Packing Trees into Planar Graphs

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Abstract. The main problem considered in this paper is the following: given two trees both with n vertices, whether it is possible to draw them on the plane with the same set of vertices without crossings and duplicated edges. We formulate this problem in terms of packing graphs and give a solution in several situations. We also solve some related problems on drawing trees and cycles.

1 Introduction

We say that the graphs $H_1, \dots, H_k$ can be *packed* into a graph G if G contains subgraphs isomorphic to $H_1, \dots, H_k$ and pairwise edge disjoint. The problem of packing graphs has been widely studied, especially when G is a complete graph. For instance, it is known that two trees with n vertices none of them equal to a star can be packed into K_n [2]. The packing of three trees presents more difficulties and has been completely solved recently [3]. A recent survey on packing graphs is [1].

We observe that when G is a planar graph, a packing into G is equivalent to a planar drawing of the graphs $H_1, \dots, H_k$ without sharing edges. In this case we obtain a simultaneous drawing of all the graphs $H_1, \dots, H_k$ on the same set of vertices. Coloring each of the graphs with a different color, one obtains a clear picture since there are no crossings and no multiple edges. This solution is a good option in some situations, since the information from a collection of k separate drawings can be difficult to integrate.

The main problem we discuss in this paper is the *tight planar packing* of two trees: given two trees T_1 and T_2 both with n vertices, none of them equal to a star, is it possible to pack them into some planar graph with n vertices? We conjecture that the answer is in the affirmative. We observe that although T_1 and T_2 can be packed into K_n as mentioned above, the union of the two graphs in K_n can certainly be non planar. We also observe that three trees with n vertices never admit a tight planar packing, since a planar graph has at most $3n - 6$ edges.

In this paper we solve the conjecture in several cases. When one of the two trees is a path we show that it can be packed with any other tree T different from a star. This provides a drawing of T together with a spanning path visiting

all the vertices and not using edges of T . We also show that the problem can be solved when one of the two trees has maximum degree $n - 2$, and that any tree T different from a star can be packed together with a copy of T .

The technique used to prove these results is based on finding special drawings of trees on the set of vertices of a convex polygon. This is the content of Section 2. In Section 3 we present the results on the packing of two trees, and in Section 4 we study related problems, particularly planar packings of cycles.

2 Drawing trees on a convex polygon

Given any tree T of order n , it is clear that T can be drawn without crossings on a convex polygon with n sides and that, in general, it can be drawn in many different ways. A property of drawings that turns out to be significant is which edges are used from the boundary of the polygon. The next two lemmas and the theorem below deal precisely with this topic.

Lemma 1. *Let T be a tree with $n > 3$ vertices and let T' be the tree obtained from T by adding a new terminal node h connected to a node v of degree $r \geq 2$. If T can be drawn on a convex polygon using only m edges on the boundary, then T' also admits a drawing with only m edges on the boundary.*

Proof. Since $r \geq 2$, there exist two neighbors w and w' of v such that the counterclockwise path from w to w' along the boundary of the polygon does not contain neither v nor any neighbor of v . Then at least one boundary edge $v_i v_{i+1}$ between w and w' does not belong to T , since otherwise there would be a cycle. Moreover, it is easy to see that it can be chosen so that h can be inserted between v_i and v_{i+1} and connected to v without creating crossings.

Let T be a tree with $p > 1$ vertices and with m terminal nodes denoted $h_1, \dots, h_m$. Let $\bar{T}$ be the tree obtained from T by adding a new terminal node $\bar{h}_i$ connected to h_i for every $i = 1, \dots, m$. Clearly $\bar{T}$ has at least four vertices and as many terminal nodes as T . Then we have the following lemma.

Lemma 2. *In any drawing of $\bar{T}$ on a convex polygon, the m edges $h_i \bar{h}_i$ lie necessarily on non-consecutive boundary edges. Moreover, $\bar{T}$ can be drawn using exactly m boundary edges.*

Proof. If $h_i \bar{h}_i$ is drawn as an internal diagonal then the remaining vertices of the polygon are split into two regions that cannot be connected without crossing $h_i \bar{h}_i$, since the degree of h_i in $\bar{T}$ is equal to two. Hence $h_i \bar{h}_i$ is a boundary edge.

We prove now the second part of the statement, by induction on m , the number of terminal nodes. Let $v_1, \dots, v_n$ be the vertices of the convex polygon, listed counterclockwise, where $n = |V(\bar{T})|$. If $m = 2$ then $\bar{T}$ is a path, which can be drawn in a zigzag way, $v_1 v_2 v_n v_3 v_{n-1} \dots$, using exactly two boundary edges.

For $m > 2$, let us consider the path C starting at $\bar{h}_m$ and ending in the first encountered node w with degree greater than 2. Let $w_{k+1} = \bar{h}_m, w_k =$

$h_m, \dots, w_2, w_1 = w$ be the nodes of C . By removing $w_2, \dots, w_{k+1}$ from $\bar{T}$ we obtain a tree T' with $m-1$ terminal nodes, $\bar{h}_1, \dots, \bar{h}_{m-1}$, among a total of $n-k$ nodes. T' can be drawn on the convex polygon $v_1 v_2 \dots v_{n-k}$, by induction, with only $h_1 \bar{h}_1, \dots, h_{m-1} \bar{h}_{m-1}$ becoming boundary edges. We can assume without loss of generality that w becomes v_{n-k} in that drawing. Now we can add the path C in a zigzag way as $v_{n-k} v_n v_{n-k+1} v_{n-1} \dots$, and the only new boundary edge will correspond precisely to $\bar{h}_m h_m$.

The two lemmas above imply clearly the following result.

Theorem 3. *Any tree T different from a star can be drawn on a convex polygon without using consecutive edges of the boundary. Moreover, the minimum number of boundary edges in any drawing of T is equal to the number of terminal nodes of the tree obtained after removing all the terminal nodes of T .*

3 Drawing two trees

We say that the graphs $H_1, \dots, H_k$ can be *packed* into a graph G if G contains subgraphs isomorphic to $H_1, \dots, H_k$ and pairwise edge disjoint.

Observe that packing a family of planar graphs $H_1, \dots, H_k$ into a planar graph G amounts to drawing simultaneously the graphs in the plane without any crossings and without multiple edges. Of course this is trivial if one does not bound the number of vertices of G , but it is a challenging problem if $|V(G)| = \max |V(H_i)|$. If this condition is fulfilled, we say that $H_1, \dots, H_k$ admit a *tight planar packing*. The following lemma justifies the study of drawings on a convex polygon presented in the preceding section.

Lemma 4. *Suppose two graphs H_1 and H_2 both with n vertices can be drawn on a convex n -gon without sharing any edge. Then they admit a tight planar packing.*

Proof. Without loss of generality we can assume that the vertices of the convex polygon lie on the equator of a sphere. Then we can draw the edges of H_1 in one hemisphere and the edges of H_2 in the opposite hemisphere. Since the two drawings were planar and did not share edges, we obtain a simple planar graph drawn on the sphere.

Our first application is to show that one can always draw a given combinatorial tree in the plane together with a spanning path without crossings or multiple edges.

Theorem 5. *Any tree T with n vertices, different from a star, admits a tight planar packing with a path P_n with n vertices.*

Proof. Let $\hat{T}$ the tree obtained from T by removing all the terminal nodes. Let us consider first the case in which $\hat{T}$ is not a single edge joining two vertices, possibility only arising when T is the so called *double-star*.

We know from Section 2 that T can be drawn on a convex polygon $v_1, \dots, v_n$, described counterclockwise, using only sides of the polygon corresponding to the edges $h_i \bar{h}_i$, $i = 1, \dots, m$, where $h_1, \dots, h_m$ are the terminal nodes of $\hat{T}$ and $\bar{h}_1, \dots, \bar{h}_m$ are terminal nodes of T respectively connected to $h_1, \dots, h_m$. Observe that this drawing does not contain edges between $\bar{h}_i$ and h_j or $\bar{h}_j$, for $j \neq i$, because $\bar{h}_i$ is a terminal node of T . Also, there is no edge between h_i and h_j as they are terminal nodes of $\hat{T}$, only adjacent if T were a double-star.

We draw now P_n using all the boundary edges different from the $h_i \bar{h}_i$, whose extremes are connected in a zigzag way by diagonals not shared with T , as proved in the preceding paragraph (see Figure 1).

If T is a double star, having two adjacent vertices with degrees $n - k$ and k , we draw T on the convex polygon using the edges $v_1 v_2, v_1 v_3, \dots, v_1 v_{k+1}$ and $v_n v_{k+1}, v_{n-1} v_{k+1}, \dots, v_{k+3} v_{k+1}, v_{k+2} v_{k+1}$. Now we can draw P_n as $v_{k+1} v_k \dots v_3 v_2 v_{k+2} v_{k+3} \dots v_n v_1$.

As a second application of our technique we show how to draw simultaneously two copies of the same tree.

Theorem 6. *Let T be any tree with n vertices different from a star. Then T and a copy of T admit a tight planar packing.*

Proof. Let P be a convex polygon with vertices $v_1, \dots, v_n$, numbered as they appear counterclockwise on its boundary. According to Theorem 3, T can be drawn on P without using consecutive edges of the boundary. A copy of this drawing of T is now obtained by rotating the first drawing one unity counterclockwise. By construction no two edges on the boundary are shared by these two drawings. On the other hand, no edge $v_i v_j$ with $j \neq i + 1$ can be shared, as this would mean that both edges $v_i v_j$ and $v_{i-1} v_{j-1}$ were present in the first drawing, which is impossible as these two diagonals cross each other.

To conclude this section we mention that a tree with n vertices and maximum degree $n - 2$ admits a tight planar packing with any tree with n vertices different from a star. We omit the proof since it does not introduce new techniques.

4 Related results

In the study of planar packings it makes sense to consider graphs other than trees or the packing of more than two graphs. In this section we present some results about packing cycles and about packing three paths of the same length. We start with the following question: when two cycles of length n admit a tight planar packing, i.e. a packing into a planar graph with n vertices.

Theorem 7. *Two cycles of length n admit a tight planar packing if, and only if, $n = 6$ or $n > 7$.*

Fig. 1. Drawing a tree on a convex polygon (i,ii,iii) and a planar packing of the tree with a path (iv,v,vi).

Proof. If $n = 3, 4$ it is clear that there is no solution. When $n = 5$ the union of two disjoint C_5 is the complete graph K_5 , which of course is not planar.

To show that there is no solution either in the case $n = 7$, we recall that a graph contractible to K_5 is non planar. Suppose G is a planar graph with 7 vertices whose edges are the disjoint union of two cycles C_7 . Then G is 4-regular and the complementary graph $\overline{G}$ is 2-regular, hence equal to $C_3 \cup C_4$ or to C_7 . If $\overline{G} = C_3 \cup C_4$ then clearly G contains $K_{3,3}$ as a subgraph and is not planar. If $\overline{G} = C_7 = v_1 v_2 \cdots v_7$, then G contains a graph contractible to K_5 that has as vertices v_1, v_2, v_4, v_6 and the contraction of $v_3 v_7$, where the path joining v_1 and v_2 goes through v_5 .

Finally, an explicit construction provides a solution in the remaining cases. The constructions for $n \geq 6$ and even, and for $n \geq 9$ and odd are exemplified in Figure 2.

Fig. 2. Tight planar packing of two cycles.

Our next result deals with the packing of a tree and a cycle. Observe that the condition on the maximum degree is necessary.

Theorem 8. *Given a tree T with $n \geq 5$ vertices and maximum degree $\Delta(T) \leq n - 3$, T and a cycle of length n admit a tight planar packing.*

Proof. By induction on n . If $n = 5$ then T is a path and the result is easily checked. Let now T' be a tree with $n + 1$ vertices and $\delta(T') \leq n - 2$. T' always contains a terminal node h such that after removing h the resulting tree T with n vertices has degree at most $n - 3$ (the only exception to this claim is the double-star with 6 vertices, and again the result is easily checked in this case). By induction T and a cycle of length n admit a tight planar packing. To fix ideas, suppose the cycle is drawn as a circle and the edges of T are drawn as internal or external chords.

Let v be the vertex adjacent to h in T' , let u and w be the two neighbors of v in the cycle, and let L be the arc of circle between u and w not containing v .

If v can be connected by an internal chord to a point x in L that is not a vertex of T without creating crossings, then we can put h in x and draw the edge vh to get a packing of T' and a cycle of length $n + 1$. Otherwise there has to be a path in T from u to w through the interior of the circle (this situation is illustrated in Figure 3.) Now we can argue in the same way trying to connect v to a point x' in L by an external chord. If this were not possible there would be also a path from u to w through the exterior of the circle, and T would contain a cycle. This completes the proof.

Fig. 3. Packing a tree and a cycle.

We observe that the above result together with the remark at the end of Section 3 actually provide an alternative proof to Theorem 5. Nevertheless the proof there was constructive instead of the inductive technique used here.

Our last results deal with the packing of three graphs. The proof of the first two is very similar to that of Theorem 7, and is omitted.

Theorem 9. *Three cycles of length $n - 2$ can be packed into a planar graph with n vertices if, and only if, $n = 6$ or $n > 7$.*

Theorem 10. *Three cycles of lengths $n - 3, n - 2$ and $n - 1$ can be packed into a planar graph with n vertices if, and only if, $n = 6$ or $n > 7$.*

An explicit construction for the case $n = 9$ is exemplified in Figure 4. Observe that the union of the three cycles shown in the picture gives rise to a maximal planar graph (a triangulation), showing that the result is best possible. This is also the case in our final result, whose proof is omitted.

Theorem 11. *For all $n \geq 3$, there is a packing of three paths of length $n - 2$ into a planar graph with n vertices.*

Fig. 4. Packing three cycles.

5 Final remarks

In this paper we have presented several results about planar packings of trees and cycles. The main tool is based on drawing trees on a convex polygon with specific properties. We believe this technique might be used to settle other cases, if not completely, of our conjecture:

Conjecture: Any two trees with $n \geq 4$ vertices different from a star admit a tight planar packing.

A case study shows that the conjecture is true for $n \leq 7$.

References

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