Series-Parallel Planar Ordered Sets Have Pagenumber Two

(Extended Abstract)

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Abstract. *The pagenumber of a series-parallel planar P is at most two.* We present an $O(n^3)$ algorithm to construct a two-page embedding in the case that it is a lattice. One consequence of independent interest, is a characterization of series-parallel planar ordered sets.

1 Introduction

A *book embedding* of a graph G consists of an embedding of its nodes along the spine of a book (i.e., a linear ordering of the nodes), and an embedding of its edges on pages so that edges embedded on the same page do not intersect. In a *book embedding* for an ordered set P the vertices of P on the spine form a linear extension (a total order $L = \{x_1 < x_2 < \cdots < x_n\}$ of the elements of P is a *linear extension* if $x < y$ in L whenever $x < y$ in P).

We say a *covers* b (or b *covered by* a) in the ordered set P, and write $a \succ b$ (or $b \prec a$), if whenever $a > c \geq b$ then $c = b$. Also, we say a is an *upper cover* of b, or b is a *lower cover* of a, or (a, b) is an edge in P. We say a is a *minimal* (respectively, *maximal*) element of P if a has no lower covers (respectively, a has no upper covers). We denote the set of all minimals (respectively, maximals) of P, $min(P)$ (respectively, $max(P)$). The *covering graph* of P, $cov(P)$, is the graph whose vertices are the elements of P, and the pair $\{a, b\}$ forms an edge in $cov(P)$ if $a \succ b$ or $a \prec b$. It is possible to orient $cov(P)$ in such a way the y-coordinate of a is less than the y-coordinate of b if $a \prec b$ and the edge (a, b)

does not pass through any other element of P . We call such drawing an *upward drawing* of P.

The *pagenumber* in both cases ($page(G)$, respectively $page(P)$) is the minimum number of pages needed taken over all linear layouts for graphs and all linear extensions for an ordered set. For instance, $page(P) = 2$ for the ordered set illustrated in Figure 1, while $page(cov(P)) = 1$. On the other hand the planar lattice in Figure 2 required three pages (this example is due to J. Czyzowicz [7]).

The pagenumber was first defined for graphs by Bernhart and Kainen [1], who conjectured that planar graphs may require an arbitrary large number of pages. In a series of attempts, it was finally established by Yannakakis [11], that $page(G) \leq 4$ for every planar graph G, and this upper bound is achieved. Fraysseix, Mendez and Pach [4] have shown that the pagenumber of any planar graph with quadrilateral faces is at most two.

The page number for ordered sets has been introduced by Nowakowski and Parker [7], who show that $page(P) = 1$ if and only if $cov(P)$ is a forest. Also, they derive a general lower bound on the page number of ordered sets and upper bounds for special classes of ordered sets. Hung [3] shows that there exists a 48-element planar ordered set which needs four pages (see Figure 3). Moreover, no planar ordered set with pagenumber five is known. Sysło [9] provides a lower bound on the page number in terms of its bump number. He also shows that, $page(P) \leq 2$ if the jump number of P is one. Ordered sets with jump number two can have an arbitrarily large page number. Later, Heath and Pemmaraju [8] gave a sequence of ordered sets each with planar covering graph and with unbounded page number. Computationally, we recently proved that finding the minimum number of pages required for a fixed linear extension of an ordered set is NP-complete.

In section 2 we study the structure of series-parallel planar lattices. In section 3 we will construct, for a series-parallel planar lattice P, an $O(n^3)$ two-page algorithm where n is the number of the elements of P. In section 4 we continue the study of the structure of series-parallel planar ordered sets. In section 5 we exploit the fact that the completion $\overline{P}$ of a series-parallel planar ordered set P is itself a series-parallel planar lattice. We use the result in section 3 to obtain a two-page linear extension $\overline{L}$ of $\overline{P}$, which we transfer to a two-page linear extension of P. In section 6 we give three open problems related to the pagenumber problem.

2 Structure of series-parallel planar lattices

The *linear sum* $P \oplus Q$ of the two disjoint ordered set P, Q is an ordered set on $P \cup Q$, that is, $a \leq b$ if

1. $a \leq b$ in P, or 2. $a \leq b$ in Q, or 3. $a \in P$ and $b \in Q$.

If we eliminate the third condition of the definition of linear sum, we will have the *disjoint sum* $P + Q$ of P, Q.

An ordered set P is *series-parallel* if P can be constructed from singletons using only the constructions of disjoint sum $+$ and linear sum $\oplus$. In other words, P can be decomposed into singletons using only disjoint sum and linear sum. For instance, the the series-parallel lattice illustrated in Figure 4 can be decomposed into

$$1\oplus(((2+6)\oplus3\oplus(4+(7\oplus(8+(10\oplus11)+12+13)\oplus9)))+(14\oplus(15+17)\oplus16))\oplus5$$

For $a \neq b$ in the ordered set P we say a is *comparable* to b if either $a < b$ or $a > b$. Otherwise, a is *noncomparable* to b, write $a \parallel b$. An *antichain* is a subset A of an ordered set P such that any two distinct elements of A are noncomparable. Dually, a *chain* of P is a subset C of P where, each pair of C are comparable.

A four-element subset $\{a, b, c, d\}$ of an ordered set P forms an $\mathbf{N}$ if the only comparabilities among them in P are $a < c$, $b < c$ and $b < d$. It is known that an ordered set is a series-parallel if and only if it contains no such $\mathbf{N}$ [10].

Fix a planar embedding of P, and let $C = \{x_1 < x_2 < ... < x_n\}$ be the left boundary chain. For each $x \in P - C$ define the interval $I(x) = (x_i, x_j)$, where

$$x_i = \max_{1\leq k<n} \{x > x_k\} \qquad\qquad x_j = \min_{1<k\leq n} \{x < x_k\}$$

Of course, $j \geq i + 1$. Notice that, $j > i + 1$ because if $j = i + 1$ then the edge (x_{i+1}, x_i) will not be an essential edge. (An edge (a, b) is not essential if there is c such that $a < c < b$.)

Notice that, every pair of these intervals is either disjoint or one contains the other. Hence the set of intervals ordered by inclusion is a forest (an ordered set P is a forest if the graph $cov(P)$ is a forest). For y, $z \in P - C$, say $y \sim z$ if $I(y) = I(z)$. It is clear that this relation is an equivalence relation. Call the equivalence classes *components.*

For example, the components of the series-parallel order in Figure 4 are:
$C_1 = \{7, 8, 9, 10, 11, 12, 13\}$ which corresponds to the interval (3,5);
$C_2 = \{6\}$ which corresponds to the interval (1,3);
$C_3 = \{14, 15, 16, 17\}$ which corresponds to the interval (1,5).
The forest obtained by ordering the intervals by inclusion is shown in Figure 6. We can show that there are no edges between the components.

Here are a few elementary terms. Fix a lattice P and fix a planar upward drawing of it. For noncomparable element $a, b \in P$ such that $a \succ c$ and $b \succ c$, we say a is *left* of b if any horizontal segment (moving from left to right) which cuts both edges, always cuts the edge (c, a) before the edge (c, b). For arbitrary noncomparable elements a and b $(a \parallel b)$ in P say that a is *left* of b, denoted $a\lambda b$, if a' is left of b', where $a \geq a' \succ inf(\{a, b\})$ and $b \geq b' \succ inf(\{a, b\})$. An element a, which does not belong to the maximal chain C is left of C if there is $b \in C$ such that $a\lambda b$. In fact, a is left of b if a is left to any maximal chain containing b. (Of course, all of these ideas are ambidextrous. If a is left of b then b is right of a, etc.)(For details see [5].)

Once equipped with the equality relation, λ becomes an order relation on P, denoted P_λ. (This result is due to J. Zilber see [2] page 32, ex. 7(c).) For example, the ordered set in Figure 5 is P_λ where P is the planar lattice in Figure 4.

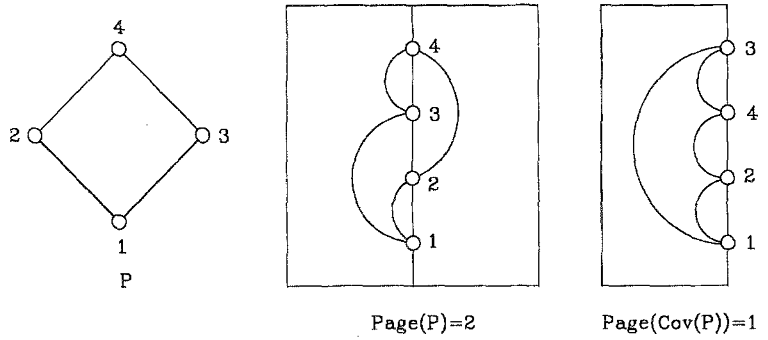

Figure 1

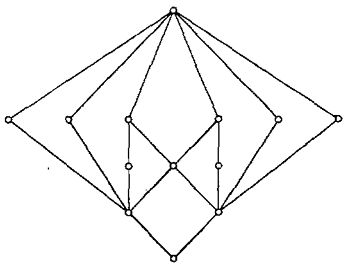

Figure 2

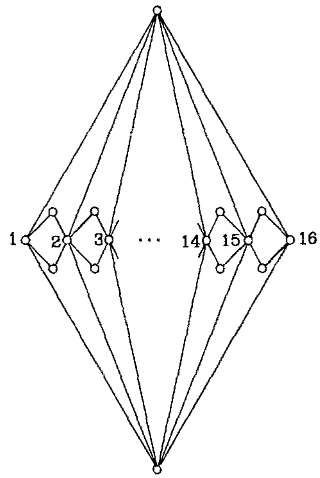

Figure 3

For a series-parallel planar lattice P, fix a planar upward drawing of P, and define the sequence of *peels* of P as follows:

$L_0 = \{x \in P : x$ belongs to the left boundary$\}$.

$L_1 = \{x \in P - L_0 :$ if y lies to the left of x, then $y \in L_0\}$.

$L_2 = \{x \in P - (L_0 \cup L_1) :$ if y lies to the left of x, then$y \in L_0 \cup L_1\}$.

$\vdots$

$L_t = \{x \in P - (L_0 \cup L_1 \cup \cdots \cup L_{t-1}) :$ if y lies to the left of x, then $y \in L_0 \cup L_1 \cup \cdots \cup L_{t-1}\}$.

We call any L_i a *peel* of P. Actually, the peels of a planar lattice P are the levels of P_λ, where P_λ the underlying set P ordered by λ.

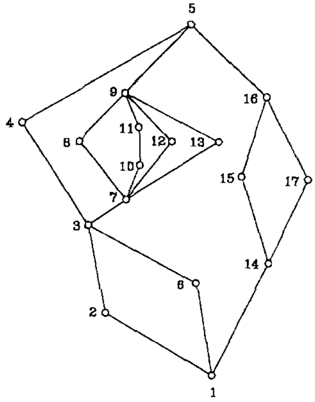

Figure 4

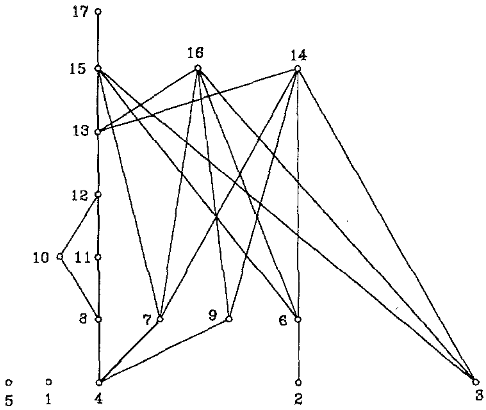

Figure 5

Thus, $L_i = \min(P_\lambda - (\bigcup_{j=0}^{i-1} L_j, 0 \le j \le t)$

Of course, t is equal the height of P_λ, where the height of an ordered set is less one than the maximum number of elements of a chain.

For example, in the series-parallel ordered set P with respect to the upward drawing shown in Figure 4

$L_0 = \{1,2,3,4,5\}$	$L_1 = \{6,7,8,9\}$	$L_2 = \{10,11\}$	$L_3 = \{12\}$
$L_4 = \{13\}$	$L_5 = \{14,15,16\}$	$L_6 = \{17\}$	

Lemma 1 *Let P be a series-parallel planar lattice. If $0 \le i \le t$, then*

1. *for any $x \in L_i, i > 0$, there exist $y \in L_{i-1}$ such that x lies to the right of y,*

2. *the peel L_i forms a chain,*

3. *the number of peels equals width(P)-1 (width(P) is the maximum size of antichain in P).*

Call a chain C in P is *saturated* if all of its covering relations, are covering relations in P. Each chain decomposes into its (maximal) saturated chains.

In a series-parallel planar lattice P each peel L_i can be decomposed into maximal saturated subchains $C_{i1}, C_{i2}, \ldots, C_{in_i}$ called the *clamped* chains for P.

For a clamped chain $C_{ij} \in L_i, i \ge 1$ define:

$$l(C_{ij}) = \{y \in L_0 \cup L_1 \cup \cdots \cup L_{i-1} : inf(C_{ij}) \succ y\}$$
$$u(C_{ij}) = \{y \in L_0 \cup L_1 \cup \cdots \cup L_{i-1} : sup(C_{ij}) \prec y\}$$

For example the table below shows the clamped chains in the series parallel planar lattice in Figure 4.

Clamped chain C_{ij}	$l(C_{ij})$	$u(C_{ij})$
$C_0 = \{1,2,3,4,5\}$	–	–
$C_{11} = \{6\}$	1	3
$C_{12} = \{7,8,9\}$	3	5
$C_{21} = \{10,11\}$	7	9
$C_{31} = \{12\}$	7	11
$C_{41} = \{13\}$	7	9
$C_{51} = \{14,15,16\}$	1	5
$C_{61} = \{17\}$	14	16

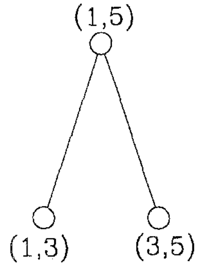

Figure 6

Lemma 2 *Let C_{ij} be a clamped chain in a series-parallel planar lattice P.*

1. *Each $l(C_{ij})$ and $u(C_{ij})$ is unique.*

2. *Each $x \in C_{ij} - \{infC_{ij}, supC_{ij}\}$ has neither lower covers nor upper covers in $L_0 \cup L_1 \cup \cdots \cup L_{i-1}$. Also, if $infC_{ij} \neq supC_{ij}$ then $infC_{ij}$ (respectively, $sup(C_{ij})$) has no lower (respectively, upper) covers in $L_0 \cup L_1 \cup \cdots \cup L_{i-1}$,*

3. *If $u(C_{ij}) \in C_{km}$, then $l(C_{ij}) \in C_{km} \cup \{l(C_{km})\}$.*

4. *$infC_{ij}$ (respectively, $supC_{ij}$) has a unique lower (respectively, upper) cover in P.*

3 Two pages are enough

In this section we will give an $O(n^3)$ two-page algorithm for a series-parallel planar lattice P, where n is the number of elements of P.

To obtain a two-page linear extension of a series-parallel planar lattice P

(i) Fix a planar upward drawing for P.

(ii) List the clamped chains of P in the following order $C_0, C_{11}, C_{12}, \cdots, C_{1n_1}, C_{21}, C_{22}, \cdots, C_{2n_2}, \cdots, C_{w1}, C_{22}, \cdots, C_{wn_w}$. We will process chain by chain according to the above order.

(iii) Put C_0 on the spine of the book. Draw the bottom edge on the right page and draw all other edges on the left page.

(iv) Suppose two pages are enough up to C_{ij-1}. For C_{ij} put all the elements of C_{ij} right below $u(C_{ij})$. Draw the edge $(inf(C_{ij}), l(C_{ij}))$ on the right page and draw all C_{ij} edges and the edge $(u(C_{ij}), sup(C_{ij}))$ on the left page.

Call this algorithm the *two-page* algorithm.

Figure 7, illustrates the steps of the two-page algorithm applied on the series-parallel planar lattice P in Figure 4.

A *greedy linear extension* of an ordered set P is a linear extension $x_1 < x_2 < \cdots < x_n$ of P such that $x_1 \in min(P)$ and, for $i \geq 1, x_{i+1} \in min(P - \{x_1, x_2, \cdots, x_i\})$ and, if possible, $x_{j+1} > x_j$. Thus, a greedy linear extension obtained by following "the rule climb as heigh as you can".

A *left greedy linear extension* of P is that greedy linear extension whose the ith element x_{i+1} is the (unique) left-most element belonging to $min(P - \{x_1, x_2, \cdots, x_i\})$

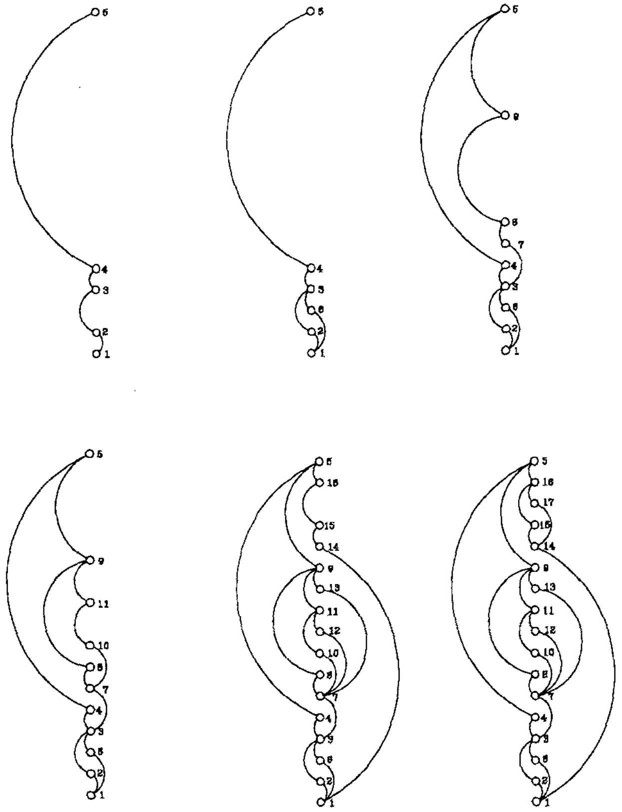

Figure 7

Lemma 3 *Let P be a series-parallel planar lattice. If L is the permutation obtained by the two-page algorithm, then*
1. L is a linear extension of P,
2.if $x \parallel y$ in P, and y lies to the left of x, then $y < x$ in L. (i.e., L is a left greedy linear extension).

Theorem 4 *The two-page algorithm for an n-element series-parallel planar lattice produces a two-page linear extension L in $O(n^3)$ time.*

For the complexity, we can find the peel C_0 by checking for each $x \in P$ if there is $y \in P - \{x\}$ such that $y \parallel x$ and y lies to the left of x. Thus, we need at most n^3 comparison operations to obtain the peels of P. To obtain the clamped chains of a certain peel we need first to sort it in $O(n \log n)$ comparisons, then determine the covering relations in this peel and that can be done in $O(n-1)$ comparisons.

Therefore, we can find all clamped chains in $O(n^2 \log n)$ comparisons. For each clamped chain C_{ij} we can find $u(C_{ij})$ by find the element in L_{i-1} which covers $inf(C_{ij})$ and this can be done in $O(n)$ comparisons. Thus, we can find $u(C_{ij})$ and $l(C_{ij})$ for all clamped chains C_{ij} in $O(n^2)$ comparisons. For the distribution of the edges among the two pages we process each edge just one time; thus, we can decide the page for each edge in $O(n^2)$ comparisons. Thus, the whole algorithm can be done in $O(n^3)$ comparisons.

4 Structure of series-parallel planar ordered sets

The completion of an ordered set P is the smallest lattice $\overline{P}$ contains P as suborder. Notice that $\overline{P}$ exists and called MacNeille completion (cf. [6].)

First, we will show that the completion $\overline{P}$ of series-parallel ordered set is series-parallel planar lattice.

The question may arise now whether we can transfer the two-page linear extension $\overline{L}$ of $\overline{P}$ (obtained by Theorem 4) to a two-page linear extension L of P ?

For example, we consider the series-parallel planar ordered set P and its completion $\overline{P}$ in Figure 8. In Figure 9, $\overline{L}$ is the two-page linear extension of $\overline{P}$ obtained by the two-page algorithm for series-parallel planar lattice. Let L be the linear extension obtained from $\overline{L}$ by removing the elements in $\overline{P}$–P. Notice that, the linear extension L needs at least three pages.

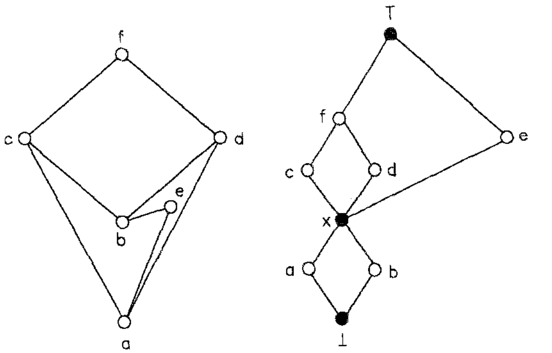

Figure 8

Figure 9

But if we redraw $\overline{P}$ in a different planar embedding as it is in Figure 12, then using the two-page algorithm for series-parallel lattices we will obtain the two-page embedding $\overline{L}$ as it illstrates in Figure 10. In Figure 10, we also, see that the linear extension L of P induced by $\overline{L}$ is a two-page linear extension.

This leads us to this question, whether we can always find a planar embedding of the completion $\overline{P}$ of the series-parallel planar ordered which can lead finally to a two-page linear extension of the ordered set ? The answer is yes.

Lemma 5 *If P is a series-parallel planar ordered set and $\overline{P}$ its completion then,*

(i) $\overline{P}$ is series-parallel, *(ii) $\overline{P}$ is a planar lattice.*

We say the ordered set P contains $K_{m,n}$, $m, n \geq 2$ if it contains a subset $\{a_1, a_2, \cdots, a_m, b_1, , b_2, \cdots, b_n\}$ satisfying $a_i \prec b_j$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. (See Figure 11). Notice that, if P contains $K_{m,n}, m, n \geq 2$ then, the sets $\{a_1, a_2, \cdots, a_m\}$ and $\{b_1, b_2, \cdots, b_n\}$ are antichins.

We say $K_{m,n} = \{a_1, a_2, \cdots, a_m, b_1, , b_2, \cdots, b_n\}$ is *maximal* in P if there is neither $a_{m+1} \neq a_i, 1 \leq i \leq m$ satisfying $a_{m+1} \prec b_j$ for every $1 \leq j \leq n$, nor $b_{n+1} \neq b_j, 1 \leq j \leq n$ satisfying $a_i \prec b_{n+1}$ for every $1 \leq i \leq m$. If a planar ordered set contains $K_{m,n}, m, n \geq 2$, then either $m = 2$ or $n = 2$.

After series of Lemma's, we conclude that if the series-parallel planar ordered set P is not a lattice, then the only obstacle to be a lattice (except the top and the bottom) is existing the maximals of $K_{2,n}$ and/or $K_{m,2}, m, n \geq 2$.

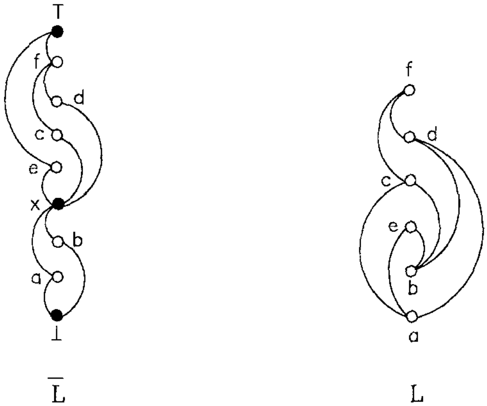

Figure 10

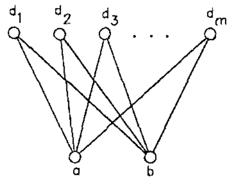

Figure 11 $K_{2,m}$

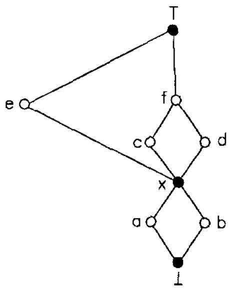

Figure 12

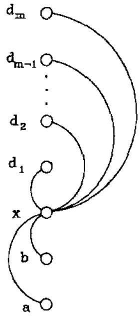

Figure 13

Lemma 6 *If the ordered set P contains $K_{2,m} = \{a, b, d_1, \ldots, d_m\}, m \geq 2$, and if P satisfies one of the following conditions, then P is not planar.*

i) There is an upper bound of some three-element subset of $\{d_1, \ldots, d_m\}$.

ii) a and b have a common lower bound and some two-element subset of $\{d_1, \cdots, d_m\}$ has a common upper bound.

iii) There are two different two-element subsets of $\{d_1, \cdots, d_m\}$ each of which has an upper bound.

As we indicate in the beginning of this section, obtaining the two-page linear extension for P depends on the planar embedding of the completion $\overline{P}$ of P. The next lemma describe such planar embedding.

Lemma 7 *Let P be a series-parallel planar ordered set. For each maximal $K_{2,m} = \{a, b, d_1, \cdots, d_m\}, m \geq 3$ and each maximal $K_{n,2} = \{d'_1, \cdot, d'_n\}, n \geq 3$ such that $\{d_1, d_2\}$ has a minimal upper bound d and $\{d'_1, d'_2\}$ has a maximal lower bound d', there is a planar upward drawing of the completion lattice $\overline{P}$ of P in which d lies to the right of $d_3, d_4, \cdots, d_m$ and d' lies to the left of $d'_3, d'_4, \cdots, d'_n$.*

5 The Main result

In this section we will prove our main result . We will first prove that two pages are enough for a series-parallel planar ordered set. As a consequence, we will give a characterization of series-parallel planar ordered sets.

Theorem 8 *If P is series-parallel planar ordered set then, page(P) ≤ 2.*

The transformation algorithm Let $\overline{P}$ be the completion of P. By Lemma 5, $\overline{P}$ is series-parallel planar lattice. Fix a planar embedding of $\overline{P}$ satisfying
1. Whenever P contains a maximal $K_{2,m} = \{a, b, d_1, \ldots, d_m\}, m \geq 3$, such that d is an upper bound of $\{d_{m-1}, d_m\}$ then, d lies to the right of $\{d_3, \ldots, d_m\}$.
2. Whenever P contains a maximal $K_{m,2} = \{d_1, \ldots, d_m, a, b\}, m \geq 3$, such that d is a lower bound of $\{d_1, d_2\}$ then d lies to the left of $\{d_3, \ldots, d_m\}$.

This is possible according to Lemma 7. If P contains either a maximal $K_{2,m}$ or a maximal $K_{m,2}, m \geq 2$, we may assume that a lies to the left of b and d_i lies to the left of d_{i+1} for $1 \leq i \leq m-1$, in $\overline{P}$.

Notice that, if P contains a maximal $K_{2,m}, m \geq 2$, then the set of the upper covers of a is $\{d_1, \ldots, d_m\}$ which also is the set of the upper covers of b. Also, the set of the lower covers of d_i is $\{a, b\}$ for each $i = 1, \ldots, m$. Dually for $K_{m,2}$.

Since $\overline{P}$ is a series-parallel parallel planar lattice, by Theorem 4 there exists a two-page linear extension $\overline{L}$ of $\overline{P}$. We will transfer it to a two-page linear extension L for P.

For a four-cycle $C = \{a < c > b < d > a\}$ in an ordered set, a *splitting element* x satisfying $a, b \leq x \leq c, d$.

If P contains a maximal $K_{2,m} = \{a, b, d_1, \ldots, d_m\}, m \geq 2$, such that x is the splitting element of $K_{2,m}$ in $\overline{P}$, then we have $a < b < x < d_1 < d_2 < \ldots < d_m$ in $\overline{L}$ and the edges distributed as in Figure 13.

Also, if P has a maximal $K_{m,2} = \{d_1, \ldots, d_m, a, b\}, m \geq 2$ such that x is the splitting element of $K_{m,2}$ in $\overline{P}$, then we have $d_1 < d_2 < \ldots < d_m < x < a < b$ in $\overline{L}$ and the edges distributed as in Figure 14.

Since P is planar, by Lemma 6, if $\{a, b\}$ has a lower (respectively, an upper) bound of $K_{2,m}$ (respectively, $K_{m,2}$) in P, then there is no subset of two elements or more of the set $\{d_1, \ldots, d_m\}$ which has an upper (respectively, a lower) bound.

To obtain a two-page linear extension L of P from $\overline{L}$

1. Remove the set $\overline{P} - P$ from $\overline{L}$ and all edges connected to its vertcies.
2. For each maximal $K_{2,m} = \{a, b, d_1, \ldots, d_m\}, m \geq 2$, in P

   i) *If* $\{a, b\}$ *has a lower bound in* P draw the edges (a, d_i) on the left page and the edges (b, d_i) on the right page (see Figure 15).

   ii) *If* $\{d_{m-1}, d_{m-1}\}$ *has an upper bound in* P draw the edges $\{(b, d_1), (a, d_i) : 1 \leq i \leq m-1\}$ on the left page and draw the edges $\{(a, d_m), (b, d_i) : 2 \leq i \leq m\}$ on the right page for each $1 \leq i \leq m$ (see Figure 16).

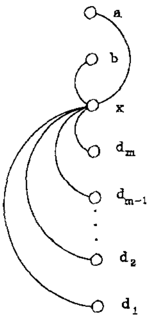

Figure 14

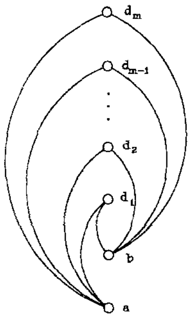

Figure 15

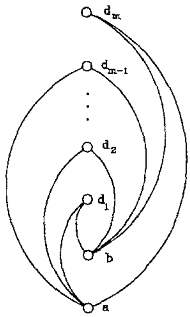

Figure 16

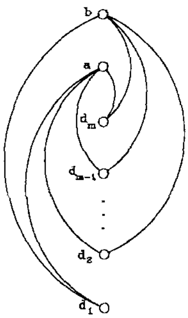

Figure 17

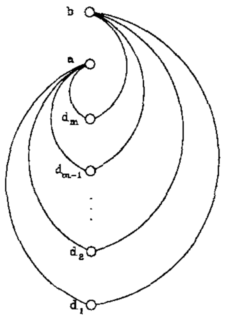

Figure 18

3. For each maximal $K_{m,2} = \{d_1, \ldots, d_m, a, b\}, m \geq 2$, in P

   i) *If* $\{d_1, d_2\}$ *has a lower bound in* P draw the edges $\{(d_1, b), (d_i, a) : 1 \leq i \leq m-1\}$ on the left page and draw the edges $\{(d_m, a), (d_i, b) : 2 \leq i \leq m\}$ on the right page (see Figure 17).

   ii) *If* $\{a, b\}$ *has an upper bound in* P draw the edges $\{(d_i, a) : 1 \leq i \leq m\}$ on the left page and the edges $\{(d_i, b) : 1 \leq i \leq m\}$ on the right page (see Figure 18).

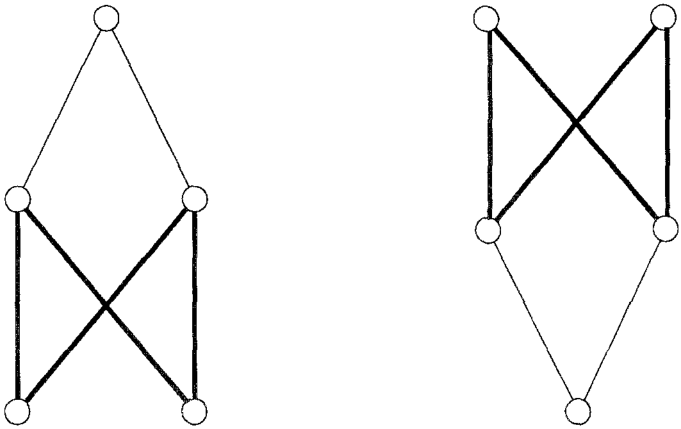

Figure 19 Simple castles.

By Lemma 3, L is greedy linear extension of P. We will show that adding the edges of the maximals $K_{2,m}$ and $K_{m,2}, m \geq 2$, do not create crossing in the same page first for $K_{2,m}$ then for $K_{m,2}$.

A *simple castle* is a covering four-cycle with the top or bottom. (The top, or bottom, need not be in a cover relation with the covering four-cycle.)(See Figure 19) A *castle* is any union of simple castles, which preserves the covering relations of each simple castle. An ordered set P *contains* a castle C if C is a subset of P and P preserves the covering relations of its simple castles.(See Figure 20)

Corollary 9 *Let P be a series-parallel planar ordered set. Then P is planar if and only P contains no $K_{3,3}$ and P contains no nonplanar castle.*

Figure 21 illustrates nonplanar ordered sets each of which contains neither $K_{3,3}$ nor a nonplanar castle. In fact, non is series-parallel.

6 Open problems

1. *Is the pagenumber for planar ordered sets bounded?*

   This question was first asked by Nowakowski and Parker [7]. Hung [3] gave a 48-element planar ordered set which requires four pages (see Figure 44). No planar ordered set required five pages is known.

2. We proved that two pages are enough to embed a series-parallel planar ordered set. Series-parallel ordered sets have dimension two. Is there a positive integer k, such that $page(P) \leq k$, for each planar ordered set of

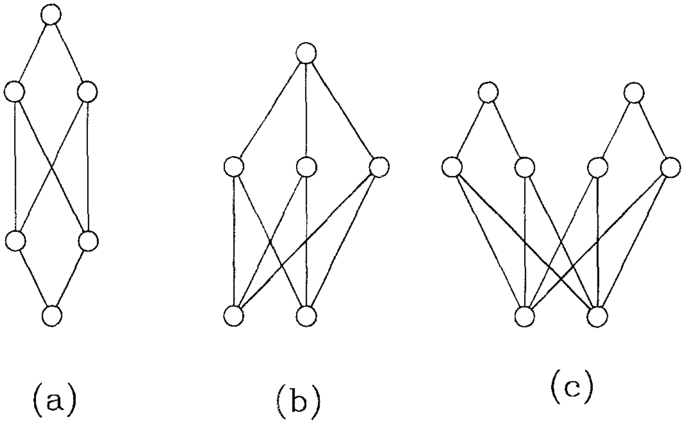

Figure 20 Minimal nonplanar castles.

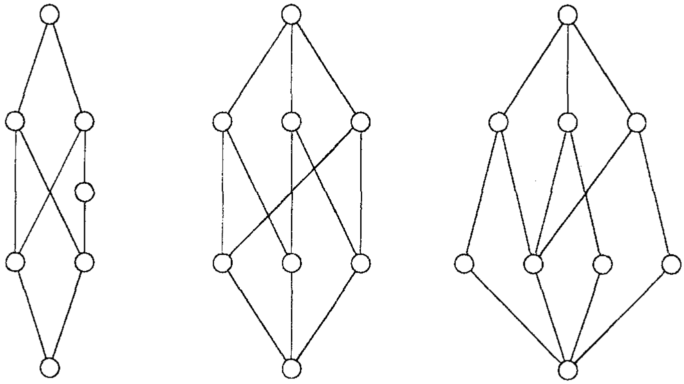

Figure 21

dimension two? ($k \geq 3$ because, $page(P) = 3$ for the planar lattice P in Figure 2). What about planar lattices?

3. Can we extend our result to (nonplanar) series-parallel ordered set?

   What is an upper bound for the (nonplanar) series-parallel ordered set P, depending on the maximal $K_{m,n}$'s in P.

   For positive integers m, n is there a function $f(m,n)$ such that for any series-parallel ordered set P
   $page(P) \leq max\ \{f(m,n) : K_{m,n}$ is a maximal in $P, m, n \geq 2\}$.

   In particular, is there a positive integer k such that
   $f(m,n) \leq min\{m,n\} + k$ for every maximal $K_{m,n}$ in P?

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