Games and Modal Mu-Calculus

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ABSTRACT We define Ehrenfeucht-Fraïssé games which exactly capture the expressive power of the extremal fixed point operators of modal mu-calculus. The resulting games have significance, we believe, within and outside of concurrency theory. On the one hand they naturally extend the iterative bisimulation games associated with Hennessy-Milner logic, and on the other hand they offer deeper insight into the logical role of fixed points. For this purpose we also define second-order propositional modal logic to contrast fixed points and second-order quantifiers.

1 Introduction

This paper further explores the technical contribution that games can make to understanding concurrency. We define Ehrenfeucht-Fraïssé games which exactly capture the expressive power of the extremal fixed point operators of modal mu-calculus. The resulting games have significance, we believe, within and outside of concurrency theory. On the one hand they naturally extend the iterative bisimulation games associated with Hennessy-Milner logic, and on the other hand they offer deeper insight into the logical role of fixed points. For this purpose we also define second-order propositional modal logic to contrast fixed points and second-order quantifiers.

There is something very appealing about trying to understand concurrency and interaction in terms of games. They are a very striking metaphor for the dialogue that a concurrent component can engage in with its environment. One example is [14] where a denotational semantics for concurrent while programs is presented whose domains are built from strategies. Another is the use of games for understanding linear logic [1]. Within process calculi, bisimulation equivalence has been pivotal. A number of authors has noted that it is essentially game theoretic [3, 18, 16] (and [15] extends this description to bisimulations that are sensitive to causality). In this paper we build on this game view of bisimulation. In previous work [17] we showed that local model checking of finite or infinite state processes can be viewed as a game, without loss of structure. In the finite state case this provides an alternative perspective from the use of automata, as it also yields fast model

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checking algorithms: furthermore, these games are definable independently of model checking as graph games which can be reduced to other combinatorial games (and in particular to the very important simple stochastic games [6]).

These concerns have practical repercussions. A guiding principle is to find clear theoretical foundations which can, at the same time, enhance tool development. Games can be naturally animated within a tool. They also offer the potential for effective machine user interaction. For instance, in the model checking case not only do they allow a user to know that a process has a property, but also *why* it has it. Games also allow a user to know *why* a process fails to have a property. In both these cases the justification can be given as a winning strategy (which is polynomial in the size of the model checking problem). Therefore, if a user believes incorrectly that a process has a property then she may become convinced otherwise by playing and losing the model checking game against the machine which holds a winning strategy. These techniques are currently being implemented in the Edinburgh Concurrency Workbench. Similar comments apply to the bisimulation game.

Section 2 is a warm up, where we present some well known concepts in a game theoretic fashion. In section 3 we describe modal mu-calculus and the notion of fixed point depth. Section 4 contains the fixed point games and the main theorem whose proof is delayed until section 6. In section 5 we present second-order modal logic and its games, and discuss its relationship with modal mu-calculus. The work reported here for modal logic has benefited from the large literature on games and logic, and in particular from [9, 18] for Ehrenfeucht-Fraïssé games for first-order logic, [4] for their extension to first-order logic with fixed points, and [8] for their extension to (monadic) second-order logic.

2 Bisimulation games

Assume a process calculus such as CCS, with the proviso[1] that all processes are built from a fixed finite set of actions $\mathcal{A}$. Let E_0 and F_0 be two such processes. We define the game $\mathcal{G}_n^0(E_0, F_0)$ as played by two participants, players I and II. Player I wants to show that E_0 and F_0 are distinguishable within n steps whereas player II wishes to demonstrate that they cannot be differentiated. (The superscript 0 will be explained in section 4.) A play of the game $\mathcal{G}_n^0(E_0, F_0)$ is a finite sequence of pairs $(E_0, F_0) \ldots (E_m, F_m)$ whose length m is at most n. If part of a play is $(E_0, F_0) \ldots (E_i, F_i)$ with $i < n$ then the next move is initiated by player I from the two possibilities in figure 1. Player I always chooses first, and then player II, with full knowledge of

[1] This proviso comes into play in Theorem 1 of section 4 (and also in section 5).

$\langle a \rangle$: Player I chooses a transition $E_i \xrightarrow{a} E_{i+1}$, and then player II chooses a transition with the same label $F_i \xrightarrow{a} F_{i+1}$.

$[a]$: Player I chooses a transition $F_i \xrightarrow{a} F_{i+1}$, and then player II chooses a transition with the same label $E_i \xrightarrow{a} E_{i+1}$.

FIGURE 1. Game moves

player I's selection, must choose a corresponding transition from the other process.

A player wins a play if her opponent becomes stuck: player I wins the play $(E_0, F_0) \dots (E_i, F_i)$, $i < n$, when she can choose a transition from E_i (or from F_i) and there is no corresponding transition from the other process F_i (or E_i), and player II wins if the processes E_i and F_i are both deadlocked. Player II also wins if the play reaches length n, for then player I has been unable to distinguish the initial processes within n steps.

A strategy for a player is a set of rules which tells her how to move depending on what has happened previously in the play. A player uses the strategy π in a play if all her moves in the play obey the rules in π. The strategy π is a *winning strategy* if the player wins every play in which she uses π. For each game $\mathcal{G}_n^0(E_0, F_0)$ one of the players has a winning strategy, and this strategy is history free in the sense that the rules do not need to appeal to moves that occurred before the current game configuration. If player II has a winning strategy for $\mathcal{G}_n^0(E_0, F_0)$ then we say that E_0 and F_0 are $(0, n)$-game equivalent, which we abbreviate as $E_0 \sim_n^0 F_0$.

Example 1 Consider the two similar vending machines

$$\begin{aligned} U &\stackrel{\text{def}}{=} \texttt{1p}.(\texttt{1p}.\texttt{tea}.U + \texttt{1p}.\texttt{coffee}.U) \\ V &\stackrel{\text{def}}{=} \texttt{1p}.\texttt{1p}.\texttt{tea}.V + \texttt{1p}.\texttt{1p}.\texttt{coffee}.V \end{aligned}$$

Although $U \sim_2^0 V$, player I has a winning strategy for the game $\mathcal{G}_3^0(U, V)$ and so $U \not\sim_3^0 V$. □

Example 2 Let $Cl_{i+1} \stackrel{\text{def}}{=} \texttt{tick}.Cl_i$, $i \geq 0$. Therefore Cl_{i+1} is a clock that ticks $i+1$ times before terminating. Let $Cl \stackrel{\text{def}}{=} \texttt{tick}.Cl$ be a clock that ticks forever. Assume that *Clock* is $\sum\{Cl_i : i \geq 1\}$ and *Clock'* is *Clock* + *Cl*. Although *Clock'* can tick forever which *Clock* cannot do, for every $n \geq 0$, $\mathit{Clock}' \sim_n^0 \mathit{Clock}$. □

Game equivalence and iterated bisimulation equivalence are intimately related. For each $n \geq 0$, let $\sim_n$ be the following relation on processes [13]:

$$\begin{array}{ll}
E \sim_0 F & \text{for all } E \text{ and F.} \\
E \sim_{n+1} F & \text{iff for each action } a \in \mathcal{A}, \\
& \text{if } E \xrightarrow{a} E' \text{ then } \exists F'.\, F \xrightarrow{a} F' \text{ and } E' \sim_n F', \text{ and} \\
& \text{if } F \xrightarrow{a} F' \text{ then } \exists E'.\, E \xrightarrow{a} E' \text{ and } E' \sim_n F'.
\end{array}$$

Fact 1 $E \sim_n^0 F$ *iff* $E \sim_n F$.

Another way of understanding these equivalences uses Hennessy-Milner logic. Let M be the following family of modal formulas where a ranges over $\mathcal{A}$:

$$\Phi \;::=\; \mathtt{tt} \;\mid\; \mathtt{ff} \;\mid\; \Phi_1 \wedge \Phi_2 \;\mid\; \Phi_1 \vee \Phi_2 \;\mid\; [a]\Phi \;\mid\; \langle a \rangle \Phi$$

The inductive stipulation below states when a process E has a modal property Φ, written $E \models \Phi$. If E fails to satisfy Φ then this is written $E \not\models \Phi$.

$$\begin{array}{lll}
E \models \mathtt{tt} & & E \not\models \mathtt{ff} \\
E \models \Phi \wedge \Psi & \text{iff} & E \models \Phi \text{ and } E \models \Psi \\
E \models \Phi \vee \Psi & \text{iff} & E \models \Phi \text{ or } E \models \Psi \\
E \models [a]\Phi & \text{iff} & \forall F.\, \text{if } E \xrightarrow{a} F \text{ then } F \models \Phi \\
E \models \langle a \rangle \Phi & \text{iff} & \exists F.\, E \xrightarrow{a} F \text{ and } F \models \Phi
\end{array}$$

The modal depth of a formula Φ, $\mathrm{md}(\Phi)$, is the maximum embedding of modal operators, and is defined as follows:

$$\begin{array}{lclcl}
\mathrm{md}(\mathtt{tt}) & = & 0 & = & \mathrm{md}(\mathtt{ff}) \\
\mathrm{md}(\Phi \wedge \Psi) & = & \max\{\mathrm{md}(\Phi), \mathrm{md}(\Psi)\} & = & \mathrm{md}(\Phi \vee \Psi) \\
\mathrm{md}([a]\Phi) & = & 1 + \mathrm{md}(\Phi) & = & \mathrm{md}(\langle a \rangle \Phi)
\end{array}$$

Assume that M_k is the sublogic $\{\Phi : \Phi \in M \text{ and } \mathrm{md}(\Phi) \leq k\}$.

Fact 2 $E \sim_n^0 F$ *iff* $\forall \Phi \in M_n.\ E \models \Phi$ *iff* $F \models \Phi$.

An easy corollary is that the relations $\sim_n^0$, $n \geq 0$, on processes constitute a genuine hierarchy.

Fact 3 *If* $m < n$ *then* $\sim_n^0 \subset \sim_m^0$.

A binary relation $\mathcal{R}$ between processes is a bisimulation relation provided that whenever $E\mathcal{R}F$, for all $a \in \mathcal{A}$:

$$\begin{array}{l}
\text{if } E \xrightarrow{a} E' \text{ then } \exists F'.\, F \xrightarrow{a} F' \text{ and } E'\mathcal{R}F', \text{ and} \\
\text{if } F \xrightarrow{a} F' \text{ then } \exists E'.\, E \xrightarrow{a} E' \text{ and } E'\mathcal{R}F'.
\end{array}$$

Two processes E and F are bisimulation equivalent, written $E \sim F$, if there is a bisimulation $\mathcal{R}$ relating them. To capture $\sim$ (instead of the iterated bismulation relations $\sim_n$) using games the notion of game is extended to encompass plays that may continue forever. Let $\mathcal{G}^0_\infty(E_0, F_0)$ be such a game. If part of a play is $(E_0, F_0) \ldots (E_i, F_i)$ then the next move is initiated by player I from the two moves in figure 1. Again a player wins if her opponent becomes stuck. Also player II wins if the play has infinite length. For each game $\mathcal{G}^0_\infty(E, F)$ one of the players has a (history free) winning strategy, and if it is player II then we write $E \sim^0_\infty F$.

Fact 4 $E \sim^0_\infty F$ *iff* $E \sim F$.

Example 3 Player I has a winning strategy for $\mathcal{G}^0_\infty(\mathit{Clock}', \mathit{Clock})$. She first chooses the $\langle \mathtt{tick} \rangle$ move, $\mathit{Clock}' \xrightarrow{\mathtt{tick}} \mathit{Cl}$, and so player II has to respond with a transition $\mathit{Clock} \xrightarrow{\mathtt{tick}} \mathit{Cl}_i$, for some $i \geq 0$. So after i further moves the game configuration becomes $(\mathit{Cl}, \mathit{Cl}_0)$, and so player I wins. This example also shows that $\sim \subset \sim^0_n$ for any n. □

3 Modal mu-calculus

Modal mu-calculus, modal logic with extremal fixed points, introduced by Kozen [12], is a very expressive propositional temporal logic with the ability to describe liveness, safety, fairness and cyclic properties of processes. Formulas of the logic, μM, given in positive form are defined as follows

$$\Phi ::= \mathtt{tt} \mid \mathtt{ff} \mid Z \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid [a]\Phi \mid \langle a \rangle \Phi \mid \nu Z.\,\Phi \mid \mu Z.\,\Phi$$

where Z ranges over a family of propositional variables, and a over $\mathcal{A}$. The binder νZ is the greatest whereas μZ is the least fixed point operator.

When E is a process let $\mathcal{P}(E)$ be the smallest transition closed set containing E: that is, if $F \in \mathcal{P}(E)$ and $F \xrightarrow{a} F'$ then $F' \in \mathcal{P}(E)$. Let $\mathcal{P}$ range over (non-empty) transition closed sets. We extend the semantics of modal logic of the previous section to encompass fixed points. Because of free variables we employ valuations $\mathcal{V}$ which assign to each variable Z a subset $\mathcal{V}(Z)$ of processes in $\mathcal{P}$. Let $\mathcal{V}[\mathcal{E}/Z]$ be the valuation $\mathcal{V}'$ which agrees with $\mathcal{V}$ everywhere except possibly Z when $\mathcal{V}'(Z) = \mathcal{E}$. The inductive definition of satisfaction below stipulates when a process E has the property Φ relative to $\mathcal{V}$, written $E \models_{\mathcal{V}} \Phi$.

$$\begin{array}{lll}
E \models_{\mathcal{V}} \mathtt{tt} & & E \not\models_{\mathcal{V}} \mathtt{ff} \\
E \models_{\mathcal{V}} Z & \text{iff} & E \in \mathcal{V}(Z) \\
E \models_{\mathcal{V}} \Phi \wedge \Psi & \text{iff} & E \models_{\mathcal{V}} \Phi \text{ and } E \models_{\mathcal{V}} \Psi \\
E \models_{\mathcal{V}} \Phi \vee \Psi & \text{iff} & E \models_{\mathcal{V}} \Phi \text{ or } E \models_{\mathcal{V}} \Psi \\
E \models_{\mathcal{V}} [a]\Phi & \text{iff} & \forall F.\,\text{if } E \xrightarrow{a} F \text{ then } F \models_{\mathcal{V}} \Phi \\
E \models_{\mathcal{V}} \langle a \rangle \Phi & \text{iff} & \exists F.\, E \xrightarrow{a} F \text{ and } F \models_{\mathcal{V}} \Phi \\
E \models_{\mathcal{V}} \nu Z.\,\Phi & \text{iff} & \exists \mathcal{E} \subseteq \mathcal{P}(E).\, E \in \mathcal{E} \text{ and } \forall F \in \mathcal{E}.\, F \models_{\mathcal{V}[\mathcal{E}/Z]} \Phi \\
E \models_{\mathcal{V}} \mu Z.\,\Phi & \text{iff} & \forall \mathcal{E} \subseteq \mathcal{P}(E).\,\text{if } (\forall F \in \mathcal{P}(E).\, F \models_{\mathcal{V}[\mathcal{E}/Z]} \Phi \text{ implies } F \in \mathcal{E}) \\
& & \qquad \text{then } E \in \mathcal{E}
\end{array}$$

The stipulations for the fixed points follow directly from Tarski-Knaster, as a greatest fixed point is the union of all postfixed points and a least fixed point is the intersection of all prefixed points. The clause for the least fixed point can be slightly simplified as follows: $E \models_{\mathcal{V}} \mu Z.\, \Phi$ iff

$$\forall \mathcal{E} \subseteq \mathcal{P}(E).\, \text{if } E \notin \mathcal{E} \text{ then } \exists F \in \mathcal{P}(E).\, F \notin \mathcal{E} \text{ and } F \models_{\mathcal{V}[\mathcal{E}/Z]} \Phi$$

A formula Φ is closed if it does not contain any free variables: in which case $E \models_{\mathcal{V}} \Phi$ iff $E \models_{\mathcal{V}'} \Phi$ for any valuations $\mathcal{V}$ and $\mathcal{V}'$. Notice that closed formulas are closed under negation.

There is a large literature on the use of μM for specifying and verifying temporal properties of processes. Here our concern is with trying to understand the role of fixed points in μM. Characterizing the expressive power of particular formulas of μM is no easy matter. We shall show that there is an algebraic characterization which generalizes the well known results of the previous section.

We define the sublogics μM_n^k to be the set of closed formulas whose modal depth is at most n and whose fixed point depth is at most k. In particular, $\mu M_n^0 = M_n$. The modal depth of Φ, $\mathrm{md}(\Phi)$, is the maximum embedding of modal operators, and extends the definition from the previous section:

$$\begin{array}{lclclcl}
\mathrm{md}(\mathtt{tt}) & = & 0 & = & \mathrm{md}(\mathtt{ff}) & = & \mathrm{md}(Z) \\
\mathrm{md}(\Phi \wedge \Psi) & = & \max\{\mathrm{md}(\Phi), \mathrm{md}(\Psi)\} & = & \mathrm{md}(\Phi \vee \Psi) & & \\
\mathrm{md}([a]\Phi) & = & 1 + \mathrm{md}(\Phi) & = & \mathrm{md}(\langle a \rangle \Phi) & & \\
\mathrm{md}(\nu Z.\, \Phi) & = & \mathrm{md}(\Phi) & = & \mathrm{md}(\mu Z.\, \Phi) & &
\end{array}$$

Similarly, the fixed point depth of Φ, written $\mathrm{fd}(\Phi)$, is the maximum embedding of fixed point operators:

$$\begin{array}{lclclcl}
\mathrm{fd}(\mathtt{tt}) & = & 0 & = & \mathrm{fd}(\mathtt{ff}) & = & \mathrm{fd}(Z) \\
\mathrm{fd}(\Phi \wedge \Psi) & = & \max\{\mathrm{fd}(\Phi), \mathrm{fd}(\Psi)\} & = & \mathrm{fd}(\Phi \vee \Psi) & & \\
\mathrm{fd}([a]\Phi) & = & \mathrm{fd}(\Phi) & = & \mathrm{fd}(\langle a \rangle \Phi) & & \\
\mathrm{fd}(\nu Z.\, \Phi) & = & 1 + \mathrm{fd}(\Phi) & = & \mathrm{fd}(\mu Z.\, \Phi) & &
\end{array}$$

Let $\mu M_n^k = \{\Phi : \Phi \in \mu M \text{ is closed and } \mathrm{fd}(\Phi) \leq k \text{ and } \mathrm{md}(\Phi) \leq n\}$.

4 Fixed point games

It is our intention to characterize the families μM_n^k in terms of games which generalize those of section 2. We define the game $\mathcal{G}_n^k(E_0, F_0)$ as played by the two participants, players I and II. For this game we assume k distinct colours $\mathcal{C}_1, \ldots, \mathcal{C}_k$. A play of the game $\mathcal{G}_n^k(E_0, F_0)$ is a finite sequence of pairs $(E_0, F_0) \ldots (E_m, F_m)$ whose length m is at most $k+n$. More precisely, there are two kinds of moves the $\langle a \rangle$ and $[a]$ moves as in figure 1, and the new μ and ν moves: a play contains no more than k fixed point moves, and

$\langle a \rangle$: If $l < n$ then player I chooses a transition $E_i \xrightarrow{a} E_{i+1}$, and then player II chooses a transition with the same label $F_i \xrightarrow{a} F_{i+1}$.

$[a]$: If $l < n$ then player I chooses a transition $F_i \xrightarrow{a} F_{i+1}$, and then player II chooses a transition with the same label $E_i \xrightarrow{a} E_{i+1}$.

ν : If $j < k$ then player I obtains the next colour C_{j+1} and paints a subset of $\mathcal{P}(E_i)$ which includes E_i the colour C_{j+1}, and then player II paints a subset of $\mathcal{P}(F_i)$ which includes F_i colour C_{j+1}, and then player I chooses $F_{i+1} \in \mathcal{P}(F_i)$ which is coloured C_{j+1}, and then player II chooses $E_{i+1} \in \mathcal{P}(E_i)$ which is coloured C_{j+1}.

μ : If $j < k$ then player I obtains the next colour C_{j+1} and paints a subset of $\mathcal{P}(F_i)$ which excludes E_i the colour C_{j+1}, and then player II paints a subset of $\mathcal{P}(E_i)$ which excludes F_i colour C_{j+1}, and then player I chooses $E_{i+1} \in \mathcal{P}(E_i)$ not coloured C_{j+1}, and then player II chooses $F_{i+1} \in \mathcal{P}(F_i)$ not coloured C_{j+1}.

FIGURE 2. Fixed point game moves

n modal moves. If part of a play is $(E_0, F_0) \ldots (E_i, F_i)$ with $i < k + n$, and the number of modal moves so far is l and the number of fixed point moves is j, then the next move is initiated by player I from the applicable moves in figure 2. In the case of the fixed point moves, player I first colours a subset of the reachable processes from one of the pair of processes in the current game configuration with the next available colour, and with full knowledge of what player I has done, player II colours a subset of the reachable processes with the same colour from the other process. There is an asymmetry in the colouring between ν and μ, as to whether the current processes are coloured. Next player I, also with full knowledge of what has been coloured so far, picks a reachable process from the set that player II was responsible for colouring: again there is an asymmetry, in the ν case she chooses a coloured process and in the μ case an uncoloured one. Finally player II, with knowledge of all the choices so far, chooses a reachable process that player I was responsible for (in the ν case a coloured process, and in the μ case an uncoloured one). Notice that a process may end up with multiple colours.

A play of $\mathcal{G}_n^k(E, F)$ involves at most k fixed point moves and n modal moves. A game is played until one of the players wins, or until there are no more available moves. The conditions for winning are given in figure 3 where we assume that $(E_0, F_0) \ldots (E_i, F_i)$ is (part of) a play with l modal moves and j fixed point moves. The important new condition is preservation of colours, as given by clause 3 for a win for player I. This means that for

Player I wins

1. If $l < n$ and $E_i \xrightarrow{a} E'$ but $\neg\exists F'.F_i \xrightarrow{a} F'$.
2. If $l < n$ and $F_i \xrightarrow{a} F'$ but $\neg\exists E'.E_i \xrightarrow{a} E'$.
3. If E_i is coloured $C_h, h \leq j$, and F_i is not coloured C_h.

Player II wins

1. If E_i and F_i are both deadlocked and condition 3 above does not hold.
2. If the play has ended, $l = n$ and $j = k$, and 3 above does not hold.

FIGURE 3. Winning conditions

player II to win, she has to make sure that whenever the game configuration reaches (E, F) then the colours of E are included in the colours of F[2].

For each game $\mathcal{G}_n^k(E, F)$ one of the players has a winning strategy (which is no longer history free, as it depends on previous colouring moves). If player II has a winning strategy for $\mathcal{G}_n^k(E, F)$ then we say that E and F are (k, n)-game equivalent, which we write as $E \sim_n^k F$.

Example 1 Player I has a winning strategy for $\mathcal{G}_1^1(\mathit{Clock}', \mathit{Clock})$. Recall the behaviour of these processes. $\mathit{Clock} \xrightarrow{\mathtt{tick}} \mathit{Cl}_i$ and $\mathit{Clock}' \xrightarrow{\mathtt{tick}} \mathit{Cl}_i$, for any i. However it is also the case that $\mathit{Clock}' \xrightarrow{\mathtt{tick}} \mathit{Cl}$ and $\mathit{Cl} \xrightarrow{\mathtt{tick}} \mathit{Cl}$. Player I's winning strategy consists of making an initial ν move. She paints Clock' and Cl with the colour C_1. Player II must respond by painting Clock and a subset $\mathcal{E} \subseteq \{\mathit{Cl}_i : i \geq 0\}$ the colour C_1. If $\mathcal{E}$ is nonempty then player I chooses the least member of it (with respect to i), and otherwise she chooses Clock. Player II must now choose either Clock' or Cl, and either way she will lose at the next step because player I will play a [`tick`] move, either $\mathit{Clock}' \xrightarrow{\mathtt{tick}} \mathit{Cl}$ or $\mathit{Cl} \xrightarrow{\mathtt{tick}} \mathit{Cl}$, and player II is either stuck or unable to avoid condition 3 for a player I win. □

The main theorem is the following which generalizes Fact 2 of section 2.

Theorem 1 *$E \sim_n^k F$ iff $\forall \Phi \in \mu M_n^k . E \models_{\mathcal{V}} \Phi$ iff $F \models_{\mathcal{V}} \Phi$.*

The proof of this result is presented in section 6, where game playing has to be extended to cope with open formulas to provide an inductive mechanism. It shows that there is an exact correspondence between game playing of length (k, n) and having the same properties in μM_n^k. A corollary (using known results [16]) is:

Fact 1 *For each k and n, $\sim \subset \sim_n^k$.*

[2]The asymmetry here between E's and F's colours is because negated variables are not allowed in the logic

We hope that Theorem 1 can be used it to provide a better understanding of how μM formulas express properties. It is possible to define for each formula Φ a signature which represents the sequences of possible moves in a game play. To understand the expressive power of Φ we need only examine those game plays that belong to its signature. This may offer a means for defining filtrations for modal mu-calulus. We hope that these games can be articulated on a machine on small processes and we look forward to examining the feasibility of implementing them. We also hope that Theorem 1 may offer deeper insight into the logical role of fixed points, and the contrast between them and second-order quantifiers. In the next section we define second-order propositional modal logic for this purpose.

An original motivation for this work on games is the issue of fixed point hierarchies. For each k we can define the set $\mu M^k = \bigcup\{\mu M^k_n : n \geq 0\}$. Is there a hierarchy of definability? For each $k \geq 1$ is there a formula $\Phi \in \mu M^k$ such that for all $\Psi \in \mu M^{k-1}$, Φ is not equivalent to Ψ?[3] Using Theorem 1 this reduces to questions about game equivalences (which appear to offer a finer analysis than automata, see [2, 11]). The hierarchy question is more interesting when k in μM^k is alternation depth, instead of fixed point depth[4]. Very recently Bradfield has shown that there is a full alternation depth hierarchy using methods from descriptive set theory [5].

5 Second-order propositional modal logic

We define second-order propositional modal logic, $2M$, as an extension of modal mu-calculus, as follows:

$$\Phi \;::=\; \mathtt{tt} \mid Z \mid \neg\Phi \mid \Phi_1 \wedge \Phi_2 \mid [a]\Phi \mid \Box\,\Phi \mid \forall Z.\Phi$$

The modality $\Box$ is the reflexive and transitive closure of the family of modalities $[a]$, $a \in \mathcal{A}$, and is included so that fixed points are definable within $2M$: this proposal for $2M$ is due to Howard Barringer. Negation is included explicitly, and we assume the expected derived operators: $\mathtt{ff} \stackrel{\text{def}}{=} \neg\mathtt{tt}$, $\Phi_1 \vee \Phi_2 \stackrel{\text{def}}{=} \neg(\neg\Phi_1 \wedge \neg\Phi_2)$, $\Phi_1 \rightarrow \Phi_2 \stackrel{\text{def}}{=} \neg\Phi_1 \vee \Phi_2$, $\langle a\rangle\Phi \stackrel{\text{def}}{=} \neg[a]\neg\Phi$, $\Diamond\Phi \stackrel{\text{def}}{=} \neg\Box\neg\Phi$, and $\exists Z.\Phi \stackrel{\text{def}}{=} \neg\forall Z.\neg\Phi$.

As with modal mu-calculus we define when a process E has a property Φ relative to $\mathcal{V}$, written $E \models_{\mathcal{V}} \Phi$, where $\mathcal{V}$ is a valuation. The semantic clauses for $\mathtt{tt}$, Z, $\wedge$ and $[a]$ are as in section 3. The new clauses are:

[3]For instance, example 1 above shows that μM^1 is more expressive than μM^0.

[4]Preliminary work suggests it is possible to characterize μM^k_n in terms of games when k is alternation depth, but so far these games are very inelegant.

$\langle a \rangle$: If $l < n$ then player I chooses a transition $E_i \xrightarrow{a} E_{i+1}$, and then player II chooses a transition with the same label $F_i \xrightarrow{a} F_{i+1}$.

$[a]$: If $l < n$ then player I chooses a transition $F_i \xrightarrow{a} F_{i+1}$, and then player II chooses a transition with the same label $E_i \xrightarrow{a} E_{i+1}$.

$\Diamond$: If $q < p$ then player I chooses $E_{i+1} \in \mathcal{P}(E_i)$, and then player II chooses $F_{i+1} \in \mathcal{P}(F_i)$.

$\Box$: If $q < p$ then player I chooses $F_{i+1} \in \mathcal{P}(F_i)$, and then player II chooses $E_{i+1} \in \mathcal{P}(E_i)$.

$\exists$: If $j < k$ then player I paints a subset of $\mathcal{P}(E_i)$ the colour $\mathcal{C}_{j+1}$, and then player II paints a subset of $\mathcal{P}(F_i)$ the colour $\mathcal{C}_{j+1}$.

$\forall$: If $j < k$ then player I paints a subset of $\mathcal{P}(F_i)$ the colour $\mathcal{C}_{j+1}$, and then player II paints a subset of $\mathcal{P}(E_i)$ the colour $\mathcal{C}_{j+1}$.

FIGURE 4. Second-order game moves

$$
\begin{array}{lll}
E \models_{\mathcal{V}} \neg\Phi & \text{iff} & E \not\models_{\mathcal{V}} \Phi \\
E \models_{\mathcal{V}} \Box\Phi & \text{iff} & \forall F \in \mathcal{P}(E).\ F \models_{\mathcal{V}} \Phi \\
E \models_{\mathcal{V}} \forall Z.\Phi & \text{iff} & \forall \mathcal{E} \subseteq \mathcal{P}(E).\ E \models_{\mathcal{V}[\mathcal{E}/Z]} \Phi
\end{array}
$$

Notice that $\Box$ is definable in μM: assuming Z is not free in Φ, the formula $\Box\Phi$ is $\nu Z.\,\Phi \wedge \bigwedge_{a\in\mathcal{A}}[a]Z$. The operator $\forall Z$ is a set quantifier, ranging over subsets of $\mathcal{P}(E)$.

There is a game theoretic characterization of $2M$, which we briefly describe. Let $2M^k_{n,p}$ be the set of closed formulas whose modal depth with respect to $[a]$ modalities is n, and whose modal depth with respect to $\Box$ is p, and whose quantifier depth is k. A play of the game $\mathcal{G}^k_{n,p}(E_0, F_0)$ is a finite sequence of pairs $(E_0, F_0) \ldots (E_m, F_m)$ whose length m is at most $k + n + p$. Again we assume k distinct colours $\mathcal{C}_1, \ldots, \mathcal{C}_k$. There are three kinds of moves, the $\langle a \rangle$ and $[a]$ moves as in figure 2, the $\Diamond$ and $\Box$ moves, and the $\exists$ and $\forall$ moves. If part of a play is $(E_0, F_0) \ldots (E_i, F_i)$ with $i < k+n+p$, and the number of $\langle a \rangle$, $[a]$ moves so far is l, and the number of quantifier moves is j, and the number of $\Diamond$, $\Box$ moves is q, then player I initiates the next move from those in figure 4. These moves are somewhat simpler than for the fixed point games[5]. A play of $\mathcal{G}^k_{n,p}(E, F)$ involves at most k quanti-

[5]Notice that the fixed point moves can almost be built from $2M$ moves: for example, the ν move is almost a $\exists$ move followed by a $\Box$ move.

Player I wins
1. If $l < n$ and $E_i \xrightarrow{a} E'$ but $\neg\exists F'.F_i \xrightarrow{a} F'$.
2. If $l < n$ and $F_i \xrightarrow{a} F'$ but $\neg\exists E'.E_i \xrightarrow{a} E'$.
3. If E_i is coloured $C_h, h \leq j$, and F_i is not coloured C_h.
4. If F_i is coloured $C_h, h \leq j$, and E_i is not coloured C_h.

Player II wins
1. If E_i and F_i are both deadlocked and the conditions 3 and 4 above both fail to hold.
2. If the play has ended, $l = n$, $j = k, q = p$, and conditions 3 and 4 above both fail to hold.

FIGURE 5. Winning conditions

fier moves, and n and p of the respective modal moves. The conditions for winning are given in figure 5 where we assume that $(E_0, F_0) \ldots (E_i, F_i)$ is (part of) a play with l $\langle a \rangle$, $[a]$ moves, q $\Diamond$, $\Box$ moves, and j quantifier moves. Notice the extra condition for a player I win, because in $2M$ quantification is permitted over negated variables.

E and F are (k, n, p)-game equivalent, written as $E \sim^k_{n,p} F$ if player II has a winning strategy for $\mathcal{G}^k_{n,p}(E, F)$. The following result, as with Theorem 1 of section 4, generalizes Fact 2 of section 2.

Theorem 1 $E \sim^k_{n,p} F$ *iff* $\forall \Phi \in 2M^k_{n,p}. E \models_{\mathcal{V}} \Phi$ *iff* $F \models_{\mathcal{V}} \Phi$.

The proof of this result follows closely that of the proof of Theorem 1 of section 4, presented in the next section.

An important question is what the relationship is between closed formulas of μM and $2M$, with respect to particular families of models. Within $2M$ we can define 3-colourability on finite connected undirected graphs. Consider such a graph. If there is an edge between two vertices E and F let $E \xrightarrow{a} F$ and $F \xrightarrow{a} E$. So in this case $\mathcal{A} = \{a\}$, and 3-colourability is given by:

$$\exists X. \exists Y. \exists Z. (\Phi \wedge \Box((X \rightarrow [a]\neg X) \wedge (Y \rightarrow [a]\neg Y) \wedge (Z \rightarrow [a]\neg Z)))$$

where Φ, which says that every vertex has a unique colour, is

$$\Box((X \wedge \neg Y \wedge \neg Z) \vee (Y \wedge \neg Z \wedge \neg X) \vee (Z \wedge \neg X \wedge \neg Y))$$

In contrast, modal mu-calculus can only express P graph properties (this follows from [10]).

First we have

Proposition 1 *μM is a sublogic of $2M$.*

Proof: There is a straightforward translation of μM into $2M$. Let Tr be this translation. The important cases are the fixed points: $\mathrm{Tr}(\nu Z.\,\Phi) = \exists Z.(Z \wedge \Box(Z \rightarrow \mathrm{Tr}(\Phi)))$ and $\mathrm{Tr}(\mu Z.\,\Phi) = \forall Z.(\Box(\mathrm{Tr}(\Phi) \rightarrow Z) \rightarrow Z)$. □

When models are restricted to binary (or n-ary, $n \geq 1$) trees, the closed formulas of $2M$ are translatable into μM. This follows because μM is then equi-expressive to tree automata [7], and $2M$ is easily codable into $S2S$. However this is not the case for processes, as pointed out by Perdita Stevens. $2M$ formulas can distinguish between bisimilar processes which μM formulas are unable to do.

6 Proof of the main theorem

In this section we prove Theorem 1 of section 4, that E and F are (k, n)-game equivalent iff they have the same μM_n^k properties. (The proof of Theorem 1 of the previous section has a similar structure.) To prove this result inductively we need to understand *open* formulas of modal mu-calculus. Therefore we let $\mu M_n^k(X_1, \ldots, X_m)$ be the set of modal mu-calculus formulas with fixed point depth k and modal depth n which also may contain occurrences of any free variable X_i, for $1 \leq i \leq m$. Because $\mathcal{A}$ is finite each $\mu M_n^k(X_1, \ldots, X_m)$ is finite up to logical equivalence.

Proposition 1 *For each k, n and m the set $\mu M_n^k(X_1, \ldots, X_m)$ is finite up to logical equivalence.*

Proof: A straightforward induction on $k + n$. □

We generalize the game $\mathcal{G}_n^k(E_0, F_0)$ to $\mathcal{G}_n^k(E_0, \overline{U}_i, F_0, \overline{V}_i)$ where $\overline{U}_i$ is a sequence of sets of processes $U_1, \ldots U_m$ and $\overline{V}_i$ is a similar sequence $V_1, \ldots V_m$. The colours for the generalized game are $\mathcal{C}_1, \ldots, \mathcal{C}_m, \ldots, \mathcal{C}_{m+k}$. The colours $\mathcal{C}_1, \ldots, \mathcal{C}_m$ are in use at the start of the game, and vertices U_i and V_i are coloured $\mathcal{C}_i$. The game is played as before (with k fixed point moves and n modal moves) but with $\mathcal{C}_{m+1}$ as the initial available colour. The winning conditions are as before: note however that condition 3 for player I's win extends to the colours in use at the start of play.

Theorem 1 *Player II has a winning strategy for $\mathcal{G}_n^k(E, \overline{U}_i, F, \overline{V}_i)$ iff $\forall \Phi \in \mu M_n^k(X_1, \ldots X_m)$. if $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \Phi$ then $F \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \Phi$.*

Proof: Suppose player II has a winning strategy for $\mathcal{G}_n^k(E_0, \overline{U}_i, F_0, \overline{V}_i)$. By induction on $k + n$ we show $\forall \Phi \in \mu M_n^k(X_1, \ldots X_m)$. if $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \Phi$ then $F \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \Phi$. The base case is when $k + n = 0$. So Φ is a boolean combination of `tt`, `ff`, and the variables X_j, $1 \leq j \leq m$. As player II has a winning strategy, we know that for each such variable X_j, if $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} X_j$ then $F \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} X_j$ (for otherwise player I would win by the winning condition 3). So the result follows. For the general case, assume it holds for

$k+n \leq l$. Assume that $k+n = l+1$. We proceed by subinduction on Φ. If Φ is $\mathtt{tt}$ or $\mathtt{ff}$ then it is clear. If it is X_j then the proof is as in the base case. The cases $\Phi_1 \wedge \Phi_2$ and $\Phi_1 \vee \Phi_2$ are routine. Suppose Φ is $[a]\Psi$, and $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \Phi$. If E is unable to perform a then, as player II has a winning strategy, F is also unable to do an a and so $F \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \Phi$. Assume that E has an a transition. Consider any transition $F \xrightarrow{a} F'$ (and there is at least one otherwise player I would win the game). Let player I choose this transition as her move which is a $[a]$ move. Player II must respond with $E \xrightarrow{a} E'$ for some E' in such a way that player II has a winning strategy for $\mathcal{G}^k_{n-1}(E', \overline{U}_i, F', \overline{V}_i)$. By the induction hypothesis, as $k+(n-1) = l$, $\forall \Phi \in \mu M^k_{n-1}(X_1, \ldots X_m)$. if $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \Phi$ then $F \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \Phi$. The formula Ψ is in $\mu M^k_{n-1}(X_1, \ldots X_m)$, and $E' \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \Psi$ and so $F' \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \Psi$. Hence for each F' such that $F \xrightarrow{a} F'$, $F' \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \Psi$, and therefore $F \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} [a]\Psi$. The case Φ is $\langle a \rangle \Psi$ is similar. Next suppose Φ is $\nu Z.\, \Psi$. As $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \nu Z.\, \Psi$, therefore $\exists \mathcal{E} \subseteq \mathcal{P}(E).\, E \in \mathcal{E}$ and $\forall E' \in \mathcal{E}.\, E' \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i][\mathcal{E}/Z]} \Psi$. Consider the game play where player I makes a ν move and colours $\mathcal{E}$ (containing E) with the next available colour $\mathcal{C}_{m+1}$. As player II has a winning strategy she can respond by colouring a set $\mathcal{F}$ (containing F) with $\mathcal{C}_{m+1}$ in such a way that for any choice $F' \in \mathcal{F}$ there is an $E' \in \mathcal{E}$ such that player II has a winning strategy for the game $\mathcal{G}^{k-1}_n(E', \overline{U}_i \mathcal{E}, F', \overline{V}_i \mathcal{F})$. Via the induction hypothesis, it follows that $\forall F' \in \mathcal{F}.\, F' \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i][\mathcal{F}/Z]} \Psi$, and as $F \in \mathcal{F}$, it follows by the semantic clause that $F \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \nu Z.\, \Psi$ as required. The final case Φ is $\mu Z.\, \Psi$ is similar.

For the other half of the theorem, suppose that $\forall \Phi \in \mu M^k_n(X_1, \ldots X_m)$. if $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \Phi$ then $F \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \Phi$. We show that player II has a winning strategy for $\mathcal{G}^k_n(E, \overline{U}_i, F, \overline{V}_i)$. It is in this half of the proof that we appeal to the restriction that $\mathcal{A}$ is a finite set. Again the proof is by induction on $k+n$. The base case is $k+n=0$. Player I can only win if E is coloured $\mathcal{C}_j$ and F is not. But this contradicts that if $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} X_j$ then $F \models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} X_j$. Suppose it holds for $k+n \leq l$. Consider the game where $k+n = l+1$ and assume that player I has a winning strategy. There are four cases according to the initial move that player I makes under her winning strategy. First, is a $\langle a \rangle$ move, and so $n \geq 1$. Suppose player I chooses $E \xrightarrow{a} E'$. If there are no available transitions from F then there is a contradiction, as $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \langle a \rangle \mathtt{tt}$ and $F \not\models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \langle a \rangle \mathtt{tt}$ (and as $n \geq 1$, $\langle a \rangle \mathtt{tt}$ $\in \mu M^k_n(X_1, \ldots X_m)$). Otherwise assume that $\{F' : F \xrightarrow{a} F'\} = \{F_1, \ldots\}$. We know that player I can win each game $\mathcal{G}^k_{n-1}(E', \overline{U}_i, F_i, \overline{V}_i)$. By the induction hypothesis there are formulas $\Phi_1, \ldots \in \mu M^k_{n-1}(X_1, \ldots X_m)$ such that $E' \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \Phi_l$ and $F_l \not\models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \Phi_l$. However there are only finitely many different $\mu M^k_{n-1}(X_1, \ldots X_m)$ formulas (up to logical equivalence). So $E' \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \bigwedge \Phi_l$ and $F_l \not\models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \bigwedge \Phi_l$ and $\bigwedge \Phi_l \in \mu M^k_{n-1}(X_1, \ldots X_m)$.

But then $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \langle a \rangle \bigwedge \Phi_l$ and $F \not\models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \langle a \rangle \bigwedge \Phi_l$ where $\langle a \rangle \bigwedge \Phi_l \in \mu M^k_n(X_1, \ldots X_m)$ which is a contradiction. So player I cannot have a winning strategy with initial $\langle a \rangle$ move. The second case, an initial $[a]$ move, is similar. The third case is that player I chooses a ν move, so $k \geq 1$, and she colours $\mathcal{E} \subseteq \mathcal{P}(E)$ (containing E) with $\mathcal{C}_{m+1}$ For every colouring choice for player II $\mathcal{F}_1, \ldots$ with $F \in \mathcal{F}_l$ player I can choose $F_l \in \mathcal{F}_l$ such that for every choice by player II of $E_{lj} \in \mathcal{E}$, player I has a winning strategy for $\mathcal{G}^{k-1}_n(E_{lj}, \overline{U}_i\mathcal{E}, F_l, \overline{V}_i\mathcal{F}_l)$. Hence by the induction hypothesis there are formulas $\Phi_{l1}, \Phi_{l2}, \ldots$ for the choice by player I of $F_l \in \mathcal{F}_l$ such that $E_{lj} \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i][\mathcal{E}/Z]} \Phi_{lj}$ and $F_l \not\models_{\mathcal{V}[\overline{V}_i/\overline{X}_i][\mathcal{F}_l/Z]} \Phi_{lj}$. Each $\Phi_{lj} \in \mu M^{k-1}_n(X_1, \ldots X_m, Z)$. There are only finitely many such formulas up to equivalence. Hence, for all $E \in \mathcal{E}$, $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i][\mathcal{E}/Z]} \bigwedge_l(\bigvee_j \Phi_{lj})$, and so, as $E \in \mathcal{E}$, $E \models_{\mathcal{V}[\overline{U}_i/\overline{X}_i]} \nu Z. \bigwedge_l(\bigvee_j \Phi_{lj})$, and by definition we know that $F \not\models_{\mathcal{V}[\overline{V}_i/\overline{X}_i]} \nu Z. \bigwedge_l(\bigvee_j \Phi_{lj})$ even though $\nu Z. \bigwedge_l(\bigvee_j \Phi_{lj}) \in \mu M^k_n(X_1, \ldots X_m)$. This contradicts that player I has a winning strategy with an initial ν move. The final case is when player I makes a μ move, and the argument is similar. □

Theorem 1 of section 4 is a corollary of this result, in the case when $m = 0$, and using the observation that closed formulas of μM are closed under negation.

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7 References

[1] Abramsky, S., and Jagadeesan, R. (1994). Games and full completeness for multiplicative linear logic. *Journal of Symbolic Logic*, **59**, 543-574.

[2] Arnold, A., and Niwinski, D. (1990). Fixed point characterization of Büchi automata on infinite trees. *J. Inf. Process. Cybern.*, **EIK 26**, 451-459.

[3] Baeten, J. and Weijland, W. (1990). *Process Algebra.* Cambridge Tracts in Theoretical Computer Science, 18.

[4] Bosse, U. (1992). An "Ehrenfeucht-Fraïssé game" for fixpoint and stratified fixpoint logic. *Lecture Notes in Computer Science*, **702**, 100-114.

[5] Bradfield, J. (1995). The modal mu-calculus alternation hierarchy is strict. *Submitted for publication.*

[6] Condon, A. (1992). The complexity of stochastic games. *Information and Computation*, **96**, 203-224.

[7] Emerson, E., and Jutla, C. (1988). The complexity of tree automata and logics of programs. Extended version from FOCS '88.

[8] Fagin, R., Stockmeyer, L., and Vardi, M. (1995). On monadic NP vs monadic co-NP. *Information and Computation*, **120**, 78-92.

[9] Hodges, W. (1993). *Model Theory.* Cambridge University Press.

[10] Immermann, N. (1986). Relational queries computable in polynomial time. *Information and Control*, **68**, 86-104.

[11] Kaivola, R. (1995). On modal mu-calculus and Büchi tree automata. *Information Processing Letters*, **54**, 17-22.

[12] Kozen, D. (1983). Results on the propositional mu-calculus. *Theoretical Computer Science*, **27**, 333-354.

[13] Milner, R. (1989). *Communication and Concurrency.* Prentice Hall.

[14] Moschovakis, Y. (1989). A game theoretic modelling of concurrency. *Procs. 4th IEEE Symposium on Logic in Computer Science.*

[15] Nielsen, M. and Clausen, C. (1994). Bisimulations, games and logic. *Lecture Notes in Computer Science*, **812**, 289-306.

[16] Stirling, C. (1993). Modal and temporal logics for processes. *Notes for Summer School in Logical Methods in Concurrency.* Department of Computer Science, Aarhus University.

[17] Stirling, C. (1995). Local model checking games. *Lecture Notes in Computer Science*, **962**, 1-11.

[18] Thomas, W. (1993). On the Ehrenfeucht-Fraïssé game in theoretical computer science. *Lecture Notes in Computer Science*, **668**.