hold:

1. All the mappings $h_i, i = 0, 1, \dots, n$, are injections.
2. σ is a perfectly zero-knowledge polynomial-time erasure scheme, and membership in S_i for all i can be checked in polynomial-time.

3. σ' is a polynomial-time erasure scheme, and membership in $\{S_{\{1,\dots,T\}}^*(m)\}$ can be checked in polynomial-time.

To prove this theorem we need the following results.

Lemma 8. *Let σ, σ' be polynomial-time threshold schemes with σ perfectly zero-knowledge, σ' a polynomial-time erasure scheme, and $h : \sigma \rightarrow \sigma'$ a polynomial-time homomorphism. Then $h(\sigma)$ is perfectly minimal-knowledge.*

Proof. Since h is a homomorphism, $h(\sigma)$ is a perfectly zero-knowledge erasure scheme (it is not necessary that $h(\sigma) = \sigma'$). We describe an expected polynomial-time simulator M^* which given any $k^* \in K^*$ simulates shares of any set of t proxies $B = \{i_1, \dots, i_{t-1}, i_t\}$ in $h(\sigma)$. Now σ is perfectly zero-knowledge, so there is a simulator M which simulates the shares in σ of the set of $t-1$ proxies $B' = \{i_1, \dots, i_{t-1}\}$. To simulate the shares of B in $h(\sigma)$, M^* first simulates the shares of B' using M , to get $(s_{i_1}, \dots, s_{i_{t-1}})$ with a distribution identical to the real one. Then M^* computes $(h_{i_1}(s_{i_1}), \dots, h_{i_{t-1}}(s_{i_{t-1}}))$. Because $h(\sigma)$ is an erasure scheme, given any $k^* \in K^*$, M^* can compute its unique extension $c_B^* \in \{S_B^*(k^*)\}$ (perfect zero-knowledge guarantees that c_B^* belongs to $\{S_B^*(k^*)\}$ – cf. the Remarks after Definition 2). Finally M^* outputs c_B^* as the simulated shares for B .

Corollary 9. *If the conditions in the Lemma are satisfied and h_0 is a one-way function then each h_i , $i = 1, \dots, n$, is one-way.*

Proof. Suppose that it is easy to invert h_j . Let $B = \{i_1, \dots, i_t\}$ be a set of t distinct proxies with $j = i_t$ and let $B' = \{i_1, \dots, i_{t-1}\}$. We describe a polynomial-time algorithm $\mathcal{A}$ which inverts h_0 , that is, given $k^* \in K^*$ computes a $k \in K$ with $h_0(k) = k^*$. As in Lemma 8, $\mathcal{A}$ chooses shares $s_{i_1}, \dots, s_{i_{t-1}}$ for B' and computes $h_{i_1}(s_{i_1}), \dots, h_{i_{t-1}}(s_{i_{t-1}})$ and their unique extension c^* . Then $\mathcal{A}$ inverts the j -th component $s_j^* \in h_j(S_j)$ of c^* to get an $s_j \in S_j$ such that $h_j(s_j) = s_j^*$, and computes $k = \mathcal{R}_B(s_{i_1}, \dots, s_{i_{t-1}}, s_j)$. Since h is a homomorphism, $h_0(k) = k^*$.

Corollary 10. *A polynomial-time threshold scheme which is a perfectly zero-knowledge polynomial-time erasure scheme is perfectly minimal knowledge.*

Proof. Take $\sigma = \sigma'$ and h the identity homomorphism in Lemma 8.

Proof of the theorem:

Completeness: This follows directly from the fact that h is a homomorphism.

Soundness: If one honest proxy accepts then all honest proxies accept. We must show that in this case the reconstruction is successful. First observe that $h(\sigma)$ is a polynomial-time erasure scheme. So $(u_1, \dots, u_T)$ has a unique extension $c^* = (s_1^*, \dots, s_n^*)$ in $h(\sigma)$ with $\mathcal{R}(c^*) = m$, which can be computed in polynomial-time. Allowing for up to T COMPLAINs and up to T faulty proxies we see that at most $2T$ of the components of c^* are in error in the majority vote count. Since $n - 2T \geq T + 1 = t$, there is a set $B = \{i_1, \dots, i_r\}$ of $r \geq t$ proxies with shares $(h_{i_1}(s_{i_1}), \dots, h_{i_r}(s_{i_r})) \in \{S_B^*(m)\}$. We are assuming that the mappings h_i are injections. It follows that $(s_{i_1}, \dots, s_{i_r}) \in \{S_B(k)\}$. Because h is a homomorphism and h_0 is injective, $k = \text{sgn}(m)$.

Secrecy: We have to simulate, essentially, tuples $(h_{\{1,\dots,T\}}(c_{\{1,\dots,T\}}), c_{B'})$, with $|B'| \leq T$, for the sharing protocol, and tuples $(h(c), c, \text{sgn}(m))$ with up to T corrupted entries in c for the reconstruction protocol. Since the sharing algorithm $\mathcal{S}$ distributes shares independently for each run of the verifiable signature sharing scheme (the dealer is honest), it is sufficient to simulate one run. Furthermore we only need to simulate the uncorrupted parts. Now σ is zero-knowledge, so $c_{B'}$ can be simulated. Therefore given the unsigned (hashed/processed) message m , $(h_{\{1,\dots,T\}}(c_{\{1,\dots,T\}}), c_{B'})$ can be simulated, because $h(\sigma)$ is a polynomial-time erasure scheme. For the reconstruction protocol we have to simulate the uncorrupted parts of $(h(c), c, \text{sgn}(m))$. In this case the simulator is also given the signature $\text{sgn}(m)$. By Corollary 10, σ is minimal-knowledge, so c can be simulated. Then $h(c)$ can be simulated, since h is polynomial-time. $\square$

Remark. For this scheme $T \leq (n - 1)/3$. The upper (exclusive) bound for the number faulty processors in a fault tolerant distributed computation in the information theoretic model is $n/3$ [2, 4]. So the scheme is optimal in this respect.

5 Applications

5.1 A Verifiable Signature Sharing scheme based on the Desmedt-Frankel zero-knowledge RSA threshold scheme

Take σ to be the zero-knowledge RSA threshold scheme in [8] with $\sigma' = \sigma$. This scheme is homomorphic (*cf.* the Remark after Definition 5) and uses polynomial interpolation for the reconstruction of the secret. In this case however the coefficients of the polynomials are not taken from a field but from a ring extension $Z[u]$ of the integers. We briefly explain this scheme.

Let N be a composite number, n be polynomially bounded in $|N|$, q a prime with $q \geq n + 1$, and $Z[u] = Z[x]/((x^q - 1)/(x - 1))$ be the ring of all polynomial expressions in u with integer coefficients taken modulo the cyclotomic polynomial $(u^q - 1)/(u - 1)$. The set of secrets K and the share sets S_i , $i = 1, \dots, n$, have a group structure. K is the multiplicative group $Z_N^*(\cdot)$ and S_i is the $Z[u]$ -module $M = Z[u] \otimes_Z Z_N^*$, the tensor product [17] of $Z[u](+)$ and $Z_N^*(\cdot)$. It is easy to see that $M \cong (Z_N^*)^{q-1} = Z_N^* \times \dots \times Z_N^*$. Each shareholder is given a share $s_i \in S_i$, and a set of $r \geq t$ shareholders $B = \{i_1, \dots, i_r\}$ reconstructs the secret by using

$$\mathcal{R}_B(s_{i_1}, \dots, s_{i_r}) = \text{proj}_1 \left(\sum_{i \in B} y_{i,B} \cdot s_i \right),$$

where proj_1 is the projection of the elements of $(Z_N^*)^{q-1}$ onto their first coordinate in Z_N^* , the sum is the internal module operation, the product is the external module operation, and the $y_{i,B}$ are the Lagrange interpolation coefficients,

$$y_{i,B} = \prod_{j \in B, j \neq i} (0 - u^j) / (u^i - u^j),$$

which are invertible elements in $Z[u]$ [8].

For our application, $h_0 : Z_N^* \rightarrow Z_N^* : x \rightarrow x^e \bmod N$, where $\gcd(e, \phi(N)) = 1$, and $h_i : (Z_N^*)^{q-1} \rightarrow (Z_N^*)^{q-1} : (x_1, \dots, x_{q-1}) \rightarrow (x_1^e, \dots, x_{q-1}^e)$, $i = 1, \dots, n$. Clearly h is injective. Also $h_i(s) = e \cdot s$, $s \in (Z_N^*)^{q-1}$, using the external module operation. Furthermore the mappings h_i commute with proj_1 and the module operations (i.e. h is a morphism), since

$$\begin{aligned} \text{proj}_1(h_i(s_i)) &= \text{proj}_1(e \cdot s_i) = h_0(\text{proj}_1(s_i)), \text{ and} \\ \sum_i y_i \cdot h_i(s_i) &= \sum_i y_i \cdot (e \cdot s_i) = e \cdot (\sum_i y_i \cdot s_i) = h_j(\sum_i y_i \cdot s_i), \end{aligned}$$

for all $s_i \in (Z_N^*)^{q-1}$, $y_i \in Z[u]$. It follows that conditions (2) in Section 2 hold, and that h is a homomorphism of σ onto σ . It is easy to prove that all the other conditions of Theorem 7 are satisfied. In particular σ is an erasure scheme. Also $\{\mathcal{S}_{\{1, \dots, T\}}(k)\} = (Z_N^*)^T$, so that one only has to check membership in each of the components Z_N^* , to check membership in $\{\mathcal{S}_{\{1, \dots, T\}}(k)\}$. Thus we have a T -resilient verifiable signature sharing scheme.

5.2 A Verifiable Signature Sharing scheme based on the Desmedt-DiCrescenzo-Burmester zero-knowledge threshold scheme

Take σ to be the perfect zero-knowledge threshold scheme in [5] with $\sigma' = \sigma$. This scheme is multiplicative and the set of secrets $K(*)$ can be any group (not necessarily abelian). The shares are sets of sub-shares which are tuples (list_a, k_a) . A set B of t distinct proxies can recompute the secret k by taking the product of appropriate k_a 's whose selection depends on B , and is determined by the lists list_a . The lists define selection functions $f_{j,B}(s_j)$, $j \in B$, which are such that equation (1) in Section 2 holds. The functions $f_{j,B}$ commute with any homomorphism of the multiplicative group $K(*)$. We take h_0 to be a one-way injective group homomorphism of a polynomial-time group $K(*)$ (for which the operation “ $*$ ” and its inverse are polynomial-time). As in the previous case we use h_0 to define injections h_i on the shares s_i (for which each (list_a, k_a) is mapped to $(\text{list}_a, h_0(k_a))$). Then conditions (2) hold, as can easily be checked, and we get a homomorphism from σ to σ . It is easy to check that σ is an erasure scheme when $t = 2$. All conditions of Theorem 7 are then satisfied. So we get a 1-resilient verifiable signature scheme. In this case the size of the shares is reduced, roughly, from $n \log N$ to $(\log n)(\log N)$.

6 Conclusion

We have shown that by using homomorphisms of secret sharing schemes we can get RSA based verifiable signature sharing schemes which are optimal with respect to the number of faulty processors. Our approach is natural, uses existing secret sharing schemes, and simplifies significantly the RSA based protocol of Franklin and Reiter [10].

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