Biomedical Applications II

Bayesian Image Reconstruction Using a High-Order Interacting MRF Model*

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Abstract. We propose a Markov Random Field image model formulated using both low-order and high-order interactions of the pixels for piecewise smooth image reconstruction. We discuss and compare the applications of two Bayes estimators, namely the Maximum a Posteriori (MAP) estimator and the Minimum Mean Squares Error (MMSE) estimator, to Positron Emission Tomography (PET) imaging. The usefulness of the high-order interactions is demonstrated especially in the reconstruction of noisy image regions with small supports. The MAP and MMSE estimates are obtained in an efficient way by making use of the noise properties of the images obtained using the Filtered Backprojection (FBP) algorithm. We illustrate the efficacy of our overall approach by simulation experiments.

1 Introduction

The use of *Markov Random Field* (MRF) or *Gibbs* image priors in the Bayesian framework of image analysis has attracted much attention in the past in many different areas since the appearance of [2, 7]. Many of the previously proposed Gibbs priors typically include only nearest neighbor correlational information. The use of higher order interactions has not been very much investigated. Most pairwise interacting models have mainly a smoothing effect on the recovered images. (Sometimes the smoothing is mitigated for sharp intensity changes, as in [8].) It has been observed that randomly sampled images from the distributions specified by such priors have typically a uniform appearance [2, 9] and that the models are often inadequate in a statistical sense [10]. This issue has been recently investigated in [10], where a statistical model with high-order interactions for a particular class of piecewise constant images was proposed. We generalize this type of model in the current paper to model piecewise smooth images, in which we use 3×3 cliques to model the borders between regions. Borders in the images have been modeled by a dual MRF composed of unobservable line elements [7]. The derivation of an equivalent energy function formulated using only the pixel values was found possible in some cases [5], but not when there are interactions between line elements [1, 5].

Using the proposed prior, we discuss and compare the applications of two Bayes estimators, namely the *Maximum a Posteriori* (MAP) estimator and the *Minimum Mean Squares Error* (MMSE) estimator, in *Positron Emission Tomography* (PET). A method to remedy the computational difficulties encountered by previously proposed algorithms for such problems will also be discussed. We illustrate the usefulness of the high-order interactions especially in the reconstruction of noisy image regions with small supports.

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2 A High-Order Markov Random Field Image Model

A MRF image model is provided by the Gibbs distribution of the general form:

$$\Pi(X) = \frac{1}{Z} e^{-\beta H(X)}. \quad (1)$$

Here $X = (x_1, \dots, x_N)$ is an image consisting of N pixels, with the gray level of the n th pixel given by x_n . $H(X)$ is an energy function to be expressed as a weighted sum of *clique* potentials. β is a parameter which controls the “peakiness” of the distribution and can be interpreted as the inverse of the *temperature* in statistical mechanics.

Following the development of an image model of a class of piecewise constant images which uses a MRF with interactions not constrained to those between pairs of pixels [4, 10], we present a generalization of the model with a similar neighborhood structure for piecewise smooth images with 256 gray levels.

First, we write the energy function as $H(X) = H_1(X) + H_2(X)$. The first term, which models smoothness, is

$$H_1(X) = -\left(\sum_{C \in \mathcal{C}_+} \lambda_1 V_C(X) + \sum_{C \in \mathcal{C}_x} \lambda_2 V_C(X) \right), \quad (2)$$

with

$$V_{\{j_1, j_2\}}(X) = \exp\left(-\frac{(x_{j_1} - x_{j_2})^2}{2\delta^2}\right), \quad (3)$$

where $\mathcal{C}_+$ denotes the set of all horizontal and vertical pair cliques, whereas $\mathcal{C}_x$ denotes that of the diagonal ones. The coefficients λ_1 and λ_2 control the strengths of each type of interaction and δ controls the degree of smoothing. The second term, which models the borders between regions, is

$$H_2(X) = - \sum_{C \in \mathcal{C}_{3 \times 3}} (\kappa_1 I_C^1(X) + \kappa_2 I_C^2(X) + \kappa_3 I_C^3(X)), \quad (4)$$

with

$$I_C^1(X) = \begin{cases} 1, & \text{if } |x_k - x_c| \leq \delta \ \forall k \in \{w, nw, n, ne, e\} \\ & \text{and } |x_k - x_s| \leq \delta \ \forall k \in \{sw, se\} \text{ and } |x_n - x_s| > \Delta, \\ \vdots & \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

(I_C^2 and I_C^3 are similarly defined as depicted in Fig. 1.) The subscripts c, n, e, s, w (denoting center, north, east, south and west, respectively) are used to index the pixels in a 3×3 clique, whereas $\mathcal{C}_{3 \times 3}$ denotes the set of all cliques with 3×3 block of pixels. The case in (5) (three others omitted), for example, is depicted by the left clique labeled “horizontal” in which the pixels labeled by ‘x’ have gray levels “similar” to those labeled by ‘a’. Each term models borders in a particular set of orientations and the coefficients κ_1, κ_2 and κ_3 control the strength of each type of interaction. Δ is a threshold for an edge and δ is defined as before. Note that by using 3×3 cliques, borders at intermediate orientations can also be modeled.

Fig. 1. Pictorial clique potential definition for borders in different orientations. Pixels labeled by ‘a’ and ‘b’ have “dissimilar” gray levels. Pixels labeled by ‘x’ in a clique all have gray levels “similar” to those labeled by either ‘a’ or ‘b’.

To encourage spatially isotropic neighbor interactions we set $\lambda_2 = \frac{1}{\sqrt{2}}\lambda_1$. For the “border elements”, we set $\kappa_1 = \kappa_2 = \frac{1}{2}\kappa_3$ since the vertical, horizontal and diagonal border elements always appear in pairs at the borders between two regions. With properly determined parameters, the model not only encourages the formation of smooth regions, but also sharp transitions at locations which make up the borders between regions.

3 Bayesian Image Reconstruction

3.1 The Posterior Distribution

We discuss the application of the proposed prior in PET image reconstruction. The measurement model in PET imaging is given by the likelihood function of the form [12]:

$$L(Y|X) = \prod_{i=1}^I \frac{l_i^{y_i} e^{-l_i}}{y_i!}, \quad (6)$$

where $l_i = \sum_j l_{ij} x_j$, J is the total number of pixels and I the number of measurements.

Under the Bayesian framework, the posterior distribution of the images is given by

$$P(X|Y) \propto \Pi(X)L(Y|X), \quad (7)$$

where $\Pi(X)$ is the prior distribution given by (1).

3.2 Bayes Estimators

We apply two Bayes estimators to the image reconstruction problem. The MAP estimate is defined by the mode of the posterior distribution, which can be obtained by maximizing (7), or equivalently, by performing the following minimization:

$$\min_X \left[\sum_{i=1}^I (l_i - y_i \ln(l_i)) + \beta H(X) \right]. \quad (8)$$

The MMSE estimate is defined by the posterior mean or the expected value (image) over the posterior distribution

$$E[X|Y] = \sum P(X|Y) \cdot X. \quad (9)$$

The MAP estimate is difficult to obtain because of the nonconvexity of $H(X)$. One approach is to use *simulated annealing* (SA) with the Gibbs Sampler [7]. However, this

is computationally expensive especially because (7) contains the likelihood function (6) whose neighborhood structure is of a large spatial extent. The calculation of the MMSE estimate is also difficult, because of the complexity of the normalizing factor of the Gibbs prior. A stochastic approach is to sample the posterior distribution using the Gibbs Sampler. The iterates $X^{(k)}$ form a Markov chain with the equilibrium distribution (7). The ergodicity of the chain guarantees that an ergodic average will converge to the MMSE estimate [7], the value of which in practice can be estimated by calculating the average of N samples $\{X^{(k)} : k = K + r \times t \text{ for } r = 1, \dots, N\}$ for some large enough values of t and K (the number of *thermalization* steps). Again this sampling process is expensive because of the involvement of the likelihood function (6).

3.3 PET Reconstruction via Restoration

In view of the above computational demands associated with the measurement model (6), instead of using the original PET measurements and this measurement model in the Bayesian estimations, we propose to use the image produced from the PET measurements by the standard *Filtered Backprojection* (FBP) algorithm [11] as input to be restored according to an appropriate model of the combined measuring-and-reconstruction process. We performed a simulation study of the properties (including both blurring and noise) of the images produced by FBP. We model the noise using the Gaussian model:

$$L'(Y|X) = \prod_{j=1}^J \exp \left(-\frac{(y_j - x_j - \mu(x_j, \bar{x}_{\eta_j}))^2}{2\sigma^2(x_j)} \right), \quad (10)$$

where x is the original value of a pixel, $\bar{x}_{\eta}$ the average value of its eight nearest neighbors, y is the value assigned by FBP to the pixel, σ is the standard deviation of the noise, and the bias (which captures the blurring) is empirically estimated using a third order polynomial $\mu(x, \bar{x}_{\eta}) = \sum_{i=1}^3 a_i x^i + \sum_{i=1}^3 b_i (\bar{x}_{\eta} - x)^i$. The coefficients a_i 's and b_i 's, and hence σ , can be determined from training data based on non-linear least-squares techniques. The computational demand associated with (7) in sampling using the Gibbs Sampler is much reduced because of the simpler (5×5) neighborhood structure of $L'(Y|X)$. Although this model is only an approximate one [13], we found it very effective in our experiments.

4 Experiments

We simulated the measurement process of PET imaging using a mathematical phantom under the same conditions as those described in [3] (involving 3,000,000 coincidence counts, 300×151 projections with independent Poisson noise). The phantom consists of an elliptical background (gray level equals to 100) with elliptical and rectangular blobby features (the gray level of each equals to either 195 or 200) in it. Fig. 2a displays the digitized phantom on the 95×95 pixel grid on which the reconstructions were performed.

We first apply the FBP algorithm to the PET measurements. We then calculate a Bayes estimate based on (7) with $L(Y|X)$ replaced by the $L'(Y|X)$ of (10). The values of $(a_1, a_2, a_3, b_1, b_2, b_3)$ are estimated from a training data set, which is produced by generating 30 sets of random projection data from the phantom and performing FBP on

Fig. 2. (a) Original phantom image (b) FBP reconstruction (c) MAP estimate with the proposed prior (d) MAP estimate with the pairwise interacting prior (e) MMSE estimate with the proposed prior. (f) MMSE estimate with the pairwise interacting prior.

each set. The coefficients obtained by a least squares fit were (0.002, -2.36e-05, -6.31e-08, 0.698, 3.39e-04, 2.02e-05).

To obtain the MAP estimate, we employed an approximate version of the Gibbs Sampler algorithm (referred to as the 5-union algorithm in [6]) in the simulated annealing, with an annealing schedule of the form $T_k = 3.5/\log(1+k)$ (k is the number of sweeps), for 500 sweeps. For the MMSE estimate, 250 thermalization steps were used and images subsequently generated (with $T_k = 1.0$) every 25th iterations (for a total of 10 samples) were averaged. The parameters δ and Δ were set to 3.0 and 75.0, respectively. (These gave consistently good results.) Furthermore, we set $\lambda_1 = 1.0$, $\kappa_1 = 0.94$ and $\beta = 1.56$ in the prior. (These choices produce piecewise smooth image realizations similar to those discussed in [10].)

Fig. 2c shows the final MAP estimate obtained using our prior, whereas Fig. 2d is obtained by replacing the prior with a pairwise interacting prior ($\kappa_i = 0.0$, $\lambda_1 = 1.0$ and $\beta = 1.60$, which produced the best result in terms of the mean squares error). The mean squares errors are 240.1 and 301.0 respectively. It can also be observed that the proposed model did a better job especially at the smaller regions where the spatial supports are small. Fig. 2e and Fig. 2f show the corresponding MMSE estimates and similar observations regarding their relative performances can be made; with the mean squares errors being 186.3 and 204.1 respectively. We also observe in each case that the MMSE estimate is better than the corresponding MAP estimate. It appears that the MMSE estimator performs better at the borders between adjacent regions.

5 Conclusions

We have proposed using a MRF prior with low-order and high-order interactions in Bayesian image reconstruction problems. The desirability of having the higher order terms in the model for modeling the borders has been demonstrated. We have discussed and compared the applications of two Bayes estimators for the PET reconstruction problem. A method which is computationally less expensive than employing stochastic methods directly was proposed for finding both the MAP and MMSE estimates. We have illustrated the efficacy of our overall approach using simulated PET data.

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