Relations as Abstract Datatypes: An Institution to Specify Relations between Algebras^{*}

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Abstract. One way to view the execution state of an imperative program is as a many sorted algebra. Program variables are (constant) functions and their types are sorts. The execution of a program defines a relation between the state of the program (algebra) before and after the execution of the program. In this paper we shall define an institution for the specification of relations between structures of some base institution (eg. the institution of equational logic or first order predicate logic). Sets of structures over a common signature, abstract datatypes, in this institution denote relations between structures of the base institution. This makes it possible to apply a rich repertoire of existing techniques for specifying abstract datatypes to the specification of relations. This paper tries to narrow the gap between algebraic specification languages like Clear, ASL or Act-One and model theoretic based specification languages like Z, VDM-SL or the Larch Interface language.

1 Introduction

States consisting of variables $x_1 \dots x_n$ of sort $s_1 \dots s_m$ can be seen as algebras over a many sorted signature $\Sigma = \langle S, Op \rangle$ having sorts $S = \{ s_i \mid i = 1 \dots m \}$ and operations $Op = \{ x_i : s_i \mid i = 1 \dots n, s_i \in S \}$, possibly satisfying some state invariants.

As an example consider the specification of a birthday book, taken from the Z reference manual [17]. The goal is to specify a system to record people's birthdays. The state of a birthday book consists of *known*, a set of names of peoples whose birthdays are recorded, and *birthday*, a set of associations of names to dates. The state invariant is that *known* is exactly the set of names occurring as a first component in the elements of *birthday*. Using a notation adapted from Z, the state of a birthday book is described by the specification *Birthdaybook*

^{*} The research described in this paper was supported by the German Ministry for Research and Technology (Bundesministerium für Forschung und Technologie) under grant ITS 9103 and by the ESPRIT Basic Research Working Group 6112 COMPASS (A Comprehensive Algebraic Approach to System Specification and Development).

Birthdaybook

Env

known : SETOFNAME

birthday : SETOFBBENTRY

$\forall n : NAME \quad n \in \text{known} \quad \text{iff}$
 $\exists \text{entry} : BBENTRY \quad \text{entry} \in \text{birthday} \quad \text{and} \quad \text{name}(\text{entry}) = n$

Birthdaybook is based on the specification *Env*

Env

BEntry

Set(*Name*)[*SETOFNAME* $\leftarrow$ *SET*]

Set(*BEntry*)[*SETOFBBENTRY* $\leftarrow$ *SET*]

and *BEntry*

BEntry

Name, Date

bbEntry : *NAME* $\times$ *DATE* $\rightarrow$ *BBENTRY*

name : *BBENTRY* $\rightarrow$ *NAME*

date : *BBENTRY* $\rightarrow$ *DATE*

name(*bbEntry*(*n*, *d*)) = *n*

date(*bbEntry*(*n*, *d*)) = *d*

and where the specifications *Name* and *Date* merely define the sorts *NAME* and *DATE*. *Set*(*Name*)[*SETOFNAME* $\leftarrow$ *SET*] is the notation for the application of the parameterized specification *Set* to the specification *Name* and then renaming the sort *SET* in the result to *SETOFNAME*.² Algebras satisfying the *Birthdaybook* specification are possible states of a birthday book.

Operations modifying the state of a system are interpreted as functions on algebras or, more generally, as relations between algebras. For example, the operation *AddBirthday*(*name*:*NAME*; *date*:*DATE*) adds a new entry to the birthday book if the birthday for *name* is not already known. More precisely, *AddBirthday*(*n*, *d*) defines, for a name *n* and a date *d*, a relation between *Birthdaybook* algebras *A* and *B* such that $\text{known}^B = \text{known}^A \cup \{n\}$, $\text{birthday}^B = \text{birthday}^A \cup \{\text{bbEntry}^A(n, d)\}$ and such that *A* and *B*, when restricted to their environment, the specification *Env*, coincide.

More formally we may define specifications

AddBirthdayIn

Birthdaybook

name? : *NAME*

date? : *DATE*

² We allow for operations to have the same name, as long as they can be distinguished by their types. For instance the *insert* operations from *Set*(*Name*)[*SETOFNAME* $\leftarrow$ *SET*] and *Set*(*BEntry*)[*SETOFBBENTRY* $\leftarrow$ *SET*] are considered different.

and

$AddBirthdayOut$ $Birthdaybook$

and consider $AddBirthday$ as a relation between $AddBirthdayIn$ algebras A and $AddBirthdayOut$ algebras B satisfying $A|_{Env} = B|_{Env}$ and the equations

$$known^B = known^A \cup \{name?^A\} \quad \text{and}$$

$$birthday^B = birthday^A \cup \{bbEntry^A(name?^A, date?^A)\}.$$

The additional constants $name?$ and $date?$ in $AddBirthdayIn$ model the input parameters of the operation $AddBirthday$. In case $AddBirthday$ is supposed to deliver a result $x!$ of sort s , a constant $x! : s$ is added to $AddBirthdayOut$. Next we observe that A and B can be extended to an algebra C , a model of

$AddBirthday$ $AddBirthdayIn$ $AddBirthdayOut[known' \leftarrow known, birthday' \leftarrow birthday]$ <hr/> $name? \notin known$ $birthday' = insert(bbEntry(name?, date?), birthday)$

such that

$$C|_{AddBirthdayIn} = A \quad \text{and} \quad C|_{AddBirthdayOut} = B.$$

Note, that there is no need to include an axiom of the form

$$known' = insert(name?, known)$$

explicitly in $AddBirthday$ because this is a consequence of the axiom

$$birthday' = insert(bbEntry(name?, date?), birthday)$$

together with the axioms of $Birthdaybook$.

On the other hand, the specification $AddBirthday$ defines a relation R between states of birthday book (models of $Birthdaybook$) by

$$A R B \quad \text{iff} \quad \exists C \quad C \models AddBirthday, \quad C|_{\iota} = A \quad \text{and} \quad C|_{\sigma} = B,$$

where ι is the inclusion of $Birthdaybook$ into $AddBirthday$ and σ is a morphism from $Birthdaybook$ to $AddBirthday$ mapping $known$ to $known'$ and $birthday$ to $birthday'$. Therefore, relations between algebras can be regarded as sets of algebras specified using standard abstract datatypes techniques.

In the rest of this paper we shall define an institution $\mathcal{R}_D$ over some base institution $\mathcal{I}$ such that sets of formulas in this institution denote relations between structures from $\mathcal{I}$, which formally are abstract datatypes in the institution $\mathcal{R}_D$. Abstract datatypes are pairs $\langle \Sigma, M \rangle$, where Σ is a signature and M a set of Σ -structures. We shall show that this institution inherits some nice properties from its base institution, like for example a cocomplete category of signatures and amalgamation, which implies extension.

2 Preliminaries

2.1 Institutions

The notion of an institution was introduced by Goguen and Burstall to define the semantics of the specification language Clear [3] independent of an underlying logic. Here we will only recall the definition of institutions, for more details see the paper by Goguen and Burstall [10]. An institution $\mathcal{I} = \langle SIGN, Mod, Sen, \models \rangle$ consists of

- a category of *signatures* $SIGN$,
- a functor $Mod: SIGN \rightarrow CAT^{op}$ assigning to each signature Σ the category of Σ -structures and to each signature morphism $\sigma: \Sigma \rightarrow \Sigma'$ a forgetful functor $Mod(\sigma): Mod(\Sigma') \rightarrow Mod(\Sigma)$,³
- a functor $Sen: SIGN \rightarrow SET$, assigning to each signature Σ the set of Σ -formulas and to each signature morphism $\sigma: \Sigma \rightarrow \Sigma'$ a translation $Sen(\sigma)$ of Σ -formulas to Σ' -formulas,⁴ and
- a family of *satisfaction relations* $(\models_{\Sigma} \subseteq Mod(\Sigma) \times Sen(\Sigma))_{\Sigma \in SIGN}$ indicating whether a Σ -formula f is valid in a Σ -structure m ($m \models_{\Sigma} f$ or for short $m \models f$)

where the *satisfaction condition* holds: for all signature morphisms $\sigma: \Sigma \rightarrow \Sigma'$, formulas $f \in Sen(\Sigma)$ and structures $m' \in Mod(\Sigma')$

$$m'|_{\sigma} \models f \quad \text{iff} \quad m' \models \sigma(f)$$

Each institution has associated two categories $PRES_{\mathcal{I}}$ and $ADT_{\mathcal{I}}$. The objects of $PRES_{\mathcal{I}}$ are pairs $\langle \Sigma, F \rangle$ where Σ is a signature from $SIGN$ and F a set of Σ -formulas. Morphisms $\sigma: \langle \Sigma, F \rangle \rightarrow \langle \Sigma', F' \rangle$ in $PRES_{\mathcal{I}}$ are signature morphisms such that $F' \models \sigma(F)$. In $ADT_{\mathcal{I}}$ objects are pairs $\langle \Sigma, M \rangle$ where M is a set of Σ -structures and morphisms $\sigma: \langle \Sigma, M \rangle \rightarrow \langle \Sigma', M' \rangle$ are signature morphisms such that $M'|_{\sigma} \subseteq M$. There are also forgetful functors $Sig^p: PRES_{\mathcal{I}} \rightarrow SIGN$ and $Sig^a: ADT_{\mathcal{I}} \rightarrow SIGN$ defined by

$$Sig^p(\langle \Sigma, F \rangle) = Sig^a(\langle \Sigma, M \rangle) = \Sigma \quad \text{and} \quad Sig^p(\sigma) = Sig^a(\sigma) = \sigma$$

It is known (eg. [15, 2]) that the functors Sig^p and Sig^a create (finite) (co)limits and that as a consequence $PRES_{\mathcal{I}}$ and $ADT_{\mathcal{I}}$ are (finitely) (co)complete if the category of signatures $SIGN$ is.

Objects in $PRES_{\mathcal{I}}$ can be thought of as *specifying* or *presenting* objects in $ADT_{\mathcal{I}}$, which are called *abstract datatypes*. Thus, there is a semantics function $sem: PRES_{\mathcal{I}} \rightarrow ADT_{\mathcal{I}}$ associating to a presentation $\langle \Sigma, F \rangle$ an abstract datatype $\langle \Sigma', M' \rangle$. Two commonly used semantic functions are *loose* and *initial*, where

$$loose(\langle \Sigma, F \rangle) = \langle \Sigma, \{ m \in Mod(\Sigma) \mid m \models F \} \rangle$$

and $initial(\langle \Sigma, F \rangle) = \langle \Sigma, M \rangle$ with M the set of all objects that are initial in the full subcategory of $Mod(\Sigma)$ whose objects satisfy F .

³ The application of $Mod(\sigma)$ to a Σ' -structure m' will be written as $m'|_{\sigma}$. If σ is understood from the context we may write $m'|_{\Sigma}$ instead.

⁴ The application of $Sen(\sigma)$ to a formula f will be abbreviated by $\sigma(f)$.

2.2 Limits and Colimits

This section defines the notations used in this paper for limits and colimits in categories. For a definition of limits and colimits refer, for example, to the book of MacLane [14].

We use the notation $\prod_{\mathbf{I}} F$ to denote the limit object of the functor $F: \mathbf{I} \rightarrow \mathbf{C}$, if it exists, and π_i^F , or π_i if there is no ambiguity, for the projections $\pi_i^F: \prod_{\mathbf{I}} F \rightarrow F(i)$. If $G: \mathbf{I} \rightarrow \mathbf{C}$ is another functor with a limit and $\tau: F \Rightarrow G$ is a natural transformation, then $\prod_{\mathbf{I}} \tau$ denotes the unique morphism from the limit object $\prod_{\mathbf{I}} F$ to the limit object $\prod_{\mathbf{I}} G$ such that for all $i \in \mathbf{I}$ the following diagram commutes

$$\begin{array}{ccc} \prod_{\mathbf{I}} F & \xrightarrow{\prod_{\mathbf{I}} \tau} & \prod_{\mathbf{I}} G \\ \pi_i^F \downarrow & & \downarrow \pi_i^G \\ F(i) & \xrightarrow{\tau_i} & G(i) \end{array}$$

The colimit object of a functor $F: \mathbf{I} \rightarrow \mathbf{C}$ is denoted by $\sum_{\mathbf{I}} F$ and ι_i^F , or ι_i , denotes the inclusions $\iota_i^F: F(i) \rightarrow \sum_{\mathbf{I}} F$. If $G: \mathbf{I} \rightarrow \mathbf{C}$ is another functor with a colimit and $\tau: F \Rightarrow G$ is a natural transformation, then $\sum_{\mathbf{I}} \tau$ denotes the unique morphism from the colimit object $\sum_{\mathbf{I}} F$ to the colimit object $\sum_{\mathbf{I}} G$ such that for all $i \in \mathbf{I}$ the following diagram commutes

$$\begin{array}{ccc} \sum_{\mathbf{I}} F & \xrightarrow{\sum_{\mathbf{I}} \tau} & \sum_{\mathbf{I}} G \\ \iota_i^F \uparrow & & \uparrow \iota_i^G \\ F(i) & \xrightarrow{\tau_i} & G(i) \end{array}$$

3 Binary Relations

Before presenting the general construction in section 4 we shall illustrate the construction using binary relations. Let us assume we are given an institution $\mathcal{I} = \langle \text{SIGN}, \text{Mod}, \text{Sen}, \models \rangle$ and let $\Theta = \langle \Sigma_1, \Sigma_2 \rangle$ be a pair of signatures from SIGN . A (binary) relation R of type Θ is a subset M of the cartesian product of $\text{Mod}(\Sigma_1)$ and $\text{Mod}(\Sigma_2)$. A morphism $\alpha: R \rightarrow S$ from a relation R of type Θ^R to a relation S of type Θ^S is a pair of signature morphisms $\alpha_1: \Sigma_1^R \rightarrow \Sigma_1^S$ and $\alpha_2: \Sigma_2^R \rightarrow \Sigma_2^S$ such that whenever $m_1 S m_2$ then $m_1|_{\alpha_1} R m_2|_{\alpha_2}$. The category REL of binary relations has relations as objects and morphisms between relations as arrows.

To specify objects from REL we need to define a notion of formulas together with a satisfaction relation between formulas and elements of the cartesian product of $\text{Mod}(\Sigma_1)$ and $\text{Mod}(\Sigma_2)$. A possible definition of such formulas is as elements of the disjoint union of $\text{Sen}(\Sigma_1)$ and $\text{Sen}(\Sigma_2)$. Then for a pair $\langle m_1, m_2 \rangle$ of $\text{Mod}(\Sigma_1) \times \text{Mod}(\Sigma_2)$ and a formula f of $\text{Sen}(\Sigma_1) + \text{Sen}(\Sigma_2)$ we define

$$\langle m_1, m_2 \rangle \models^{\text{REL}} f \quad \text{iff} \quad \begin{cases} m_1 \models f & \text{if } f \in \text{Sen}(\Sigma_1) \\ m_2 \models f & \text{if } f \in \text{Sen}(\Sigma_2) \end{cases}$$

However, with this definition it is not possible to write a formula similar to

$$birthday' = insert(bbEntry(name?, date?), birthday)$$

from the *AddBirthday* example because this formula is neither a formula over the signature of *AddBirthdayIn* nor a formula over the signature of *AddBirthdayOut*, instead, it is a formula over the disjoint union of the signatures of *AddBirthdayIn* and *AddBirthdayOut*.

Therefore, we are interested in formulas f from $Sen(\Sigma_1 + \Sigma_2)$. Now we have the problem to define the satisfaction relation between pairs $\langle m_1, m_2 \rangle$ and formulas f . In the institution of first order logic, and many other interesting institutions, there is a one to one correspondence between pairs of algebras from $Mod(\Sigma_1) \times Mod(\Sigma_2)$ and algebras from $Mod(\Sigma_1 + \Sigma_2)$. A $\Sigma_1 + \Sigma_2$ -algebra m defines a pair of Σ_1 - and Σ_2 -algebras by $\langle m|_{\Sigma_1}, m|_{\Sigma_2} \rangle$ and each pair $\langle m_1, m_2 \rangle$ defines an algebra $m_1 + m_2$ such that the interpretation of a symbol s (a sort, function or predicate symbol) from $\Sigma_1 + \Sigma_2$ in $m_1 + m_2$ is the interpretation of s in m_1 if s originates from Σ_1 or the interpretation of s in m_2 otherwise. This makes it possible to define

$$\langle m_1, m_2 \rangle \models^{REL} f \quad \text{iff} \quad m_1 + m_2 \models f.$$

Given a set of formulas F from $Sen(\Sigma_1 + \Sigma_2)$ a pair $Sp = \langle \Theta, F \rangle$ defines a relation by

$$m_1 Sp m_2 \quad \text{iff} \quad \forall f \in F : \langle m_1, m_2 \rangle \models^{REL} f.$$

The category of specifications of binary relations, *RELSPEC*, has as objects pairs $Sp = \langle \Theta, F \rangle$ and as arrows $\alpha: Sp_R \rightarrow Sp_S$ signature morphisms $\alpha: \Sigma_1^R + \Sigma_2^R \rightarrow \Sigma_1^S + \Sigma_2^S$ such that $F^S \models^{REL} \alpha(F^R)$.

There is a functor *Rel* from *RELSPEC* to *REL* mapping specifications of binary relations to binary relations defined by

$$Rel(\langle \Theta, F \rangle) = \langle \Theta, \{ \langle m_1, m_2 \rangle \mid \langle m_1, m_2 \rangle \models^{REL} F \} \rangle \quad \text{and}$$

$$Rel(\alpha: Sp_R \rightarrow Sp_S) = \alpha: Rel(Sp_R) \rightarrow Rel(Sp_S).$$

The domain of a binary relation $R = \langle \Theta, M \rangle$ in *REL* is given by

$$dom(R) = \{ m_1 \mid \langle m_1, m_2 \rangle \in M \}$$

and its codomain by

$$cod(R) = \{ m_2 \mid \langle m_1, m_2 \rangle \in M \}.$$

3.1 Binary Relations as Abstract Datatypes

In this section we shall define an institution $\mathcal{R}_2$ such that the category of abstract datatypes⁵ ADT_2 (cf. section 2.1) in this institution is the category of binary relations REL and the category of presentations $PRES_2$ is the category of specifications of binary relations $RELSPEC$. The construction of $\mathcal{R}_2$ is parameterized by an institution $\mathcal{I}$. To define the satisfaction relation of $\mathcal{R}_2$ we require that $SIGN$ has coproducts and that Mod preserves them. Now the institution

$$\mathcal{R}_2 = \langle R\text{SIGN2}, R\text{Mod2}, R\text{Sen2}, \models^{\mathcal{R}_2} \rangle$$

is given by:

$$R\text{SIGN2} = \text{SIGN}^2,$$

$$R\text{Mod2}(\Theta: \mathbf{2} \rightarrow \text{SIGN}) = \prod_{\mathbf{2}} \Theta; \text{Mod} \quad R\text{Mod2}(\alpha: \Theta^R \Rightarrow \Theta^S) = \prod_{\mathbf{2}} \alpha; \text{Mod}$$

$$R\text{Sen2}(\Theta) = \text{Sen}(\sum_{\mathbf{2}} \Theta) \quad R\text{Sen2}(\alpha) = \text{Sen}(\sum_{\mathbf{2}} \alpha)$$

$$m \models^{\mathcal{R}_2} f \quad \text{iff} \quad m \models f$$

The category $\mathbf{2}$ is the discrete category having only 1 and 2 as objects and the identities as arrows.

Objects in ADT_2 are pairs $R = \langle \Theta, M \rangle$, where Θ is an element of $R\text{SIGN2}$ and M is a subset of $R\text{Mod2}(\Theta)$. Θ is a functor from $\mathbf{2}$ to SIGN and thus, a pair of signatures $\langle \Theta(1), \Theta(2) \rangle$ from $\text{SIGN} \times \text{SIGN}$. Since $R\text{Mod2}(\Theta)$ is $\text{Mod}(\Theta(1)) \times \text{Mod}(\Theta(2))$, elements of $M \subseteq R\text{Mod2}(\Theta)$ are pairs $\langle m_1, m_2 \rangle$.

An arrow $\alpha: \Theta^R \Rightarrow \Theta^S$ in $R\text{SIGN2}$ is a natural transformation between functors from $\mathbf{2}$ to SIGN and thus, just a pair of signature morphisms α_i from $\Theta^R(i)$ to $\Theta^S(i)$ ($i = 1, 2$). The application of $R\text{Mod2}$ to an arrow α of $R\text{SIGN2}$ yields the pair of functors $\langle \text{Mod}(\alpha_1), \text{Mod}(\alpha_2) \rangle$.

Thus an arrow $\alpha: \langle \Theta^R, M^R \rangle \rightarrow \langle \Theta^S, M^S \rangle$ in ADT_2 is a morphism α from Θ^R to Θ^S in $R\text{SIGN2}$ such that $\langle m_1|_{\alpha_1}, m_2|_{\alpha_2} \rangle \in M^R$ for all $\langle m_1, m_2 \rangle \in M^S$ or, whenever $m_1 \text{ } S \text{ } m_2$ then $m_1|_{\alpha_1} \text{ } R \text{ } m_2|_{\alpha_2}$.

Since SIGN has coproducts, the object $\sum_{\mathbf{2}} \Theta = \Theta(1) + \Theta(2)$ exists for any object Θ in $R\text{SIGN2}$. Moreover, since Mod preserves coproducts

$$\text{Mod}(\Theta(1) + \Theta(2)) = \text{Mod}(\Theta(1)) \times \text{Mod}(\Theta(2)) = R\text{Mod2}(\Theta).$$

Therefore each m in $R\text{Mod2}(\Theta)$ is also in $\text{Mod}(\Theta(1) + \Theta(2))$ and each formula f in $R\text{Sen2}(\Theta)$ is an element of $\text{Sen}(\Theta(1) + \Theta(2))$. Thus the satisfaction relation $m \models^{\mathcal{R}_2} f$ iff $m \models f$ is well defined.

⁵ To avoid subscripts of subscripts we write ADT_2 instead of $ADT_{\mathcal{R}_2}$.

4 An Institution to Specify Relations Between Algebras

The institution $\mathcal{R}_2$ is based on the discrete category **2**. However, the construction of $\mathcal{R}_2$ did not depend on the specific properties of **2**. In fact, the same construction can be used to define an institution

$$\mathcal{R}_{\mathbf{D}} = \langle \mathit{RSIGND}, \mathit{RModD}, \mathit{RSenD}, \models^{\mathcal{R}_{\mathbf{D}}} \rangle$$

based on an arbitrary category **D**, provided that *SIGN* is cocomplete with respect to diagrams of shape **D** and that *Mod* preserves the colimit of diagrams of shape **D**:

$$\begin{aligned} \mathit{RSIGND} &= \mathit{SIGN}^{\mathbf{D}} \\ \mathit{RModD}(\Theta: \mathbf{D} \rightarrow \mathit{SIGN}) &= \prod_{\mathbf{D}} \Theta; \mathit{Mod} & \mathit{RModD}(\alpha: \Theta^R \Rightarrow \Theta^S) &= \prod_{\mathbf{D}} \alpha; \mathit{Mod} \\ \mathit{RSenD}(\Theta) &= \mathit{Sen}(\sum_{\mathbf{D}} \Theta) & \mathit{RSenD}(\alpha) &= \mathit{Sen}(\sum_{\mathbf{D}} \alpha) \\ m \models^{\mathcal{R}_{\mathbf{D}}} f & \quad \text{iff} \quad m \models f \end{aligned}$$

The following easy facts hold:

- $\mathcal{R}_{\mathbf{D}}$ is an institution,
- *RSIGND* is (finitely) cocomplete if *SIGN* is and
- *RModD* preserves (finite) colimits if *Mod* preserves them.

Objects in $\mathit{ADT}_{\mathbf{D}}$ are typed relations of arity **D**. Let, for example, **D** be the discrete category $\mathbf{K}_n$ generated by the set $\{1, \dots, n\}$. An object Θ of RSIGNK_n is an n -tuple $\Sigma_1, \dots, \Sigma_n$ of signatures from *SIGN* and objects in $\mathit{RModK}_n(\Theta)$ are n -tuples $m_1, \dots, m_n$ such that $m_i \in \mathit{Mod}(\Sigma_i)$. Thus an object $R = \langle \Theta, M \rangle$ in $\mathit{ADT}_{\mathbf{K}_n}$ is an n -ary relation $R \subseteq \mathit{Mod}(\Sigma_1) \times \dots \times \mathit{Mod}(\Sigma_n)$.

Now let **D** have an arrow $g: d \rightarrow d'$ other than the identity. Together with signatures $\Theta(d) = \Sigma_d$ and $\Theta(d') = \Sigma_{d'}$ objects Θ in *RSIGND* also provide a signature morphism $\Theta(g) = \sigma: \Sigma_d \rightarrow \Sigma_{d'}$. This imposes on objects m from $\mathit{RModD}(\Theta)$ the restriction that

$$\pi_{d'}(m)|_{\sigma} = m_{d'}|_{\sigma} = m_d = \pi_d(m)$$

5 Relations with Shared Parts

We are interested in binary relations R , such that, if $m_1 R m_2$ then m_1 and m_2 are persistent extensions of some algebra, like an algebra providing some base types such as *bool* and *nat*. This type of relation can be modeled by a three place relation R given by a set $M \subseteq \mathit{Mod}(\Sigma_1) \times \mathit{Mod}(\Sigma_0) \times \mathit{Mod}(\Sigma_2)$ with the requirement that for all $\langle m_1, m_0, m_2 \rangle$ from M

$$m_1|_{\sigma_1} = m_0 = m_2|_{\sigma_2},$$

where $\sigma_i: \Sigma_0 \rightarrow \Sigma_i$ ($i = 1, 2$) define the parts that Σ_1 and Σ_2 have in common.

The type of R is given by a functor Θ from $\mathbf{3}$ to $SIGN$, where $\mathbf{3}$ is the category freely generated by the diagram

such that $\Theta(1) = \Sigma_1$, $\Theta(0) = \Sigma_0$, $\Theta(2) = \Sigma_2$, $\Theta(g_1) = \sigma_1$ and $\Theta(g_2) = \sigma_2$. The set M is a subset of the objects of the pullback $\prod_{\mathbf{3}} \Theta; Mod$ of the following diagram in CAT

Since the above diagram commutes in CAT , for any object m and morphism $h: m \rightarrow m'$ in $\prod_{\mathbf{3}} \Sigma; Mod$ we have

$$\pi_1(m)|_{\sigma_1} = m_1|_{\sigma_1} = m_0 = m_2|_{\sigma_2} = \pi_2(m)|_{\sigma_2} \quad \text{and}$$

$$\pi_1(h)|_{\sigma_1} = h_1|_{\sigma_1} = h_0 = h_2|_{\sigma_2} = \pi_2(h)|_{\sigma_2}.$$

For objects $R = \langle \Theta, M \rangle$ in the category of abstract datatypes of the institution $\mathcal{R}_{\mathbf{3}}$ we define

$$m_1 R m_2 \quad \text{iff} \quad \exists m \in M : \pi_1(m) = m_1 \text{ and } \pi_2(m) = m_2$$

which gives us that whenever $m_1 R m_2$ then $m_1|_{\Theta(g_1)} = m_2|_{\Theta(g_2)}$.

Formulas in $RSen_{\mathbf{3}}(\Theta)$ are formulas over the pushout $\sum_{\mathbf{3}} \Theta$ of the following diagram in $SIGN$

Here we have to require that $SIGN$ has all pushouts and that Mod preserves them, which is equivalent for the institution $\mathcal{I}$ to have amalgamation [7, 2, 5].

As an example the *AddBirthday* specification is given as an object of $PRES_{\mathbf{3}}$ for first order logic with equality as the base institution. The following functor $\Theta: \mathbf{3} \rightarrow SIGN$ is the signature of *AddBirthday*:

$\Theta(0) = \text{sorts } NAME, DATE, BBENTRY,$
 $\text{SETOFNAME, SETOFBBENTRY}$
ops $bbentry: NAME \times DATE \rightarrow BBENTRY$
 $name: BBENTRY \rightarrow NAME$
 $date: BBENTRY \rightarrow DATE$
 $insert: NAME \times SETOFNAME \rightarrow SETOFNAME$
 $insert: NAME \times SETOFBBENTRY \rightarrow SETOFBBENTRY$
 $\dots$
 $\Theta(1) = \Theta(0) +$
ops $known: SETOFNAME$
 $birthday: SETOFBBENTRY$
 $name?: NAME$
 $date?: DATE$
 $\Theta(2) = \Theta(0) +$
ops $known: SETOFNAME$
 $birthday: SETOFBBENTRY$
 $\Sigma_3 \Theta = \Theta(0) +$
ops $known: SETOFNAME$
 $birthday: SETOFBBENTRY$
 $known': SETOFNAME$
 $birthday': SETOFBBENTRY$
 $name?: NAME$
 $date?: DATE$

$\Theta(g_1)$, $\Theta(g_2)$ and ι_1 are the corresponding inclusion of signatures and ι_2 maps $birthday$ to $birthday'$ and $known$ to $known'$ and is the identity on all other symbols. The axioms of *AddBirthday* are formulas in $R\text{Sen}3(\Theta) = \text{Sen}(\Sigma_3 \Theta)$:

$name(bbEntry(n, d)) = n$
 $date(bbEntry(n, d)) = d$
 $\dots$ (all the axioms for $Set(NAME)$ and $Set(BBEntry)$)
 $\forall n: NAME \quad n \in known \quad \text{iff}$
 $\quad \exists entry: BBENTRY \quad entry \in birthday \quad \text{and} \quad name(entry) = n$
 $\forall n: NAME \quad n \in known' \quad \text{iff}$
 $\quad \exists entry: BBENTRY \quad entry \in birthday' \quad \text{and} \quad name(entry) = n$
 $name? \notin known$
 $birthday' = insert(bbEntry(name?, date?), birthday)$

For an algebra m of $R\text{Mod}3(\Theta)$ satisfying *AddBirthday* it holds that, among others,

$$birthday'^m = insert^m(bbEntry^m(name?^m, date?^m), birthday^m).$$

Since m is a persistent extension of m_1 and m_2 , which in turn are persistent extensions of m_0 , this is equivalent to

$$birthday^{m_2} = insert^{m_0}(bbEntry^{m_0}(name?^{m_1}, date?^{m_1}), birthday^{m_1}).$$

When writing specifications of relations it is convenient to use instead of functors $\Theta: \mathbf{D} \rightarrow \text{SIGN}$ functors $\Theta: \mathbf{D} \rightarrow \text{PRES}_{\mathcal{I}}$ as the type of relations. We have

done this in section 1, where we gave *AddBirthdayIn*, *Env* and *AddBirthdayOut* as specifications instead of just giving their signatures and including their axioms (eg. the axioms for the environment or the state invariant of *BirthdayBook*) in the set of formulas defining *AddBirthday*, as we have done in this section. This can be achieved by using the institution $\mathcal{PI} = \langle PRES_{\mathcal{I}}, PMod, Sig^p; Sen, \models \rangle$ built from an institution $\mathcal{I}$ as the base institution of $\mathcal{R}_{\mathbf{D}}$ instead of $\mathcal{I}$ itself. Here $PMod(\langle \Theta, F \rangle)$ yields the full subcategory of $Mod(\Theta)$ whose objects satisfy F .

6 Specifications with Auxiliary Symbols

In some cases we would like to use auxiliary sort or function symbols in the specification of relations. For example let $\Theta(1)$ and $\Theta(2)$ be enrichments of $\Theta(0)$ by a new sort s . To define how the interpretation of sort s in m_1 is related to the interpretation of sort s in m_2 an auxiliary function $f: \iota_1(s) \rightarrow \iota_2(s)$ may be needed. For example to require that s^{m_1} is isomorphic to a subset of s^{m_2} we provide the formula $\forall x, y : f(x) = f(y) \Rightarrow x = y$. However, this formula can not be written in any of the institutions $\mathcal{R}_2$ and $\mathcal{R}_3$ because formulas in these institutions are only allowed to use the symbols from $\Theta(1)$ and $\Theta(2)$.

A solution to this problem is to use $\mathcal{R}_4$, where $\mathbf{4}$ is the category freely generated by

and the equation $g_1; g_4 = g_2; g_3$. Now the function $f: \iota_1(s) \rightarrow \iota_2(s)$ can be defined in $\Theta(3)$. It is easy to prove that for a functor $\Theta: \mathbf{4} \rightarrow \mathbf{SIGN}$ from $R\mathbf{SIGN4}$ the colimit $\sum_{\mathbf{4}} \Theta$ is isomorphic to $\Theta(3)$. This holds for any category $\mathbf{D}$ with a terminal object $d_{\top}$. Thus by providing $\Theta(3)$ one defines exactly the symbols one can use in the specification of relations. Most likely $\Theta(3)$ will be an extension of the pushout of $\Theta(g_1)$ and $\Theta(g_2)$ by auxiliary sort and function symbols, but it is also possible that $\Theta(3)$, $\Theta(1)$ and $\Theta(2)$ coincide.

Objects $Sp = \langle \Theta, F \rangle$ in $PRES_4$ define binary relations by

$$m_1 Sp m_2 \quad \text{iff} \quad \exists m \models^{\mathcal{R}_4} F : \pi_1(m) = m_1 \text{ and } \pi_2(m) = m_2.$$

6.1 Composition of Binary Relations

Using $\mathcal{R}_4$ has also the advantage that $PRES_4$ is closed under the composition of binary relations. If Sp_R and Sp_S are viewed as binary relations and $Sp_R; Sp_S$ is defined in the usual set theoretic way then there exists an object Sp_T of $PRES_4$ defining the same binary relation as $Sp_R; Sp_S$. This, depending on the expressiveness of the base institution $\mathcal{I}$, does not hold, in general, for the composition of relations in $PRES_{2/3}$ (see below).

Let $Sp_R = \langle \Theta^R, F^R \rangle$ and $Sp_S = \langle \Theta^S, F^S \rangle$ be specifications in $PRES_4$. If $\Theta^R(g_2) = \Theta^S(g_1)$ then the composition $Sp_R; Sp_S = \langle \Theta^T, F^T \rangle$ is defined. The type of $Sp_R; Sp_S$ is given by the following diagram in $SIGN$

and $F^T = \sigma_1(F^R) \cup \sigma_2(F^S)$.

Theorem. $m_1 Sp_R; Sp_S m_2$ iff $\exists n : m_1 Sp_R n$ and $n Sp_S m_2$

As an example consider the specifications Sp_R and Sp_S such that $\Theta^R(g_2) = \Theta^S(g_1)$ and $\Theta^R(1) = \Theta^S(1) = \Theta^S(2)$ is an enrichment of $\Theta(0)$ by a constant $x : s$. Let F^R contain the formula $x = f(x')$ and F^S the formula $x'' = g(x')$, where f and g are functions defined in $\Theta^R(0)$. For $Sp_R; Sp_S = \langle \Theta^T, F^T \rangle$ as an object of $PRES_4$ the type Θ^T contains the auxiliary symbol $x' : s$ in $\Theta^T(3)$ together with $x : s$ from $\Theta^T(1) = \Theta^R(1)$ and $x'' : s$ from $\Theta^T(2) = \Theta^S(2)$ and F^T equals $\{x = f(x'), x'' = g(x')\}$.

However, to define $Sp_R; Sp_S = \langle \Theta^T, F^T \rangle$ in $PRES_3$ we do not have the possibility to use the auxiliary symbol $x' : s$, therefore we have to find a set of formulas equivalent to the conjunction of $x = f(x')$ and $x'' = g(x')$ only using the symbols x , x'' , f and g . Provided that the base institution $\mathcal{I}$ allows to use existential quantifiers we can use $\{\exists x' : x = f(x') \wedge x'' = g(x')\}$. In general, it may not be possible to find such a set of formulas.

7 Semantics

Which relation $\langle \Theta, M \rangle$ from ADT_i is denoted by the presentation $\langle \Theta, F \rangle$ in $PRES_i$ ($i = 2, 3, 4$)? This is a question about the semantics function

$$sem : PRES_i \rightarrow ADT_i.$$

Common semantics are *initial* and *loose* from section 2. However, *initial* is not appropriate since all models of $initial(Sp)$ are isomorphic, and thus, if $m_1 initial(Sp) m_2$ and $m'_1 initial(Sp) m'_2$, then m_1 is isomorphic to m'_1 and m_2 is isomorphic to m'_2 .

Another possible semantics is *free*. If $free(\langle \Theta, F \rangle) = \langle \Theta, M \rangle$ then structures m in M are required to be free extensions of $m|_{\iota_1}$ (or $m|_{\Theta(g_4)}$ in case

of $\mathcal{R}_4$) with respect to the full subcategory of $Mod(\sum_i \Theta)$ ($i = 2, 3, 4$) given by $\{m \mid m \models F\}$ (cf. Goguen and Burstall [10] and Baumeister [2]). $free(Sp)$ can be considered as denoting a partial function from structures of $Mod(\Theta(1))$ to structures of $Mod(\Theta(2))$ in the sense that, if for some m_1 there exists m_2 such that $m_1 \text{ free}(Sp) m_2$ then m_2 is unique up to isomorphism. If $\sum_i \Theta$ does not define any new sorts with respect to $\Theta(1)$, m_2 is unique.

$loose(Sp)$ (or Rel in section 3) in general yields a relation between algebras which can be interpreted as a non-deterministic, partial function on algebras. However, for certain types of F the loose semantics of Sp yields a (partial) deterministic function on algebras. This is, for example, the case when F consists of formulas of type $x' = t[x]$ for all symbols x' not in $\Theta(1)$, where $t[x]$ is a term over the signature $\Theta(1)$, and $\sum_i \Theta$ does not contain new sorts with respect to $\Theta(1)$. Thus loose semantics allows to specify non deterministic functions as well as deterministic functions.

8 Related Work

It is not new or surprising to view states as algebras. For example, Ganzinger [9] and Gurevich [11] use this view to define an algebraic semantics of programming languages. Ganzinger defines the semantics of while-programs to be the composition of a free functor and a forgetful functor on algebras representing the state of the program. In principle we have three specifications. $Spec_{base}$ is the specification of some base types defining the domain over which the variables range. $Spec_X$ is a conservative extension of $Spec_{base}$ containing constants of the form $x : s$ denoting the variables of the program. Each program p determines a specification $Spec_{X \cup X'}^p$, which is an enrichment of $Spec_X$ that contains for each $x : s$ in $Spec_X$ a constant $x' : s$ and contains for example in the case where p is the assignment statement $x := t$ the formula $x' = t$. The semantics of a program p is given by $T_p; U_\sigma$, where $T_p : Alg(Spec_X) \rightarrow Alg(Spec_{X \cup X'}^p)$ is the left adjoint to the forgetful functor U with respect to the inclusion of $Spec_X$ in $Spec_{X \cup X'}^p$, and $U_\sigma : Alg(Spec_{X \cup X'}^p) \rightarrow Alg(Spec_X)$ is the forgetful functor for the signature morphism σ that takes each $x : s$ in $Spec_X$ to $x' : s$ in $Spec_{X \cup X'}^p$. In general T_p will be a persistent functor, however, if p does not terminate then $(T_p; U)(A) \neq A$. This approach corresponds to using the *free* semantics of section 7 with objects $Sp_p = \langle \Theta, F \rangle$ from $PRES_3$ such that $\Theta(0) = Spec_{base}$, $\Theta(1) = \Theta(2) = Spec_X$ and $\langle \sum_3 \Theta, F \rangle = Spec_{X \cup X'}^p$. The only difference is that in our approach, in the case of non-termination, $free(Sp_p)$, seen as a function, is undefined; this is the reason why *free* is defined by using free extensions instead of a free functor. Ganzinger's results can be used in two ways. First, we have new operations at hand to build relations from relations, like the conditional and while construct; on the other hand, the semantics of while-programs have the same semantic domain as their specifications, objects in ADT_3 . This provides us with a connection of specifications of state transformations to their implementation by imperative programs within the same framework.

In the evolving algebra approach of Gurevich [11] a transformation of an algebra A to an algebra B over the same signature is given by a set of guarded

local function updates of the form **if** b **then** $f(t) := t_0$. The problem in translating these updates in our framework to formulas over $RSen2(\Theta)$ for some object $Sp = \langle \Theta, F \rangle$ from $PRES_2$ is the treatment of conflicting updates, eg. $f(a) := true$ and $f(a) := false$. Gurevich chooses one of the updates non deterministically and discards the others, while in our approach all updates have to be satisfied. This means that some values have to be equated (eg. $true = false$) or the relation, viewed as a (non-deterministic) function, is not defined for some algebras. The algebras of Gurevich have only one sort called the superuniverse containing two distinguished elements $true$ and $false$. Different sorts (universes) are modeled by using characteristic functions on the superuniverse. This allows to model the evolution of sorts just by changing the interpretation of their characteristic functions without having to change the interpretation of the superuniverse.

In the implicit state approach of Dauchy and Gaudel [4] states are enrichments of the datatypes used in the system by access functions. Modifiers are functors on states built from elementary modifiers by sequential and indifferent composition and using conditionals. Elementary modifiers $\mu\text{-}ac(\pi, t)$, for some accessor function ac and terms π and t over the signature of states, can be translated into our framework as the formula

$$\forall x(\exists y : (\pi = x \Rightarrow ac'(x) = t) \quad \vee \quad \forall y : (\pi \neq x \Rightarrow ac'(x) = ac(x))),$$

where y is the vector of variables occurring free in π and t . This is a formula in F , where $Sp = \langle \Theta, F \rangle$ is an object of $PRES_3$, $\Theta(1) = \Theta(2)$ contains the accessor function ac and $\Theta(0)$ is the specification of the used datatypes. Now, for a structure m satisfying Sp , it holds that

$$ac^{m_2}(v) = \begin{cases} \rho^*(t) & \text{if } \rho^*(\pi) = v \text{ for a variable assignment } \rho: X \rightarrow m_1 \\ ac^{m_1}(v) & \text{else} \end{cases}$$

This means, that $loose(Sp)$ defines a function on algebras corresponding to the semantics of the elementary modifiers of Dauchy and Gaudel. Sequential composition of modifiers is given by the composition of relations in $PRES_3$ and ADT_3 and indifferent composition can be described by colimits in $PRES_3$ and ADT_3 , respectively.

In all the above approaches the stress is on *constructing* transformations of algebras rather than on specifying these transformations in a possibly *non-constructive* way. In our approach we are able to specify how the states before and after a transformation are related and not necessarily how this relation is established (eg. by while-programs, local function updates or modifiers).

D-oids are a model theoretic approach by Astesiano and Zucca [1] to define dynamic datatypes. D-oids consist of a set of algebras over some signature together with a set of dynamic operations. A dynamic operation associates to an algebra A and a set of values from A an algebra B , a family of functions from the carrier sets of A to the carrier sets of B , the tracking map $f: A \Rightarrow B$, and possibly a return value from B . However, Astesiano and Zucca do not have a method to specify their dynamic datatypes yet.

All the previous approaches keep the interpretation of sorts fixed, allowing only for the interpretation of function symbols to change. This is not required by

the d-oids approach. The tracking map f is responsible for relating the elements of the sort whose interpretation has changed in B and also for the elements of those sorts that did not change. In our approach there is the possibility to include new sorts in $\Theta(1)$ and $\Theta(2)$ and functions f_i in $\Theta(3)$, $f_i: s_i \rightarrow s'_i$, for new sorts s_i in $\Theta(1)$ and $\Theta(2)$, and $f_i: s_i \rightarrow s_i$, for sorts s_i in $\Theta(0)$. The functions f_i can be seen as defining the tracking map of dynamic operations.

An approach similar to d-oids is the informal proposal for dynamic abstract datatypes by Ehrig and Orejas [8]. The instant structure specification, a conservative extension of some value type specification, is intended to model class sorts and attribute functions of objects. A dynamic operation denotes for a model A of the instant structure specification, together with a set of values from A , a set of transformations from A to some other instant algebra B . Given a specification $Sp = \langle \Theta, F \rangle$ in $PRES_4$, a value type specification corresponds to $\Theta(0)$, an instant structure specification, roughly, to $\Theta(1)$ and $\Theta(2)$ and certain classes of transformations could be modeled using function symbols $f: s \rightarrow s'$ and $f: s \rightarrow s$ in $\Theta(3)$. Then a dynamic operation is an object $R = \langle \Theta^R, M^R \rangle$ of ADT_4 such that $M^R \models_{\mathcal{R}_4} F$.

9 Conclusion

In this paper we have defined a family of institutions in which abstract datatypes have an interpretation as relations between structures from some base institution. We have achieved two things. First we have a model theory of typed relations, which are objects in ADT_2 , and for typed relations respecting a common environment, objects in ADT_3 . Since relations are abstract datatypes in certain institutions methods for the construction of abstract datatypes in an arbitrary institution, like eg. various forms of parameterization, can be used to also construct relations, see for example the work of Goguen and Burstall [3, 10], Sannella, Wirsing and Tarlecki [16, 15], Ehrig and others [6]. However, further investigation is needed into whether these constructions are appropriate for constructing relations and which additional constructions, eg. the composition of relations, are needed.

The second achievement of this paper is a specification method for relations between algebras within the theory of abstract datatypes resembling the methods used in specification languages like Z [17], VDM-SL [13] and Larch [12]. Future research has to show how far the resemblance between the presented method and Z, VDM-SL and Larch extends. One could image that the two tiered approach of Larch for some programming language can be given full a semantics within this framework. The base institution is given by the Larch Shared Language; Larch Interface Language specifications are objects in $PRES_4$, which have a denotation as objects in ADT_3 , and the results of Ganzinger [9] can be used to define the semantics of the programs again as objects of ADT_3 , which can then be checked to see if they implement the specification.

Colimits in $SIGN$ and therefore in ADT_D and $PRES_D$ allow us to combine smaller states to larger ones and to extend relations from the smaller states to relations on the larger ones. Also colimits allow us to combine several relations on one state as one relation on the same state. This corresponds to a parallel

composition of relations instead of a sequential one defined by the composition of relations in section 6.1.

The interpretation of abstract data types as relations fails when it comes to refinement. The refinement of abstract datatypes does not correspond to the refinement of relations if relations are viewed as non-deterministic functions. The refinement of one abstract datatype by another is an abstract datatype morphism, while R refines to S with respect to a signature morphism $\alpha: \Theta^R \rightarrow \Theta^S$ iff. $\text{dom}(R) \subseteq \text{dom}(S)|_\alpha$ and $\alpha: R \rightarrow S|_{\alpha(\text{dom}(R))}$ is a morphism between relations. $S|_{\alpha(\text{dom}(R))}$ denotes the restriction of S to the domain $\alpha(\text{dom}(R))$.

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