A Note on the Hash Function of Tillich and Zémor

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Extended Abstract¹

At Crypto 1994, Tillich and Zémor proposed the following hash function [1], based on the group $SL_2(\mathbb{F}_{2^n})$:

We denote the field with 2^n elements with $\mathbb{F}_{2^n}$. Let

$$A := \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B := \begin{pmatrix} \alpha & \alpha + 1 \\ 1 & 1 \end{pmatrix}$$

be elements of $SL_2(\mathbb{F}_{2^n})$, the group of 2×2 -matrices with determinant 1 over $\mathbb{F}_{2^n}$. Let $\alpha \in \mathbb{F}_{2^n}$ denote the zero of a generating polynomial $p(x)$ of the field $\mathbb{F}_{2^n} \simeq \mathbb{F}_2[x]/p(x)$. The hash value $h \in \mathbb{F}_{2^n}^{2 \times 2}$ of a binary stream $a_0 \dots a_m$ is calculated as the product $M_0 \dots M_m$ with

$$M_i := \begin{cases} A & \text{if } a_i = 0, \\ B & \text{if } a_i = 1. \end{cases}$$

We analyse this hash function by embedding the two matrix rings generated by A and B respectively, as subrings of $\mathbb{F}_{2^n}^{2 \times 2}$. Using this technique, clashing sequences can be found by calculating discrete logarithms in the fields $\mathbb{F}_{2^n}$ or $\mathbb{F}_{2^{2n}}$. These clashing sequences have the special structure of very long sequences of zeros and ones; thus this “attack” can not produce reasonable collisions. Nevertheless this result exposes an additional structure of the underlying ring and thus reduces the security of this hash function.

1 Embeddings

Properties of the matrices A and B will be discussed for A only. For B similar relations hold. We employ the matrix representation of $\mathbb{F}_{(2^n)^2}$ over $\mathbb{F}_{2^n}$ to show that

$$V := \mathbb{F}_{2^n} \cdot A + \mathbb{F}_{2^n} \cdot I$$

is a field with $(2^n)^2$ elements, if $x^2 + \alpha x + 1 \in \mathbb{F}_{2^n}[x]$ is irreducible. Here, $I \in \mathbb{F}_{2^n}^{2 \times 2}$ denotes the identity matrix.

If $(x + \beta) \left(x + \frac{1}{\beta}\right) = x^2 + \alpha x + 1 \in \mathbb{F}_{2^n}[x]$ is reducible, then we define an equivalence relation on the algebra V by:

¹ The full paper appears in “Proceedings of Cryptography and Coding; Cirencester; 1995” [2]

Lemma 1. Let $M_1 = \lambda_1 A + \mu_1 I$, $M_2 = \lambda_2 A + \mu_2 I \in V$. Then

$$M_1 \simeq M_2 : \Longleftrightarrow \lambda_1 \cdot \beta + \mu_1 = \lambda_2 \cdot \beta + \mu_2$$

defines an equivalence relation on the algebra V .

With this relation it is easy to show that $V/\simeq$ is a field with 2^n elements. In both the reducible and irreducible case, A and its powers can now be regarded as elements of a finite field. The main properties are summarised in the following

Theorem 2. Let $A := \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{F}_{2^n})$ and let $\hat{A}(x) := x^2 + \alpha x + 1 \in \mathbb{F}_{2^n}$ factor into $\hat{A}(x) = (x + \beta)(x + \frac{1}{\beta})$. Then a matrix $M \in \mathbb{F}_{2^n}^{2 \times 2}$ is a power of A , iff the following conditions hold:

- $M = \lambda A + \mu I$ for some $\lambda, \mu \in \mathbb{F}_{2^n}$,
- $\det(M) = 1$ and
- $M^{\text{ord}(A)} = 1$.

2 Clashing sequences

With the results of the previous section it is very easy to find clashing sequences for the SL_2 hash function:

- Choose $C \in SL_2(\mathbb{F}_{2^n})$ with a given factorization into A 's and B 's and a small integer $l (= 3, 4)$, and assume

$$A^{i_1} B^{i_2} \dots A^{i_l} = (\lambda_1 A + \mu_1 I) \cdot (\lambda_2 B + \mu_2 I) \cdot \dots \cdot (\lambda_l A + \mu_l I) = C.$$

- This gives 4 polynomial equations of degree l . Add the l equations for the determinants, i.e. $\det(\lambda_i A + \mu_i I) = 1$, and solve this system of polynomial equations over $\mathbb{F}_{2^n}$.
- Check the conditions $(\lambda_i A + \mu_i I)^{\text{ord}(A)} = 1$ or $(\lambda_i B + \mu_i I)^{\text{ord}(B)} = 1$. If they are not fulfilled, try again, otherwise calculate the discrete logarithms of the matrices calculated.

I have implemented this method in AXIOM and for the small example $\mathbb{F}_{2^{21}}$ with defining polynomial $p(x) = x^{21} + x^2 + 1$, I have found lots of clashing sequences. As expected the sequences are all very long; one of the shorter ones is: $A^{79670} = B^3 \cdot A^7 \cdot B^{69216} \cdot A \cdot B^{88234}$, where $\text{ord}(A) = 2097151$ and $\text{ord}(B) = 699051$.

References

1. J-P. Tillich, G. Zémor; *Hashing with SL_2* ; Proceedings of CRYPTO'94, Y. Desmedt (Ed.), LNCS Vol 839, Springer, pp. 40–49, 1994.
2. W. Geiselmann; *A Note on the Hash Function of Tillich and Zémor*, Proceedings of the 5th IMA Conference on Cryptography and Coding, C. Boyd (Ed.), LNCS Vol 1025, Springer, pp. 257–263, 1995.