Nonfinite Axiomatizability of Shuffle Inequalities

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Abstract. There is some set of inequations $t \leq t'$ whose models are the algebras in the variety of ordered algebras generated by the algebras $\mathcal{L}_\Sigma = (P_\Sigma, \cdot, \otimes, 1)$, where P_Σ consists of all subsets of the free monoid Σ^* , $B \cdot C = \{uv : u \in B, v \in C\}$, and $B \otimes C$ is the shuffle product of the two languages. We show that there is no finite set of such inequations.

1 Introduction

In a previous paper [4], the authors found concrete descriptions of the free algebras in varieties generated by languages equipped with various sets of operations all involving the shuffle operation. For example, we considered the variety generated by the structures $\mathcal{L}_\Sigma = (P_\Sigma, \cdot, \otimes, 1)$, where P_Σ consists of all subsets of the free monoid Σ^* , $B \cdot C = \{uv : u \in B, v \in C\}$, and $B \otimes C$ is the shuffle product of the two languages. The free algebras in this variety are labeled series-parallel posets. The free algebras in the variety of *ordered* algebras generated by the structures $(\mathcal{L}_\Sigma, \subseteq)$ are again the labeled series-parallel posets equipped with an ordering we call the trace ordering. (In [7], labeled posets with the “subsumption order” were shown to be the free ordered algebras in a finitely axiomatizable variety.) “Closed subsets” of the labeled posets are the free algebras in the variety generated by the structures $(P_\Sigma, \cdot, \otimes, +, *, 0, 1)$, where $+$ is binary union and $*$ is the Kleene star operation. These results seem to show that two different models of parallelism are equivalent: the so-called language or interleaving model, and a pomset model.

What the result in [4] does not establish is precisely what the valid inequations or equations are in each of the varieties considered. One would like to have an

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axiomatic description of the set of valid equations and inequations, preferably via a simple finite set of axioms. In this paper, we consider the variety $\mathbf{Lg}_{\leq}$ of ordered algebras generated by the language structures $(P_{\Sigma}, \cdot, \otimes, 1, \subseteq)$ and show that there is no finite set of inequations which axiomatize the inequations valid in $\mathbf{Lg}_{\leq}$.

The axiomatization of the varieties generated by the language structures $(P_{\Sigma}, \cdot, \otimes, +, 0, 1)$ and $(P_{\Sigma}, \cdot, \otimes, +, *, 0, 1)$ is considered in the paper [5].

2 Review

In this section, we review the necessary definitions and results from [4].

Definition 1. A **bimonoid** $M = (M, \cdot, \otimes, 1)$ consists of a monoid $(M, \cdot, 1)$ and a commutative monoid $(M, \otimes, 1)$ with a common neutral element. A **bimonoid morphism** $M \rightarrow M'$ is a function $M \rightarrow M'$ which preserves the unit and the two binary operations.

Definition 2. An **ordered bimonoid** $(M, \leq)$ is a bimonoid $(M, \cdot, \otimes, 1)$ whose underlying set M is equipped with a partial ordering $\leq$ such that for all $x, y, a, b \in M$,

$$x \leq a, y \leq b \Rightarrow x \cdot y \leq a \cdot b \text{ and } x \otimes y \leq a \otimes b.$$

A **morphism of ordered bimonoids** is an order preserving bimonoid morphism.

Three motivating examples of ordered bimonoids are the language structures $\mathcal{L}_{\Sigma} := (P_{\Sigma}, \cdot, \otimes, 1)$, $\mathcal{R}_{\Sigma} := (R_{\Sigma}, \cdot, \otimes, 1)$, $\mathcal{F}_{\Sigma} := (F_{\Sigma}, \cdot, \otimes, 1)$, where P_{Σ} is the collection of all subsets, R_{Σ} consists of the regular subsets, and F_{Σ} consists of the finite subsets of the free monoid Σ^* . The ordering on each bimonoid is that of set inclusion. The set $B \cdot C$ is $\{uv : u \in B, v \in C\}$, and $B \otimes C$ is the shuffle product of languages:

$$B \otimes C := \{u_1 v_1 \cdots u_n v_n : u_1 \cdots u_n \in B, v_1 \cdots v_n \in C, u_i, v_i \in \Sigma^*, n \geq 0\}$$

If $f : A \rightarrow B$ is a morphism of ordered bimonoids, we say f is **order-reflecting** if $xf \leq yf \Leftrightarrow x \leq y$. Note that an order-reflecting morphism is necessarily injective. Say that the ordered bimonoid A is an ordered subalgebra of the ordered bimonoid B if there is an order-reflecting morphism $A \rightarrow B$. A **variety** $\mathcal{V}$ of **ordered bimonoids** is a collection of ordered bimonoids closed under products (ordered componentwise), ordered subalgebras and order preserving morphic images. Equivalently, (see [3]), a variety of ordered bimonoids is the collection of all ordered bimonoids which satisfy a set of inequations $t \leq t'$, for certain bimonoid terms t, t' . Any collection of ordered bimonoids is contained in a least variety of ordered bimonoids.

One variety of interest is that generated by the language bimonoids.

Definition 3. $\mathbf{Lg}$ denotes the variety of bimonoids generated by the language bimonoids $\mathcal{L}_{\mathcal{L}}$, and $\mathbf{Lg}_{\leq}$ denotes the variety of ordered bimonoids generated by the ordered bimonoids of languages $(\mathcal{L}_{\mathcal{L}}, \subseteq)$.

Remark 4. The variety $\mathbf{Lg}_{\leq}$ is generated by the ordered bimonoids of finite languages $\mathcal{F}_{\mathcal{L}}$ and thus by the ordered bimonoids $\mathcal{R}_{\mathcal{L}}$ of regular languages [4].

Labeled posets with serial and parallel product form another example of a bimonoid. An A -labeled poset $P = (P, \leq, \ell)$ consists of a poset $(P, \leq_P)$, sometimes written just $(P, \leq)$, and an assignment of an element $v\ell$ in A to each vertex v in P . (Here P denotes both the poset and the underlying set of elements or “vertices” of P . Thus the expression “ $v \in P$ ” is meaningful.) When A is understood, we will say only “labeled poset”. A **morphism** $f : P \rightarrow Q$ of A -labeled posets is a function $P \rightarrow Q$ which preserves the ordering and the labeling. We agree to identify isomorphic labeled posets, without further mention. (Many authors call an isomorphism class of a labeled poset a “pomset”.) *To save space, we assume “poset” means “finite poset”.* We denote the empty poset by $\mathbf{1}$. Two operations on labeled posets are important here, sequential, or serial product $P \cdot Q$ and parallel or shuffle product $P \otimes Q$. Given labeled posets P, Q , with disjoint underlying sets,

$$P \cdot Q := (P \cup Q, \leq_{P \cdot Q})$$

$$P \otimes Q := (P \cup Q, \leq_{P \otimes Q}),$$

where for $v, v' \in P \cup Q$,

$$v \leq_{P \cdot Q} v' \Leftrightarrow v \leq_P v' \text{ or } v \leq_Q v' \text{ or } \\ v \in P \text{ and } v' \in Q.$$

$$v \leq_{P \otimes Q} v' \Leftrightarrow v \leq_P v' \text{ or } v \leq_Q v'.$$

The labeling is extended to $P \otimes Q$ and $P \cdot Q$ in the obvious way.

Definition 5. We let $\mathbf{SP}_A$, for “series-parallel” or “shuffle product”, denote the least class of posets containing the empty poset $\mathbf{1}$, the singleton posets a , labeled a , for each $a \in A$, closed under the operations $P \cdot Q$, $P \otimes Q$. The posets in $\mathbf{SP}_A$ will be called “series-parallel” posets. We let $\mathbf{SP}(A)$ be the following bimonoid:

$$\mathbf{SP}(A) := (\mathbf{SP}_A, \cdot, \otimes, \mathbf{1}).$$

Remark 6. Grabowski [8] and Valdes [10] found the following “N-condition” to characterize the series-parallel posets. A poset P is series-parallel iff there is no four-element subset $\{a, b, c, d\} \subseteq P$ whose only nontrivial order relationships are given by

$$a < c, \quad a < d, \quad b < d.$$

A nice proof of this fact is given in [7]. It follows from this characterization that any subset of a series-parallel poset, ordered with the inherited order, is itself series-parallel. This fact will be used several times without mention.

Theorem 7. [7] $\mathbf{SP}(A)$ is freely generated in the variety of all bimonoids by the set A .

We will impose an ordering on $\mathbf{SP}(A)$ making use of the alphabet $\Sigma(A)$, defined below. Given the set A , let $\bar{A}$ be a set disjoint from A , and let

$$a \mapsto \bar{a}$$

be a bijection. Define the sets A_N and $\bar{A}_N$ by:

$$\begin{aligned} A_N &:= A \times \mathbf{N} \\ \bar{A}_N &:= \bar{A} \times \mathbf{N} \end{aligned}$$

where $\mathbf{N} = \{0, 1, \dots\}$. We denote elements in A_N as a_i, b_j , etc., and elements in $\bar{A}_N$ as $\bar{a}_i, \bar{b}_j$. Let $\Sigma(A)$ be the infinite alphabet

$$\Sigma(A) := A_N \cup \bar{A}_N.$$

Let $h_0 : \mathbf{SP}(A) \rightarrow \mathcal{L}_{\Sigma(A)}$ be the unique bimonoid morphism satisfying

$$ah_0 := \{a_1\bar{a}_1, a_2\bar{a}_2, \dots, a_n\bar{a}_n, \dots\},$$

each $a \in A$.

Remark 8. An *expansion* of a poset $P \in \mathbf{SP}(A)$ is a poset P' in $\mathbf{SP}(\Sigma(A))$ obtained from P by replacing each vertex v of P by a two element chain $v(1) < v(2)$. The ordering in P' is defined as follows:

$$v(i) \leq v'(j) \Leftrightarrow v < v' \text{ in } P \text{ or } v = v' \text{ and } i \leq j \in [2].$$

If v is labeled a in P , then for some $i \geq 1$, $v(1)$ is labeled a_i and $v(2)$ is labeled $\bar{a}_i$. Suppose that

$$v_1 v_2 \dots v_m$$

is a topological run of P' (i.e., a surjection $v : [m] \rightarrow P$ such that $v_i \leq_P v_j \Rightarrow i \leq j$). The *trace* of this run is the word in $\Sigma(A)^*$ formed by concatenating the labels of the vertices in the run. In [4] it is shown that for any poset $P \in \mathbf{SP}(A)$, a word u belongs to Ph_0 iff u is a trace of an expansion of P .

One of the main results of [4] is the following.

Theorem 9. The morphism h_0 is injective. It follows that $\mathbf{Lg}$ is the variety of all bimonoids. $\square$

For any labeled poset P , the words u in Ph_0 satisfy at least the following conditions:

- (i) $|u|_a = |u|_{\bar{a}}$, for each $a \in A$. ($|u|_a$ is the number of occurrences of the letter a in the word u .)
- (ii) If w is a prefix of u , $|w|_a \geq |w|_{\bar{a}}$, for each $a \in A$.

We say a word in $\Sigma(A)^*$ is balanced if it satisfies both conditions (i) and (ii).

Definition 10. A monoid endomorphism φ of $(A_N \cup \overline{A}_N)^*$ is **admissible** if, for each $a \in A$, $i \in \mathbb{N}$, there is some $j \in \mathbb{N}$ with

$$\begin{aligned} a_i \varphi &= a_j, \quad \text{and} \\ \overline{a}_i \varphi &= \overline{a}_j. \end{aligned}$$

Thus, admissible morphisms only change subscripts on letters. For example, an admissible morphism may identify a_1 and a_2 .

Definition 11. The preorder $\sqsubseteq$ on the set of (balanced) words in $\Sigma(A)^*$ is the least reflexive and transitive relation which satisfies the following conditions. $u \sqsubseteq u'$ if, for some words w, w' and distinct letters a_i, b_j ,

$$\begin{aligned} u &= wa_i b_j w' \text{ and } u' = wb_j a_i w'; \quad \text{or} \\ u &= w\overline{a}_i \overline{b}_j w' \text{ and } u' = w\overline{b}_j \overline{a}_i w'; \quad \text{or} \\ u &= w\overline{a}_i b_j w' \text{ and } u' = wb_j \overline{a}_i w'; \quad \text{or} \\ u &= u' \varphi, \quad \text{for some admissible } \varphi. \end{aligned}$$

Definition 12. For A -labeled posets P, Q in $\mathbf{SP}(A)$, $P \leq Q$ iff for each word $u \in Ph_0$ there is a word $u' \in Qh_0$ with $u \sqsubseteq u'$. We call the order $\leq$ the **trace ordering** on $\mathbf{SP}(A)$ (see Remark 8).

We note some easy consequences of Definition 12.

Proposition 13. For posets $P, Q \in \mathbf{SP}(A)$, the following are equivalent.

1. $P \leq Q$
2. $Ph_0 \subseteq Qh_0$. □

Since h_0 is injective, the relation $\leq$ is a partial order on $\mathbf{SP}(A)$.

Remark 14. If u, v are words and $v \in Ph_0$, for some $P \in \mathbf{SP}(A)$, and if $u \sqsubseteq v$, then $u \in Ph_0$. It follows that for posets $P, Q \in \mathbf{SP}(A)$, $P \leq Q$ iff for each word u which is maximal in Ph_0 there is some word v which is maximal in Qh_0 with $u \sqsubseteq v$. Note that by the last clause in Definition 11, each letter occurs at most once in any maximal word.

Proposition 15. Suppose that in $\mathbf{SP}(A)$, $P \leq P_1$ and $Q \leq Q_1$. Then

$$P \cdot Q \leq P_1 \cdot Q_1 \quad \text{and} \quad P \otimes Q \leq P_1 \otimes Q_1.$$

Thus, $(\mathbf{SP}(A), \leq)$ is an ordered bimonoid in $\mathbf{Lg}_{\leq}$. □

Theorem 16. [4] For each set A , $(\mathbf{SP}(A), \leq)$ is the free ordered bimonoid in the variety of ordered bimonoids $\mathbf{Lg}_{\leq}$. $\square$

For later use, we note several other properties of the trace order.

Lemma 17. *Suppose that $P \leq Q$ in $\mathbf{SP}(A)$. Then*

1. *For each $a \in A$, the number of vertices of P labeled a , $|P|_a$, is the same as $|Q|_a$;*
2. *The number of minimal elements in Q is at least the number of minimal elements in P ;*
3. *The number of maximal elements in Q is at least the number of maximal elements in P ;*
4. *The height of P is at least the height of Q ;*
5. *The width of Q is at least the width of P .* $\square$

3 A Metatheorem

In this section, we present a method to show that the variety $\mathbf{Lg}_{\leq}$ of ordered bimonoids defined above is not finitely axiomatizable. The method depends upon finding a sequence of posets P_n, P'_n satisfying certain properties.

Suppose that A is the singleton set $\{a\}$.

Theorem 18. Suppose that for each integer $n \geq 1$ there are posets P_n and P'_n in $\mathbf{SP}(A)$ with the following properties:

- (P1) P'_n is a successor of P_n with respect to the trace ordering, so that $P_n < P'_n$ and for all P in $\mathbf{SP}(A)$, $P_n \leq P < P'_n$ implies $P = P_n$.
- (P2) $|P_n| > n$.
- (P3) P'_n does not have a nontrivial series decomposition.
- (P4) There are no nonempty posets $U \leq U'$, $V \leq V'$ with $P_n = U \otimes V$ and $P'_n = U' \otimes V'$.
- (P5) If

$$t(x_1, \dots, x_k) \leq t'(x_1, \dots, x_k)$$

is a valid inequation in $\mathbf{Lg}_{\leq}$ and

$$P_n = t(R_1, \dots, R_k)$$

$$P'_n = t'(R_1, \dots, R_k),$$

for some posets $R_1, \dots, R_k$ in $\mathbf{SP}(A)$, then $|t| > n$.

Then the variety $\mathbf{Lg}_{\leq}$ is not finitely axiomatizable.

The number $|t|$ in condition (P5) denotes the number of occurrences of variables in t . Note that since $\mathbf{Lg}_{\leq} \models t \leq t'$, it follows that for each variable x_i , the number of occurrences of x_i in t is the same as the number of occurrences of x_i in t' . Thus $|t| = |t'|$. Note also that condition (P2) follows from (P5) and (P1): Let $t(x), t'(x)$ be terms such that $t(a) = P_n$ and $t'(a) = P'_n$.

The remainder of this section is devoted to the proof of Theorem 18.

Let E be a finite set of inequations between bimonoid terms with $\mathbf{Lg}_{\leq} \models E$. Let

$$n := \max\{|t|, |t'| : t \leq t' \in E\}$$

and $m := |P_n|$. Using the properties of the posets, we construct an ordered bimonoid B_E and terms $p(x)$ and $p'(x)$ such that:

1. $B_E \models E$;
2. $B_E \not\models p(x) \leq p'(x)$;
3. $\mathbf{Lg}_{\leq} \models p(x) \leq p'(x)$.

To construct B_E , consider the free bimonoid $\mathbf{SP}(A)$. The relation $\sim$ defined by

$$P \sim Q \Leftrightarrow |P|, |Q| > m \text{ or } P = Q,$$

is a bimonoid congruence relation on $\mathbf{SP}(A)$. The quotient $\mathbf{SP}(A)/\sim$ may be conveniently represented as the bimonoid whose elements are the labeled posets $P \in \mathbf{SP}(A)$ with $|P|$ at most m , and the element 0 which represents the congruence class of the posets P with $|P| > m$. The operations are defined in a straightforward manner.

Now we introduce a partial order on the bimonoid $\mathbf{SP}(A)/\sim$. For P, Q in $\mathbf{SP}(A)/\sim$, we define

$$P \sqsubseteq Q$$

iff $P = Q$, or $P, Q \neq 0$, $P \leq Q$, and either $P \neq P_n$ or $Q \neq P'_n$. Thus, for nonzero elements we keep the ordering in $\mathbf{SP}(A)$, except P_n and P'_n become unrelated. The element 0 is not comparable to any other element. Clearly, $\sqsubseteq$ is reflexive and antisymmetric.

Proposition 19. *If $P \sqsubset Q \sqsubset R$, then $P \sqsubset R$. Thus, $\sqsubseteq$ is a partial order.*

Proof. The only possible concern is $P = P_n$ and $R = P'_n$. But this is impossible, since P'_n is a successor of P_n in the trace ordering. $\square$

The resulting structure, which we denote B_E , is an ordered bimonoid, since the fact that the operations are monotonic is immediate from (P3) and (P4).

Claim: B_E satisfies the inequations in E .

Suppose

$$t(x_1, \dots, x_k) \leq t'(x_1, \dots, x_k)$$

is in E . Since $\mathbf{Lg}_{\leq} \models E$, the variables which actually appear in t are the same as that appearing in t' . We may assume that these are the variables $x_1, \dots, x_k$. Moreover, for each x_i , the number of occurrences of x_i in t is the same as the number of occurrences of x_i in t' , in notation: $|t|_{x_i} = |t'|_{x_i}$. We need to show that

$$t(R_1, \dots, R_k) \sqsubseteq t'(R_1, \dots, R_k), \quad (1)$$

for all $R_1, \dots, R_k$ in B_E . But the left hand side is 0 iff the right hand side is 0. If the two sides are nonzero, then the R_i 's are nonzero, and the two sides are evaluated in B_E in the same way as in $\mathbf{SP}(A)$. Thus,

$$t(R_1, \dots, R_k) \leq t'(R_1, \dots, R_k).$$

But, by (P5) and the choice of the integer m , it is not possible that the left hand side is P_n and the right hand side is P'_n , proving (1).

Claim: There is some inequation $p(x) \leq p'(x)$ which holds in $\mathbf{Lg}_{\leq}$ but not in B_E . Indeed, let $p(x)$ and $p'(x)$ be terms with $p(a) = P_n$ and $p'(a) = P'_n$ in $\mathbf{SP}(A)$. Since $P_n \leq P'_n$ in $\mathbf{SP}(A)$, $p(x) \leq p'(x)$ holds in $\mathbf{Lg}_{\leq}$, but fails in B_E , by construction.

Thus we have proved: If there are posets P_n, P'_n satisfying conditions (P1)–(P5), then for each finite set E of inequations which are valid in $\mathbf{Lg}_{\leq}$ there exists a finite ordered bimonoid B_E and a unary inequation $p(x) \leq p'(x)$ which holds in $\mathbf{Lg}_{\leq}$ but not in B_E .

Remark 20. It follows easily from the compactness theorem that there is no finite first order axiomatization of $\mathbf{Lg}_{\leq}$ either.

4 The Construction

In this section, we define the sequence of posets P_n and P'_n and prove a number of facts about the ordering of the posets in $\mathbf{SP}(A)$ in order to show the sequence has the properties (P1)–(P5) of Theorem 18. All posets are assumed to belong to $\mathbf{SP}(A)$, where A is a singleton set, say $A = \{a\}$.

The collection of posets in $\mathbf{SP}(B)$ with a unique maximal vertex and a unique minimal vertex is denoted $\mathbf{SP}_1(B)$.

For any poset $R \in \mathbf{SP}(B)$, where B contains the letter a , define the “top-bottom” of R , $tb(R)$ as follows:

$$tb(R) := a \cdot R \cdot a. \quad (2)$$

Thus, $tb(R)$ has a unique maximal and minimal element.

Definition 21. Define the following sequence of posets in $\mathbf{SP}(A)$.

$$P_1 = a \cdot a \quad (3)$$

$$P'_1 = a \otimes a. \quad (4)$$

Assume P_n, P'_n defined so that P'_n is a successor of P_n .

$$P_{n+1} := (a \cdot tb(P_n)) \otimes (a \cdot tb(P_n)) \otimes (a \cdot tb(P'_n)) \quad (5)$$

$$P'_{n+1} := a \cdot (tb(P_n) \otimes tb(P'_n)) \otimes (a \otimes a) \cdot tb(P'_n). \quad (6)$$

The fact that the posets P_n, P'_n satisfy properties (P2),(P3) and (P4) of Theorem 18 is either immediate or follows easily from Lemma 17. Thus, it remains to establish properties (P1) and (P5).

It is clear that P'_1 is the unique successor of P_1 . From Lemma 30 below, it follows that P'_n is a successor of P_n , for all $n \geq 1$.

Suppose that R is a poset in $\mathbf{SP}(A)$ having the minimal elements $x_1, x_2, \dots$. Define the subposet R_i to consist of those vertices above only the one minimal vertex x_i :

$$R_i := \{v \in R : x_j \leq v \Rightarrow j = i\}. \quad (7)$$

Lemma 22. *The subposet R_i of R contains a maximal element of R iff*

$$R = R_i \otimes (R - R_i).$$

Proof. Clearly, if $R = R_i \otimes (R - R_i)$, then any vertex maximal in R_i is maximal in R . We prove the converse. Suppose that R_i contains the maximal vertex y_i . If $v \notin R_i$, then $v \geq x_j$ for some $j \neq i$. Thus for any $v' \in R_i$, it is impossible that $v' \geq v$. Conversely, if $v \geq v'$, the subposet of R consisting of y_i, x_i, v, x_j shows that R violates the “N-condition”, and cannot belong to $\mathbf{SP}(A)$. (See Remark 6 for the “N-condition”.) $\square$

Lemma 23. *Suppose that the posets P_i, Q_i , $i \in [3]$, all belong to $\mathbf{SP}_1(A)$ and each has $N \geq 1$ vertices. Let $P := \otimes_{i=1}^3 P_i$, $Q := \otimes_{i=1}^3 Q_i$. If*

$$P \leq R \leq Q,$$

then there are posets $R_i, i \in [3]$ in $\mathbf{SP}_1(A)$ such that $R = \otimes_{i=1}^3 R_i$, and each R_i has size N .

Proof. We may assume $N > 1$ or else $P = R = Q$. Now by Lemma 17, R has 3 minimal and 3 maximal vertices. Suppose that $\{x_1, x_2, x_3\}$ are the minimal vertices and that $\{y_1, y_2, y_3\}$ are the maximal vertices in R . The poset R has no isolated vertex. (Indeed, if S, T are any posets in $\mathbf{SP}(A)$ such that $a \otimes S \leq T$ then $T = a \otimes T'$, for some $T' \in \mathbf{SP}(A)$.) Thus, the vertices x_i, y_j are all distinct. For an element $i \in [3]$, the set R_i was defined in (7).

Sublemma 24. – *Some set R_i has at least N vertices.*

- *No set R_i can have more than N vertices.*
- *One of the sets R_i which has exactly N vertices contains one of the maximal vertices y_j .*
- *None of the sets R_i can contain 2 maximal vertices.*

Proof. We show first that some set R_i has at least N vertices. Indeed, the word

$$u := a_1^3 \bar{a}_1 (a_2 \bar{a}_2)^{N-1} \bar{a}_1 (a_2 \bar{a}_2)^{N-1} \bar{a}_1 (a_2 \bar{a}_2)^{N-1}$$

belongs to Ph_0 and hence to Rh_0 . The vertices in R of any run with trace u (see Remark 8) corresponding to the first occurrence of $\bar{a}_1$ and the first $N-1$ a_2 's belong to some set R_i . Hence this R_i has at least N vertices.

But no set R_i can have more than N vertices. Otherwise, there is a word

$$a_1^3 \bar{a}_1 (a_2 \bar{a}_2)^k w$$

in Rh_0 where $k \geq N$, but there is no such word in Qh_0 . Thus, some set R_i contains exactly N vertices.

One of the sets R_i which has exactly N vertices must contain one of the maximal vertices y_j . Indeed, otherwise, the word

$$a_1^3 \bar{a}_1 (a_2 \bar{a}_2)^{N-2} a_3 \bar{a}_1 (a_2 \bar{a}_2)^{N-2} a_3 \bar{a}_1 (a_2 \bar{a}_2)^{N-2} a_3 \bar{a}_3 \quad (8)$$

does not belong to Rh_0 even though it does belong to Ph_0 .

Now we show none of the sets R_i can contain 2 maximal vertices. Indeed, if $|R_i| = k + 3 \leq N$ and R_i contains two of the y_j , then there is a word

$$a_1^3 \bar{a}_1 (a_2 \bar{a}_2)^k a_3 a_3 v \bar{a}_3 \bar{a}_3$$

in Rh_0 , where v does not contain $\bar{a}_3$. But there is no such word in Qh_0 . □

Now suppose that R_1 has N vertices and contains the maximal element y_1 , say. Then by Lemma 22,

$$R = R_1 \otimes (R - R_1).$$

Let $R' = (R - R_1)$. We now show $R' = R_2 \otimes R_3$. Note that $R_2 \cup R_3 \subseteq R'$. We know R' contains the two maximal vertices y_2, y_3 and the two minimal vertices x_2, x_3 . Suppose, without loss of generality, that $y_2 > x_2$. Then it is impossible that $y_2 > x_3$. Otherwise, if $y_3 > x_2$ (or $y_3 > x_3$), the “N-condition” implies that both y_2 and y_3 are greater than both x_2 and x_3 . But then, the word u of (8) is not in Rh_0 even though it belongs to Ph_0 . Thus, $y_2 \in R_2$, showing that

$$R' = R_2 \otimes (R' - R_2),$$

by Lemma 22. But $R_3 = R' - R_2$, since R has only 3 minimal vertices. Thus,

$$R = R_1 \otimes R_2 \otimes R_3.$$

Since $|R_1| = N$, and $|R_2|, |R_3| \leq N$ and $|R| = 3N$, it follows that $|R_2| = |R_3| = N$, completing the proof of Lemma 23. □

Remark 25. Lemma 23 holds when 3 is replaced by any positive integer.

Lemma 26. *Suppose that Q is a successor of P , where $P, Q \in \mathbf{SP}_1(A)$, and that R_1, R_2, R_3 are posets in $\mathbf{SP}_1(A)$ with $|P| = |Q| = |R_1| = |R_2| = |R_3|$. If*

$$P \otimes P \otimes Q \leq R_1 \otimes R_2 \otimes R_3 \leq P \otimes Q \otimes Q,$$

then, for each $i \in [3]$, $R_i \in \{P, Q\}$ and either exactly one or exactly two of the R_i are equal to P .

Proof. First we show $P \leq R_i$, for each $i \in [3]$. Otherwise, there is a maximal word $a_1\bar{a}_1ua_2\bar{a}_2$ in Ph_0 which is not in R_ih_0 . Then

$$a_1^3\bar{a}_1ua_2\bar{a}_2\bar{a}_1ua_2\bar{a}_2\bar{a}_1ua_2\bar{a}_2$$

is in $(P \otimes P \otimes Q)h_0$ but not in $(R_1 \otimes R_2 \otimes R_3)h_0$.

Now we show $R_i \leq Q$, for each $i \in [3]$. Otherwise, choose a word $a_1\bar{a}_1ua_2\bar{a}_2$ in R_ih_0 not in Qh_0 such that $a_1, \bar{a}_1$ do not occur in u . Since $P \leq Q$, this word does not occur in Ph_0 either. Then there is a word

$$a_1^3\bar{a}_1ua_2\bar{a}_2v$$

in $(R_1 \otimes R_2 \otimes R_3)h_0$ not in $(P \otimes Q \otimes Q)h_0$.

Since Q is a successor of P , it follows that either two of the R_i are P or exactly one R_i is P . $\square$

Corollary 27. *Suppose that $P, Q \in \mathbf{SP}_1(A)$. If Q is a successor of P , then $P \otimes Q \otimes Q$ is a successor of $P \otimes P \otimes Q$.*

Proof. Suppose that

$$P \otimes P \otimes Q \leq R \leq P \otimes Q \otimes Q.$$

By Lemma 23, $R = R_1 \otimes R_2 \otimes R_3$, where P, Q and the R_i satisfy the hypotheses of Lemma 26. Hence, by that Lemma, $R = P \otimes Q \otimes Q$ or $R = P \otimes Q \otimes Q$. $\square$

Lemma 28. *Suppose that P, Q belong to $\mathbf{SP}(A)$, and that Q is a successor of P . Then $\text{tb}(Q)$ is a successor of $\text{tb}(P)$, and both $\text{tb}(P)$ and $\text{tb}(Q)$ belong to $\mathbf{SP}_1(A)$. $\square$*

Lemma 29. *Suppose that P, Q belong to $\mathbf{SP}(A)$ and that $P \leq Q$. Let P', Q' be defined by*

$$\begin{aligned} P' &:= (a \cdot P) \otimes (a \cdot P) \otimes (a \cdot Q) \\ Q' &:= (a \cdot (P \otimes Q)) \otimes ((a \otimes a) \cdot Q). \end{aligned}$$

Then $P' \leq Q'$.

Proof. Up to an admissible bijection, each maximal word in $P'h_0$ is in either U_1 or U_2 , where $K = Ph_0$ and $L = Qh_0$ and where

$$\begin{aligned} U_1 &:= a_1 a_2 a_3 \bar{a}_1 (K \otimes \bar{a}_2 K \otimes \bar{a}_3 L) \\ U_2 &:= a_1 a_2 a_3 \bar{a}_1 (L \otimes \bar{a}_2 K \otimes \bar{a}_3 K). \end{aligned}$$

But, $K \subseteq L$, so that

$$(K \otimes \bar{a}_2 K \otimes \bar{a}_3 L) \subseteq (K \otimes \bar{a}_2 L \otimes \bar{a}_3 L).$$

The inequation

$$(x \cdot z) \otimes (y \cdot z) \leq z \otimes ((x \otimes y) \cdot z)$$

is valid in $\mathbf{SP}(A)$ (see [7]). Hence,

$$(K \otimes \bar{a}_2 L \otimes \bar{a}_3 L) \subseteq (K \otimes L \otimes (\bar{a}_2 \otimes \bar{a}_3)L).$$

Thus,

$$\begin{aligned} U_1 &\subseteq a_1 a_2 a_3 \bar{a}_1 (K \otimes L \otimes (\bar{a}_2 \otimes \bar{a}_3)L) \\ &\subseteq Q'h_0. \end{aligned}$$

Similarly,

$$\begin{aligned} L \otimes \bar{a}_2 K \otimes \bar{a}_3 K &\subseteq L \otimes K \otimes (\bar{a}_2 \otimes \bar{a}_3)K \\ &\subseteq K \otimes L \otimes (\bar{a}_2 \otimes \bar{a}_3)L, \end{aligned}$$

so that

$$\begin{aligned} U_2 &\subseteq a_1 a_2 a_3 \bar{a}_1 (K \otimes L \otimes (\bar{a}_2 \otimes \bar{a}_3)L) \\ &\subseteq Q'h_0. \quad \square \end{aligned}$$

Lemma 30. Suppose that P, Q belong to $\mathbf{SP}_1(A)$ and that Q is a successor of P . Let P', Q' be defined by

$$\begin{aligned} P' &:= (a \cdot P) \otimes (a \cdot P) \otimes (a \cdot Q) \\ Q' &:= (a \cdot (P \otimes Q)) \otimes ((a \otimes a) \cdot Q). \end{aligned}$$

Then Q' is a successor of P'

Proof. Lemma 29 showed that $P' \leq Q'$. Suppose that

$$P' \leq R \leq Q'.$$

R must have 3 minimal vertices. Let R_1 denote the poset in $\mathbf{SP}(A)$ obtained from R by deleting the minimal vertices. Let $P_1 := P \otimes P \otimes Q$ and $Q_1 := P \otimes Q \otimes Q$.

We claim that $P_1 \leq R_1 \leq Q_1$. Indeed, if $R_1 \not\leq Q_1$, there is a word $u \in R_1 h_0 - Q_1 h_0$. Then $a_1^3 \bar{a}_1^3 u \in R h_0 - Q' h_0$, a contradiction. Similarly, $P_1 \leq R_1$. But, since Q_1 is a successor of P_1 by Lemma 27, it follows that $R_1 = P_1$ or $R_1 = Q_1$. Thus, R_1 has 3 connected components, each having a unique minimal element.

Suppose that the minimal elements of R_1 are y_1, y_2, y_3 . But R itself has three minimal elements, say x_1, x_2, x_3 . Each y_i is a successor of at least one x_j . Up

to isomorphism, there are only two possible posets determined by the order relations among the x_i 's and y_j 's:

$$(x_1 \cdot y_1) \otimes (x_2 \cdot y_2) \otimes (x_3 \cdot y_3) \quad (9)$$

or

$$((x_1 \otimes x_2) \cdot y_1) \otimes (x_3 \cdot (y_2 \otimes y_3)). \quad (10)$$

Indeed, if, for example, the ordering were

$$((x_1 \otimes x_2) \cdot (y_1 \otimes y_2)) \otimes (x_3 \cdot y_3),$$

then the word

$$a_1^3 \bar{a}_1 (a_2 \bar{a}_2)^K \bar{a}_1 (a_2 \bar{a}_2)^K \bar{a}_1 (a_2 \bar{a}_2)^K$$

would be in $P'h_0 - Rh_0$, where $K = |P| \geq 2$.

Note that by the "N-condition" (see Remark 6), in case (9) it is impossible that x_i is below any vertex above x_j , when $i \neq j$. In case (10), x_3 is not below any vertex above both x_1 and x_2 , and neither x_1 nor x_2 is below any vertex above x_3 .

We now consider the two cases separately.

Case (9). In this case, either

$$R = (a \cdot P) \otimes (a \cdot P) \otimes (a \cdot Q) = P', \text{ or} \quad (11)$$

$$R = (a \cdot P) \otimes (a \cdot Q) \otimes (a \cdot Q). \quad (12)$$

We show that (12) is impossible. Indeed, if v is a word in $Qh_0 - Ph_0$ and u is a word in Ph_0 , neither of which contain the letters $a_1, \bar{a}_1$, then

$$a_1^3 \bar{a}_1 v \bar{a}_1 u \bar{a}_1 u \in Rh_0 - Q'h_0.$$

Case (10). In this case, either

$$R = a \cdot (P \otimes Q) \otimes (a \otimes a) \cdot Q = Q', \text{ or} \quad (13)$$

$$R = a \cdot (Q \otimes Q) \otimes (a \otimes a) \cdot P, \text{ or}, \quad (14)$$

$$R = a \cdot (P \otimes Q) \otimes (a \otimes a) \cdot P, \text{ or}, \quad (15)$$

$$R = a \cdot (P \otimes P) \otimes (a \otimes a) \cdot Q. \quad (16)$$

We show each of the last three possibilities leads to a contradiction. Choose maximal words v in $Qh_0 - Ph_0$ and $u \in Ph_0$, neither of which contain the letters $a_1, \bar{a}_1$.

(15) and (16) are impossible. Otherwise $P' \not\leq R$. Indeed, if (15) holds,

$$a_1^3 \bar{a}_1 u \bar{a}_1 u \bar{a}_1 v \in P'h_0 - Rh_0.$$

And if (16) holds,

$$a_1^3 \bar{a}_1 v \bar{a}_1 u \bar{a}_1 u \in P'h_0 - Rh_0.$$

(14) is impossible: otherwise $R \not\leq Q'$. Indeed, writing $v = a_i \bar{a}_i v'$,

$$a_1^3 \bar{a}_1 a_i \bar{a}_i v' \bar{a}_i v' \bar{a}_1 u \in Rh_0 - Q'h_0.$$

The proof of Lemma 30 is complete. $\square$

Corollary 31. For each $n \geq 1$, P'_n is a successor of P_n . Thus property (P1) of Theorem 18 is established. $\square$

We now establish property (P5) in the following form.

Lemma 32. Suppose that B is any set containing the letter 'a'. Suppose that $Q \leq Q' \in \mathbf{SP}(B)$ and $\varphi : \mathbf{SP}(B) \rightarrow \mathbf{SP}(A)$ is an ordered bimonoid morphism such that $Q\varphi = P_n$ and $Q'\varphi = P'_n$. Then $|Q| > n$.

Proof. We may assume that $b\varphi \neq 1$, all $b \in B$, and that $b\varphi = a \Rightarrow b = a$. We use induction on n . The case $n = 1$ is clear. Suppose that $Q \leq Q'$ and that $Q\varphi = P_{n+1}$ and $Q'\varphi = P'_{n+1}$. It is easy to see that

$$\begin{aligned} Q &= (a \cdot Q_1) \otimes (a \cdot Q_2) \otimes (a \otimes Q_3) \\ Q' &= (a \cdot (Q'_1 \otimes Q'_2)) \otimes ((a \otimes a) \cdot Q'_3), \end{aligned}$$

where

$$\begin{aligned} Q_i\varphi &= tb(P_n), \quad i = 1, 2 \quad \text{and} \quad Q_3\varphi = tb(P'_n) \\ Q'_1\varphi &= tb(P_n) \quad \text{and} \quad Q'_j\varphi = tb(P'_n), \quad j = 2, 3. \end{aligned}$$

The posets Q_i, Q'_i , $i \in [3]$, have a unique maximal and minimal element.

Sublemma 33. Suppose that R_i, R'_i , $i = 1, 2$, are posets in $\mathbf{SP}_1(B)$ such that

$$R_1 \otimes R_2 \leq R'_1 \otimes R'_2.$$

Then either $R_1 \leq R'_1$ or $R_2 \leq R'_2$.

Proof. In order to obtain a contradiction, assume that u_i is a maximal word in $R_i h_0 - R'_i h_0$, $i = 1, 2$. We may assume that

$$\begin{aligned} u_1 &= x\bar{x}v_1y\bar{y} \\ u_2 &= x'\bar{x}'v_2y'\bar{y}' \end{aligned}$$

where x, y, x', y' are distinct letters in $\Sigma(B)$ and where v_1 and v_2 have no letter in common and do not contain the symbols $x, y, x', y', \bar{x}, \bar{y}, \bar{x}', \bar{y}'$. Then

$$xx'\bar{x}v_1y\bar{x}'v_2y'\bar{y}\bar{y}' \in R_1 h_0 \otimes R_2 h_0 - (R'_1 h_0 \otimes R'_2 h_0). \quad \square$$

Note that by symmetry, with the assumptions of Sublemma 33, also $R_1 \leq R'_1$ or $R_2 \leq R'_2$.

Sublemma 34. Suppose that R_i, R'_i , $i \in [3]$, belong to $\mathbf{SP}_1(B)$. Let

$$\begin{aligned} R &:= (a \cdot R_1) \otimes (a \cdot R_2) \otimes (a \cdot R_3) \\ R' &:= (a \cdot (R'_1 \otimes R'_2)) \otimes ((a \otimes a) \cdot R'_3). \end{aligned}$$

Then if $R \leq R'$, then

$$\begin{aligned} R_1 \otimes R_2 &\leq R'_1 \otimes R'_2 \\ R_3 &\leq R'_3, \quad \text{and} \\ |R_1| &= |R_2| = |R_3|. \end{aligned}$$

Of course, by symmetry, also many more relations hold, e.g.,

$$\begin{aligned} R_1 \otimes R_3 &\leq R'_1 \otimes R'_2 \\ R_2 &\leq R'_3. \end{aligned}$$

Proof of Sublemma 34. First, we show $|R_1| + |R_2| = |R'_1| + |R'_2|$. Let $u_i \in R_i h_0$ be maximal, $i \in [3]$, and assume the three words have no letter in common and do not contain the letter a_1 . Write

$$u_i = u'_i y_i \bar{y}_i,$$

where $y_i, i \in [3]$, are letters. Since the word

$$w := a_1^3 \bar{a}_1 u'_1 y_1 \bar{a}_1 u'_2 y_2 \bar{a}_1 u'_3 y_3 \bar{y}_1 \bar{y}_2 \bar{y}_3$$

is in Rh_0 and since $R \leq R'$, we know $w \in R'h_0$. This is possible only if $|R_1| + |R_2| = |R'_1| + |R'_2|$. By symmetry, $|R_1| + |R_3| = |R'_1| + |R'_2|$ and $|R_2| + |R_3| = |R'_1| + |R'_2|$. Hence,

$$|R_1| = |R_2| = |R_3| = N, \text{ say.}$$

Then, $|R'_1| + |R'_2| = 2N$ and $|R'_3| = N$. But then

$$R_1 \otimes R_2 \leq R'_1 \otimes R'_2 \quad \text{and} \quad R_3 \leq R'_3.$$

Indeed, let $u \in (R_1 \otimes R_2)h_0$ be a word not containing the symbol a_1 . Let $v \in R_3 h_0$ have no letter in common with u . Then

$$w := a_1^3 \bar{a}_1 \bar{a}_1 u \bar{a}_1 v \in Rh_0,$$

so that $w \in R'h_0$. But this can happen only if $u \in (R'_1 \otimes R'_2)h_0$ and $v \in R'_3 h_0$. $\square$

Proof of Lemma 32, completed. By Sublemmas 34 and 33, it follows that $Q_1 \leq Q'_2$ or $Q_2 \leq Q'_2$. Suppose that $Q_2 \leq Q'_2$. Since $Q_2 \varphi = tb(P_n)$ and $Q'_2 \varphi = tb(P'_n)$, we have $Q_2 = tb(\bar{Q}_2)$ and $Q'_2 = tb(\bar{Q}'_2)$, where $\bar{Q}_2 \varphi = P_n$ and $\bar{Q}'_2 \varphi = P'_n$. By the induction assumption, $|\bar{Q}_2| > n$, so that $|Q_2| > n + 2$, showing $|Q| > n + 1$. $\square$

Remark 35. With a different argument, we can show that the posets P_n and P'_n satisfy the following stronger version of (P5). For all terms $t(x_1, \dots, x_k)$ and $t'(x_1, \dots, x_k)$ such that each variable x_i occurs the same number of times in t as in t' , if $P_n = t(Q_1, \dots, Q_k)$ and $P'_n = t'(Q_1, \dots, Q_k)$, for some posets Q_j in $\mathbf{SP}(A)$, then $|t| > n$.

5 Final Remarks

Although in the last section we formulated most of the lemmas only for the case that A is a singleton, several of them hold for larger alphabets. The assumption that A is a singleton made some proofs notationally simpler.

After proving the metatheorem Theorem 18, the nontrivial task was to find the actual posets P_n, P'_n satisfying (P1)–(P5). The difficulty was finding the balance

among the various properties. According to (P1), P_n and P'_n should be close to one another. But, by (P4) and (P5), their structure should be very different. By (P4), they cannot have a parallel decomposition into nontrivial posets related by the trace ordering, again a property that says they are far from one another. The particular construction we used was not forced. In particular, we might have made the following definitions:

$$\begin{aligned} P_{n+1} &:= ((a \otimes a) \cdot tb(P_n)) \otimes ((a \otimes a) \cdot tb(P'_n)) \\ P'_{n+1} &:= (a \cdot tb(P'_n)) \otimes ((a \otimes a \otimes a) \cdot tb(P'_n)). \end{aligned}$$

The proof for this choice is similar.

The lemmas proved in Section 4 do not give a full account of the order structure of $\mathbf{SP}(A)$, but are sufficient for the present paper. We plan to give a systematic account of this topic, together with some other results, in a future paper.

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