Exponential Vector Field Tomography

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Abstract. In the exponential Radon transform in $\mathbb{R}^2$, the integrals of a scalar function f over lines, with exponential weight functions, are determined. In this paper we demonstrate how to define two different kinds of exponential Radon transforms for vector fields in $\mathbb{R}^2$ in a natural way. It is shown that having data from these transforms it is possible to reconstruct the vector field uniquely.

The motivation to study this problem is ultrasound measurements of flows, from which velocity spectra along lines can be determined. The first moment of these can be interpreted by means of one of the exponential vectorial Radon transforms.

1 Introduction

It is well known that a scalar function can be reconstructed from all its line integrals, X-ray transform and from all its integrals over hyperplanes, Radon transform. There exist several different numerical approaches, when only a finite number of integrals are known. See e.g. [4] for details. These transforms have a wide use in computerised tomography. However, sometimes a finer modelling is required. One such example is emission tomography, where the signal detected is not only dependent on the material along the beam, but also on the distance between the source of emission and the detector. In these cases it is often natural to study the exponential Radon transform.

The theory of how to define X-ray and Radon transforms for vector fields and reconstruction from these transforms has been addressed in among others [3], [1], [7], [5], [6], [8], [9] and [10]. Given one X-ray like transform the solenoid (rotational) part of the flow can be found. To reconstruct the complete flow another Radon like transform is needed, which gives the potential part. From the solenoid part and potential part it is possible to reconstruct the field using Helmholtz decomposition.

One situation where the vectorial X-ray transform appears is when making Doppler measurements on flows using continuous ultrasound, see [3], [2] and [7]. When a continuous ultrasound beam bounces against a particle a signal is backscattered. This signal obtains a Doppler frequency shift, proportional to the speed of the particle along the beam. This phenomenon appears for each particle along the beam, giving rise to a Doppler velocity spectra. Using only part of the information in this spectra, namely the first moment, the vectorial X-ray transform appears, see [10]. One problem that has been ignored earlier, when studying this problem, is that the Doppler signals originate from points at different distances from the ultrasound detector, and due to ‘shadowing’, signals originating from points further away from the detector will have lower amplitude than signals originating from points closer to the detector.

The problem of defining transforms describing this problem and reconstructing from them is addressed in this paper, in the case of $\mathbb{R}^2$, assuming the shadowing to be described by an exponential decay of the signal amplitude. It is shown that it is indeed possible to reconstruct the vector field having both the exponential vectorial Radon transform and the exponential normal flow Radon transform. However in this case each of the transforms appear for each of the inversions of the solenoid and potential part.

2 The non-exponential case

We begin by recalling some classical definitions. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions.

Definition 1. Let $Z = S^{n-1} \times \mathbb{R}$. For $f \in \mathcal{S}(\mathbb{R}^n)$, the *Radon transform* is

$$Z \ni (\theta, s) \mapsto \mathcal{R}f(\theta, s) = \int_{\theta^\perp} f(s\theta + y) dy = \int_{x \cdot \theta = s} f(x) dx. \quad (1)$$

Let $T = \{(\theta, x) \mid \theta \in S^{n-1}, x \in \theta^\perp\}$. The *X-ray transform* is

$$T \ni (\theta, x) \mapsto \mathcal{P}f(\theta, x) = \int_{-\infty}^{\infty} f(x + t\theta) dt. \quad (2)$$

In the Radon transform, integrations are over hyperplanes, whereas in the X-ray transform integrations are over lines. In $\mathbb{R}^2$ they differ only on parametrisation.

As mentioned in the introduction, the following transform describes information extractable from ultrasound Doppler measurements of flows.

Definition 2. For $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^n)$, the *vectorial X-ray transform* is

$$T \ni (\theta, x) \mapsto \mathcal{V}f(\theta, x) = \mathcal{P}(f \cdot \theta)(\theta, x) = \int_{\mathbb{R}^n} f(x + t\theta) \cdot \theta dt.$$

This transform gives the solenoid part of the flow. If the field is not divergence free, as for $n = 2$, when regarding slices of 3-D flows, another transform is needed.

Definition 3. For $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^n)$, the *normal flow Radon transform* is

$$Z \ni (\theta, s) \mapsto \mathcal{V}^\perp f(\theta, s) = \mathcal{R}(f \cdot \theta)(\theta, s) = \int_{\theta^\perp} f(s\theta + y) \cdot \theta dy.$$

For $n = 2$, it is proven in [1] that this transform together with the vectorial X-ray transform is enough to determine the field uniquely. This transform is an ‘inner product probe’ transform as in [6].

Given both the vectorial X-ray transform and the normal flow Radon transform it is possible to reconstruct the vector field uniquely. To see this, we first need a Helmholtz

decomposition theorem. Let $\mathcal{D}'$ be the set of distributions and $\mathcal{E}'$ the set of distributions with compact support. Let $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathcal{E}'(\mathbb{R}^n, \mathbb{R}^n)$. For $I = (i_1, i_2, i_3)$, let

$$\text{curl}_I \mathbf{f}_I = \begin{vmatrix} \hat{e}_{i_1} & \hat{e}_{i_2} & \hat{e}_{i_3} \\ \frac{\partial}{\partial x_{i_1}} & \frac{\partial}{\partial x_{i_2}} & \frac{\partial}{\partial x_{i_3}} \\ f_{i_1} & f_{i_2} & f_{i_3} \end{vmatrix}.$$

Let $\sum_I'$ mean the sum over all $I = (i_1, i_2, i_3)$, such that $1 \leq i_1 < i_2 < i_3 \leq n$. Let E

be the fundamental solution of the Laplace operator. In $\mathbb{R}^2$, let $\text{curl}_\perp \mathbf{f} = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}$.

The proof of the following Helmholtz decomposition theorem can be found in [10].

Theorem 4. Let $\mathbf{q} \in \mathcal{E}'(\mathbb{R}^n)$ and let $\mathbf{r}_{i,j} \in \mathcal{E}'(\mathbb{R}^n, \mathbb{R}^n)$, $i, j = 1 \dots n$, be given.

– For $n \geq 3$: If the $\mathbf{r}_{i,j}$:s obey the Jacobi identities

$$\frac{\partial}{\partial x_k} r_{i,j} + \frac{\partial}{\partial x_j} r_{k,i} + \frac{\partial}{\partial x_i} r_{j,k} = 0, \quad 1 \leq i < j < k \leq n,$$

then there is a unique $\mathbf{f} \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n)$ such that $\mathbf{f}$ is a harmonic function outside $\text{supp } \mathbf{p} \cup \text{supp } \mathbf{q}$, $\mathbf{f} \rightarrow 0$ when $|x| \rightarrow \infty$, and

$$\text{div } \mathbf{f} = \mathbf{q}, \quad \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = r_{i,j} \quad i, j = 1, \dots, n, \quad i \neq j.$$

The field $\mathbf{f}$ is given by

$$\mathbf{f} = E * \left(\frac{1}{n-2} \sum_I' \text{curl}_I (\hat{e}_{i_1} r_{i_2, i_3} + \hat{e}_{i_2} r_{i_3, i_1} + \hat{e}_{i_3} r_{i_1, i_2}) + \nabla \mathbf{q} \right).$$

– For $n = 2$: If $r_{1,2} = -r_{2,1}$, then there is a unique $\mathbf{f} \in \mathcal{D}'(\mathbb{R}^2, \mathbb{R}^2)$ such that $\mathbf{f}$ is harmonic outside $\text{supp } \mathbf{p} \cup \text{supp } \mathbf{q}$, $\mathbf{f} \rightarrow 0$ when $|x| \rightarrow \infty$, and

$$\text{div } \mathbf{f} = \mathbf{q}, \quad \text{curl}_\perp \mathbf{f} = r_{1,2}.$$

The field $\mathbf{f}$ is given by

$$\mathbf{f} = E * \left(\frac{\partial \mathbf{q}}{\partial x_1} - \frac{\partial \mathbf{p}}{\partial x_2}, \frac{\partial \mathbf{p}}{\partial x_1} + \frac{\partial \mathbf{q}}{\partial x_2} \right).$$

In among others [3] and [10] the following theorem was proved.

Theorem 5. If $\mathbf{f} \in \mathcal{S}(\mathbb{R}^2, \mathbb{R}^2)$, then

$$\frac{\partial}{\partial s} \mathcal{V} \mathbf{f}(\theta, s) = \mathcal{P} \text{curl}_\perp \mathbf{f}(\theta, s). \quad (3)$$

Consider a vector field $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^n)$. For any i, j , $i \neq j$, the vectorial X-ray transform is used to compute the quantity $r_{i,j} = \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}$, by considering the lines in the subspace where all coordinates except x_i, x_j are kept constant.

Doing this for all possible choices of such constants, and for all $i \neq j$, and making a 2-D reconstruction according to Theorem 5, Theorem 4 determines the solenoid part of the vector field.

Especially this can be used in $\mathbb{R}^3$ to find the 3-D curl. In [3] it is shown how knowledge about the curl can be used to recover the field uniquely in the case of Neumann conditions at the boundary of a bounded region. In [5] a similar situation is regarded.

Theorem 6. *If $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^n)$, then*

$$\frac{\partial}{\partial s} \mathcal{V}^\perp f(\Theta, s) = \mathcal{R} \operatorname{div} f(\Theta, s). \quad (4)$$

This theorem can be found in [8] and in [6]. Once the divergence is known the potential part of the vector field is obtained from Theorem 4.

To summarise, if the values of both the vectorial X-ray transform and normal flow Radon transform are known, Theorem 5 and Theorem 6 combined with Theorem 4 gives a way of reconstructing f uniquely.

3 The exponential case

As discussed earlier the vectorial X-ray transform models e.g. ultrasound Doppler measurements. In the following we will model the case where we have ‘shadowing’ of exponential type.

First the theory for the *scalar* exponential Radon transform will be discussed, as it is the basis also for the theory developed for the vectorial case.

Definition 7. For $f \in C_0^\infty(\mathbb{R}^2)$ and $\mu \in \mathbb{R}$, the *exponential Radon transform* is

$$Z \ni (\Theta, s) \mapsto \mathcal{T}_\mu f(\Theta, s) = \int_{x \cdot \Theta = s} e^{\mu x \cdot \Theta^\perp} f(x) dx = \int_{-\infty}^{\infty} e^{\mu t} f(s\Theta + t\Theta^\perp) dt.$$

For $g \in C_0^\infty(Z)$, the *reciprocal exponential Radon transform* is

$$\mathbb{R}^2 \ni x \mapsto \mathcal{T}_\mu^\# g(x) = \int_{S^1} e^{\mu x \cdot \Theta^\perp} g(\Theta, x \cdot \Theta) d\Theta.$$

Let $\hat{f}$ be the Fourier transform of f . The Fourier slice theorem reads (see [4]):

Theorem 8. *If $f \in C_0^\infty(\mathbb{R}^2)$, then $(\mathcal{T}_\mu f)^\wedge(\Theta, \sigma) = (2\pi)^{1/2} \hat{f}(\sigma\Theta + i\mu\Theta^\perp)$.*

As can be seen from the theorem, the 1-dimensional Fourier transform of $\mathcal{T}_\mu$ gives the 2-dimensional Fourier transform of f on a two dimensional surface of $\mathbb{C}^2$. It is not easy to reconstruct f from these data. It is also possible to state an analogue of the filtered back projection theorem, [4].

Theorem 9. *Let $f \in C_0^\infty(\mathbb{R}^2)$ and $g \in C_0^\infty(Z)$. Then*

$$\mathcal{T}_{-\mu}^\# g * f = \mathcal{T}_{-\mu}^\# (g * \mathcal{T}_\mu f).$$

By choosing $\mathcal{T}_{-\mu}^{\#}g$ as an approximate δ -function it is possible to reconstruct a regularisation of f , by a so called ‘filtered back projection’ algorithm. g is often called a filter or a tomography kernel. There is also an exact reconstruction formula for the exponential Radon transform, [4]. However, now we will turn vector fields. Set $\mathbf{f} = (f_1, f_2)$.

Definition 10. For $\mathbf{f} \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and $\mu \in \mathbb{R}$, the *exponential vectorial X-ray transform* is

$$Z \ni (\Theta, s) \mapsto \mathcal{V}_\mu \mathbf{f}(\Theta, s) = \mathcal{T}_\mu(\mathbf{f} \cdot \Theta^\perp)(\Theta, s) = \int_{-\infty}^{\infty} e^{\mu t} \mathbf{f}(s\Theta + t\Theta^\perp) \cdot \Theta^\perp dt.$$

The *exponential normal flow Radon transform* is

$$Z \ni (\Theta, s) \mapsto \mathcal{V}_\mu^\perp \mathbf{f}(\Theta, s) = \mathcal{T}_\mu(\mathbf{f} \cdot \Theta)(\Theta, s) = \int_{-\infty}^{\infty} e^{\mu t} \mathbf{f}(s\Theta + t\Theta^\perp) \cdot \Theta dt.$$

For $\mu = 0$ these coincide with the transforms dealt with in Section 2. A simple use of the Fourier slice theorem, Theorem 8, gives

$$(\mathcal{V}_\mu \mathbf{f})^\wedge(\Theta, \sigma) = (2\pi)^{1/2} \hat{\mathbf{f}}(\sigma\Theta + i\mu\Theta^\perp) \cdot \Theta^\perp, \quad (5)$$

$$(\mathcal{V}_\mu^\perp \mathbf{f})^\wedge(\Theta, \sigma) = (2\pi)^{1/2} \hat{\mathbf{f}}(\sigma\Theta + i\mu\Theta^\perp) \cdot \Theta. \quad (6)$$

We will now demonstrate how the transforms defined above can be inverted. We have, from (5) and (6), with $\Theta = (\theta_1, \theta_2)$,

$$\begin{aligned} & i\sigma(\mathcal{V}_\mu \mathbf{f})^\wedge(\Theta, \sigma) + \mu(\mathcal{V}_\mu^\perp \mathbf{f})^\wedge(\Theta, \sigma) = \\ & = (2\pi)^{1/2} \left(\hat{\mathbf{f}}(\sigma\Theta + i\mu\Theta^\perp) \cdot i\sigma\Theta^\perp + \hat{\mathbf{f}}(\sigma\Theta + i\mu\Theta^\perp) \cdot i\mu\Theta \right) = \\ & = (2\pi)^{1/2} \left(\hat{f}_1(\sigma\Theta + i\mu\Theta^\perp)(-i\sigma\theta_2) + \hat{f}_2(\sigma\Theta + i\mu\Theta^\perp)i\sigma\theta_1 \right) + \\ & \quad + (2\pi)^{1/2} \left(\hat{f}_1(\sigma\Theta + i\mu\Theta^\perp)\mu\theta_1 + \hat{f}_2(\sigma\Theta + i\mu\Theta^\perp)\mu\theta_2 \right). \end{aligned}$$

Introducing the coordinates $\xi_1 = \sigma\theta_1 - i\mu\theta_2$ and $\xi_2 = \sigma\theta_2 + i\mu\theta_1$, this can be rewritten

$$i\sigma(\mathcal{V}_\mu \mathbf{f})^\wedge(\Theta, \sigma) + \mu(\mathcal{V}_\mu^\perp \mathbf{f})^\wedge(\Theta, \sigma) = (2\pi)^{1/2} \left(-\hat{f}_1(\xi_1, \xi_2)i\xi_2 + \hat{f}_2(\xi_1, \xi_2)i\xi_1 \right).$$

Rewriting the right-hand side as $\text{curl}_\perp$, by means of the Fourier slice theorem for the scalar exponential Radon transform, Theorem 8, we obtain

$$i\sigma(\mathcal{V}_\mu \mathbf{f})^\wedge(\Theta, \sigma) + \mu(\mathcal{V}_\mu^\perp \mathbf{f})^\wedge(\Theta, \sigma) = (\mathcal{T}_\mu \text{curl}_\perp \mathbf{f})^\wedge(\Theta, \sigma).$$

In the same manner it follows that

$$-\mu(\mathcal{V}_\mu \mathbf{f})^\wedge(\Theta, \sigma) + i\sigma(\mathcal{V}_\mu^\perp \mathbf{f})^\wedge(\Theta, \sigma) = (\mathcal{T}_\mu \text{div } \mathbf{f})^\wedge(\Theta, \sigma).$$

An inverse Fourier transform yields

Theorem 11. *If $f \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and $\mu \in \mathbb{R}$, then*

$$\mathcal{T}_\mu \operatorname{curl}_\perp f(\Theta, s) = \frac{\partial}{\partial s} \mathcal{V}_\mu f(\Theta, s) + \mu \mathcal{V}_\mu^\perp f(\Theta, s), \quad (7)$$

$$\mathcal{T}_\mu \operatorname{div} f(\Theta, s) = -\mu \mathcal{V}_\mu f(\Theta, s) + \frac{\partial}{\partial s} \mathcal{V}_\mu^\perp f(\Theta, s). \quad (8)$$

These two formulas can be inserted into the exact reconstruction formula for the scalar exponential Radon transform, to give $\operatorname{curl}_\perp f$ and $\operatorname{div} f$. Then f_1 and f_2 can be determined via Helmholtz decomposition, proving that the exponential transforms are invertible, but also that in each of the reconstructions both transforms appear. For the ordinary vectorial transforms, $\mu = 0$, they appear separately. It is also possible to formulate filtered back projection algorithms. Insert (7) into the formula of Theorem 9, $\mathcal{T}_{-\mu}^\# g * f = \mathcal{T}_{-\mu}^\# (g * \mathcal{T}_\mu f)$, and use the rules for derivatives of convolutions. Analogously for $\operatorname{div} f$.

Theorem 12. *If $f \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$, $g \in C_0^\infty(Z)$ and $\mu \in \mathbb{R}$, then*

$$\mathcal{T}_{-\mu}^\# g * \operatorname{curl}_\perp f = \mathcal{T}_{-\mu}^\# \left(\frac{\partial g}{\partial s} * \mathcal{V}_\mu f + \mu g * \mathcal{V}_\mu^\perp f \right), \quad (9)$$

$$\mathcal{T}_{-\mu}^\# g * \operatorname{div} f = \mathcal{T}_{-\mu}^\# \left(-\mu g * \mathcal{V}_\mu f + \frac{\partial g}{\partial s} * \mathcal{V}_\mu^\perp f \right). \quad (10)$$

The computational complexity for the reconstruction will thus be the same as for ordinary filtered back projection up to a constant.

4 Simulations and experiments

Choosing $\hat{w}(\sigma)$ as $|\sigma|$ for $|\mu| < |\sigma| < b$, we have $\mathcal{T}_{-\mu}^\# w \rightarrow 2(2\pi)^{3/2} \delta$ as $b \rightarrow \infty$, see [4]. By varying b we can adjust the degree of regularisation. A big b gives many fine details but high sensitivity to noise, a small b vice versa.

We have simulated flows in a straight pipe, with a parabolic flow speed profile. The direction of the flow is along the pipe starting in the upper part of the pipe and then going downwards. $\mathcal{V}_\mu f$ and $\mathcal{V}_\mu^\perp f$ were computed for $\mu = -0.1$ on MATLAB. $\operatorname{curl}_\perp$ is a rotationally symmetric operator, so we expect it to look like the first derivative of the one-dimensional parabolic function describing the flow speed variation across the pipe, that is to be N-shaped. We expect to find a ‘N-ribbon’ going along the pipe. The divergence should be positive at sources and negative at sinks. We expect a source at the upper end of the pipe and a sink at the lower end. In the figures the value at each point is represented with a gray level, where white means a positive and black a negative one.

Usually only $\mathcal{V}_\mu$ is available. Reconstruction will not be possible straight away. However, if we assume that there is no divergence in a region, then

$$\mathcal{T}_{-\mu}^\# \left(\frac{\partial g}{\partial s} * \mathcal{V}_\mu^\perp f \right) = \mathcal{T}_{-\mu}^\# (-\mu g * \mathcal{V}_\mu f)$$

Fig. 1. To the left: Amplitude of original flow. Middle: Reconstruction of $\text{curl}_\perp$ using Theorem 12. To the right: Reconstruction of div using Theorem 12

in that region. After replacing g by $\frac{\partial g}{\partial s}$ and inserting the result into (9), we receive

$$(\mathcal{T}_{-\mu}^\# \frac{\partial g}{\partial s}) * \text{curl}_\perp f = \mathcal{T}_{-\mu}^\# \left(\left(\frac{\partial^2 g}{\partial s^2} + \mu^2 g \right) * \mathcal{V}_\mu f \right).$$

Let $\frac{\partial g}{\partial s} = w$, then $\frac{\partial^2}{\partial s^2} g + \mu^2 g =$

$$2 \frac{1}{s^3} \left(s^2 (b^2 - \mu^2) \cos(sb) - 2(sb \sin(sb) + \cos(sb) - \cos(s|\mu|) - s|\mu| \sin(s|\mu|)) \right).$$

This is used as tomography kernel.

A reconstruction of $\text{curl}_\perp$ under the assumption of no divergence, when using $\mu = -0.1$ in the algorithm above, is shown. In the central parts of the pipe, where we expect no divergence, this should be a good assumption. To demonstrate the effect of attenuation, a reconstruction of $\text{curl}_\perp$ by an algorithm where $\mu = 0$ is included.

Fig. 2. To the left: Amplitude of flow-speed. Middle: Reconstruction of $\text{curl}_\perp$ from simulation, using $\mu = -0.1$. To the right: Reconstruction if μ is erroneously set to 0.

Presently we are trying to measure flows in plastic tubes submerged in a water bath. The measurements, performed by the Department of Electrical Measurements, Lund Institute of Technology, were done on a circular phantom, see [2]. The plastic pipe was chosen rotationally symmetric in order to avoid unfavoured directions, due to reflections. The ultimate goal is that, in future, these methods may give rise to new ways of detecting e.g. cancer tumours in the female breast.

Fig. 3. Experiment. 3-D versions of the reconstructions. To the left: $\mu = 0$ Middle: $\mu = -0.05$ To the right: $\mu = -0.12$

In the reconstructions we have assumed that there is no divergence. As can be seen, there is a small, but visible, difference if the reconstruction is made with or without attenuation. As the correct flow is not known, the error has to be estimated subjectively. Ideally the correct flow should be a 'N-ribbon', bent to a circle. The reconstruction with $\mu = 0$ is 'bumpier' than the one with $\mu = 0.05$, precisely as demonstrated with the simulations of the straight pipe above. When μ is too large the reconstruction again becomes 'bumpier'.

5 Conclusions

In this paper it has been shown that for $\mathbb{R}^2$ the vectorial X-ray transform and the normal flow Radon transforms can be extended to exponential transforms. Inversion formulas and filtered back projection formulas have been developed. Simulations show that these filtered back projection formulas work well. Real data from flow measurements confirm that the reconstruction result can be improved using these new methods.

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