Semi-Trace Morphisms^{*} and Rational Transductions

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Abstract. We investigate trace and semi-trace morphisms from an algebraic point of view thanks to rational transductions. The main result is a characterization of (semi-) trace morphisms which are equivalent to some rational transduction. Within this result we easily characterize context-free trace morphisms.

Introduction

The free partially commutative monoids study was initiated by Cartier and Foata [CF69] whose aim was to solve some combinatorial problems. Trace languages, which are subsets of a free partially commutative monoid, were proposed by Mazurkiewicz [Maz77] as tools for the description of concurrent program behaviour. Since then a lot of work has been done in this theory, the major contributions to this theory are presented in the *Book of Traces* [DR95].

A partially commutative alphabet is a couple (A, θ) in which A is an alphabet, and θ , the independence relation, is a symmetric and irreflexive binary relation over A . Associated with the independence relation θ , an application $f_\theta : 2^{A^*} \rightarrow 2^{A^*}$ is defined as follows: $f_\theta(L)$ is the set of the words which are equivalent to some word of L for the congruence generated by θ . Thus f_θ is a unary operation over languages which is named *partial commutation function associated with θ* . M. Clerbout and M. Latteux [CL87] introduced the notion of semi-commutation which generalizes the notion of commutation: a semi-commutation is an irreflexive independence relation over A . Thanks to non symmetric rules it is easy to model a program behaviour on a shared memory machine as, for instance, the producer/consumer system: let p and c denote respectively the producer and the consumer, the non-symmetric rule $cp \longrightarrow pc$ means that for all program $A = ucpv$, the program $B = upcv$ has got the same result.

Since the well known model of concurrent systems, the Petri-nets, are just collections of appropriately integrated producer/consumer systems, semi-commutations are useful for modelling behaviours of Petri-nets [HK89, Och90, Och92].

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There is no need to prove the importance of morphisms within mathematics. There is, moreover, an important reason for investigations of morphisms on semi-commutations. Given a machine M_A with an instruction set A and a machine M_B with an instruction set B , the instruction set A may represent high level operations and B may represent basic operations. Such a correspondence between A and B is called the refinement of actions.

Trace morphisms were already investigated, in particular injective trace morphisms were studied by Chrobak et al., Ochmański [CR87, Och88] and more recently by Bruyère et al. and Diekert et al. [BDFG94, DMR94]. Recognizable trace morphisms (i.e. for which the image of any recognizable trace language is recognizable) were characterized by Duboc [Dub86a] and conditions were given for recognizable semi-commutation morphisms [CGL⁺92, OW93].

This paper deals with semi-trace morphisms which are equivalent to some rational transduction. We are not going to introduce special rational transductions for semi-traces. Our goal is just to use rational transductions as a tool to examine semi-trace morphisms from an algebraic point of view. Therefore we work in the free monoid: a semi-trace $[t]_\theta$ is associated with a set of word $f_\theta(t)$ and a semi-trace morphism $\tilde{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ is associated with a word morphism $\varphi : A^* \mapsto B^*$ and $\tilde{\varphi}$ is equivalent to $\tilde{\varphi} = f_\gamma \circ \varphi$.

The question to decide whether a semi-trace morphism is equivalent to some rational transduction can be reduced to the following one: given a semi-trace morphism $\tilde{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$, does there exist a rational transduction $\tau : A^* \mapsto 2^{B^*}$ such that for any word $u \in A^*$ we have $\tilde{\varphi}(u) = f_\gamma \circ \varphi(u) = \tau \circ f_\theta(u)$? In this paper we characterize semi-trace morphisms which are equivalent to some rational transduction. Thanks to this characterization we are able to decide whether or not a trace morphism is a context-free trace morphism, that is, given a trace morphism $\tilde{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$, is the image of any context-free language of $\mathbf{M}(A, \theta)$ by $\tilde{\varphi}$ equal a context-free language of $\mathbf{M}(B, \gamma)$?

After recalling basic notions, we precise the notion of semi-trace morphisms. In Section 3, we introduce a model of rational transduction and show the limits of the model. In Section 4, within this model we characterize semi-trace morphisms which are equivalent to some rational transduction. Then, thanks to this result we characterize context-free trace morphisms and give a new proof of a result of Clerbout and Roos [CR90].

1 Background

We assume that the reader is familiar with the formal language theory.

For $w \in A^*$, $\text{alph}(w)$ is the set of letters which occur in the word w , $|w|$ means the length of w .

We recall some basic notions about directed graphs: a *digraph* G is a pair (V_G, A_G) where V_G is a finite set of *vertices* and $A_G \subseteq V_G \times V_G$ is a finite set of *arcs*. There exists a *path* from a vertex u to a vertex v if there are arcs $(u_0, u_1), (u_1, u_2) \dots (u_n, u_{n+1})$ in A_G with $u_0 = u$ and $u_{n+1} = v$. A digraph G is *strongly connected* if, for each pair of vertices v, w there is a path in G from

v to w . A digraph G is *connected* if, for each pair of vertices v, w there is an undirected path in G from v to w .

Morphisms and rational transductions (from [Ber79])

Let A and B be two alphabets. Any mapping $\varphi : A \mapsto B^*$ extended to $\varphi : A^* \mapsto B^*$ with $\varphi(uv) = \varphi(u)\varphi(v)$ is said to be a *morphism*.

The mapping $\varphi^{-1} : B^* \mapsto 2^{A^*}$ defined by $\varphi^{-1}(u) = \{w \in A^* \mid \varphi(w) = u\}$ is named the *inverse morphism* associated with φ .

A morphism $\psi : A^* \mapsto 2^{B^*}$ is said to be a *substitution*.

The *projection* $\Pi_B(w) : A^* \mapsto B^*$, for $B \subset A$ is the morphism defined by: $\Pi_B(a) = a$ if $a \in B$ else $\Pi_B(a) = \varepsilon$.

Let A and B be alphabets. A *rational relation* R defined over $A^* \times B^*$ is a rational subset of $A^* \times B^*$.

A *transduction* τ from A^* into B^* is a function from A^* into 2^{B^*} , the *graph* of τ is the relation R defined by $R = \{(u, v) \mid v \in \tau(u)\}$.

A transduction $\tau : A^* \mapsto B^*$ is rational iff its graph R is a rational relation over A and B .

The next theorem, due to Nivat, characterized rational transduction:

Theorem 1 [Niv68]. *The following assertions are equivalent:*

- τ is a rational transduction.
- There exist an alphabet Z , a rational language $K \subseteq Z^*$ and a couple (φ, ψ) of morphisms (a bimorphism) $\varphi : Z^* \mapsto A^*$ and $\psi : Z^* \mapsto B^*$ such that $\tau(u) = \psi(\varphi^{-1}(u) \cap K)$.

As a corollary we have: the families of recognizable languages and context-free languages are closed under rational transductions.

An other important theorem, due to Elgot and Mezei, is that the composition of two rational transductions is again a rational transduction.

Semi-commutations

A *semi-commutation* (A, θ) is a finite alphabet A equipped with an *irreflexive semi-commutation relation* $\theta \subseteq A \times A - \{(x, x) \mid x \in A\}$.

The *converse* of θ is the relation $\theta^{-1} = \{(b, a) \mid (a, b) \in \theta\}$. A symmetrical semi-commutation (i.e. such that $\theta^{-1} = \theta$) is named a *partial commutation*.

The *non-commutation graph* of (A, θ) is the pair $(A, \bar{\theta})$, where $\bar{\theta} = A \times A - \theta$ is the *non-commutation relation* associated with θ .

We say that a word $w \in A^*$ is (*strongly*) *connected* iff the subgraph $(\text{alph}(w), \bar{\theta})$ is (strongly) connected.

With each semi-commutation (A, θ) is associated a rewriting system $S = \langle A, P \rangle$ named *semi-commutation system* in which P is the set $\{xy \rightarrow yx \mid (x, y) \in \theta\}$. We write $u \xrightarrow{\theta} v$ if there is a rule $xy \rightarrow yx$ in P and two words w

and w' such that $u = xwyw'$ and $v = wyxw'$. We write $\xrightarrow[\theta]{*}$ for the reflexive and transitive closure of $\xrightarrow{\theta}$.

Each semi-commutation (A, θ) induces a *semi-commutation function* $f_\theta : 2^{A^*} \mapsto 2^{A^*}$ as follows: $f_\theta(L) = \{u \in A^* \mid \exists w \in L, w \xrightarrow[\theta]{*} u\}$.

We say that a language $L \subseteq A^*$ is θ -closed iff $f_\theta(L) = L$. A word m in A^* is *rigid* for θ if $f_\theta(m) = \{m\}$.

We will use the following theorem proved by Clerbout and Latteux (see also [CLRZ92]):

Theorem 2 [CL87]. *If R_1 and R_2 are two recognizable θ -closed languages, then $R_1 \cup R_2$, $f_\theta(R_1 R_2)$ are recognizable. Moreover, if any word in R_1 is strongly connected w.r.t. $\bar{\theta}$ then $f_\theta(R_1^*)$ is recognizable.*

Corollary 3 [CL87]. *Let $w \in A^*$. The language $f_\theta(w^*)$ is recognizable iff w is strongly connected w.r.t. $\bar{\theta}$.*

2 Semi-trace morphisms

Let (A, θ) be a semi-commutation. A semi-trace monoid is the 3-tuple $\mathbf{M}(A, \theta) = (\mathbf{M}, \cdot, 1)$ where $\mathbf{M} = \{[u]_\theta \mid u \in A^*\}$, the concatenation is defined $[u]_\theta \cdot [v]_\theta = [uv]_\theta$ and the neutral element $1 = [\varepsilon]_\theta$. An element $[u]_\theta$ in $\mathbf{M}(A, \theta)$ is called a *semi-trace*, it is equal to the set of words $f_\theta(u)$.

When (A, θ) is a partial commutation, $\mathbf{M}(A, \theta)$ is a trace monoid and an element of $\mathbf{M}(A, \theta)$ is called a trace.

Let (A, θ) and (B, γ) be semi-commutations. A word morphism $\varphi : A^* \mapsto B^*$ defines a *semi-trace morphism* $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ if the image of a semi-trace is a semi-trace: we must have $[\bar{\varphi}([u.v]_\theta)]_\gamma = [\bar{\varphi}([u]_\theta) \cdot \bar{\varphi}([v]_\theta)]_\gamma$. They are such that $ab \xrightarrow[\theta]{*} ba \Rightarrow \varphi(ab) \xrightarrow[\gamma]{*} \varphi(ba)$.

For instance, let $A = \{a, b\}$ and $\varphi : A^* \mapsto A^*$ be the identity morphism, $\bar{\varphi} : \mathbf{M}(A, \{(a, b)\}) \mapsto \mathbf{M}(A, \emptyset)$ is not a semi-trace morphism: indeed $\bar{\varphi}([ab]_\theta) = [ab]_\emptyset + [ba]_\emptyset = ab + ba$ and $\bar{\varphi}([a]_\theta) \cdot \bar{\varphi}([b]_\theta) = [ab]_\emptyset = ab$.

Since rational transductions are defined over free monoids, we are not going to work directly with semi-traces: we consider that a semi-trace is a set of words. From this point of view $\bar{\varphi}$ is equivalent to $\bar{\varphi} : A^* \mapsto 2^{B^*}$ defined by $\bar{\varphi} = f_\gamma \circ \varphi$.

2.1 Recognizable trace morphisms

In [Dub86a] Duboc characterized recognizable trace morphisms i.e. such that the image of any recognizable trace language by the morphism is a recognizable trace language. For this purpose she defined the notion of connected trace morphism: *a trace morphism is connected iff the image of any connected trace (w.r.t. $\bar{\theta}$) is a connected trace (w.r.t. $\bar{\gamma}$).*

Proposition 4 [Dub86a]. *A trace morphism is recognizable iff it is connected.*

The next proposition characterizes trace morphisms which are connected, it is based on the characterization of the solutions of the equation $f_\theta(xy) = f_\theta(yx)$ obtained by Cori, Métivier [CM85] and generalized by Duboc [Dub86b].

Fact 5 [Och88]. Let (A, θ) and (B, γ) be partial commutations. A connected word morphism $\varphi : A^* \mapsto B^*$ defines a trace morphism $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ iff $\forall (a, b) \in \theta$:

- either $\text{alph}(\varphi(a)) \times \text{alph}(\varphi(b)) \subseteq \gamma$
- or there exist two integers α and β , and a word t such that
$$\bar{\varphi}(a) = f_\gamma(t^\alpha) \quad \text{and} \quad \bar{\varphi}(b) = f_\gamma(t^\beta)$$

2.2 Recognizable semi-trace morphisms

Definition 6. A semi-trace morphism $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ is recognizable if for any θ -closed recognizable language $L \subseteq A^*$ the language $\bar{\varphi}(L) \subseteq B^*$ is recognizable.

As we have already seen, when we speak about semi-commutations we deal with strongly connectivity rather than connectivity. We say that a semi-trace morphism is strongly connected iff the image of any strongly connected semi-trace (w.r.t. $\bar{\theta}$) is a strongly connected semi-trace (w.r.t. $\bar{\gamma}$). It is easy to see that a recognizable semi-trace morphism is strongly connected, but the converse is false. The problem of characterizing recognizable semi-trace morphism is still open ([CGL⁺92, OW93]).

In order to characterize strongly connected semi-trace morphisms we have to check for any $(x, y) \in \theta$ if the inclusion $\bar{\varphi}(yx) \subseteq \bar{\varphi}(xy)$ (i.e. $\bar{\varphi}(yx) \subseteq \bar{\varphi}(xy)$) holds. We will use the following lemma:

Lemma 7 [Roo89]. Let (A, θ) be a semi-commutation and u be a strongly connected word of A^* . If $m_1 w m_2 \in f_\theta(u^*)$ with $|w|_y > 2|u|_y * \text{Card}(\text{alph}(u))$ for some letter y , then $\text{alph}(w) = \text{alph}(u)$.

Fact 8. Let (A, θ) and (B, γ) be semi-commutations. A strongly connected word morphism $\varphi : A^* \mapsto B^*$ defines a strongly connected semi-trace morphism $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ iff $\forall (x, y) \in \theta$ we have $\bar{\varphi}(yx) \subseteq \bar{\varphi}(xy)$ that is:

- either $\text{alph}(\varphi(x)) \times \text{alph}(\varphi(y)) \subseteq \gamma$
- or there exist two integers α and β , and a word t such that
$$\bar{\varphi}(x) = f_\gamma(t^\alpha) \quad \text{and} \quad \bar{\varphi}(y) = f_\gamma(t^\beta)$$

Proof. We focus on the case $(x, y) \in \theta$ and $\text{alph}(\varphi(x)) \times \text{alph}(\varphi(y)) \not\subseteq \gamma$. We are going to show that if $\bar{\varphi}$ is a strongly connected semi-trace morphism then there exist two positive integer p and q such that $\bar{\varphi}(x^p) = \bar{\varphi}(y^q)$.

Set $u = \varphi(x)$ and $v = \varphi(y)$, let $a \in \text{alph}(u)$ and $b \in \text{alph}(v)$ be such that $(a, b) \in \bar{\gamma}$. If $\text{alph}(u) \cap \text{alph}(v) = \emptyset$ then $f_\gamma(\Pi_{ab}(uv)) = a^b b^j$ and $\Pi_{ab}(vu) = b^j a^i$ therefore $\bar{\varphi}(yx) \not\subseteq \bar{\varphi}(xy)$ i.e. $\bar{\varphi}$ is not a semi-trace morphism.

Let $a \in \text{alph}(u) \cap \text{alph}(v)$, and choose $p > 0$ and q such that $|u^p|_a = |v^q|_a$.

As $vu \in f_\gamma(uv)$ we get $v^q u^p \in f_\gamma(u^p v^q)$, and from Levi's Lemma for semi-trace there exist words u_1, u_2, v_1, v_2 such that: $u_1 u_2 \in f_\gamma(u^p)$, $v_1 v_2 \in f_\gamma(v^q)$, $\text{alph}(u_2) \times \text{alph}(v_1) \subseteq \gamma$, $v^q \in f_\gamma(u_1 v_1)$ and $u^p \in f_\gamma(u_2 v_2)$.

Since $|u^p|_a = |v^q|_a$ we must have $|u_2 v_1|_a = 0$.

Suppose that $u_2 \neq \varepsilon$: observe that for any z in $\text{alph}(u_2)$ we have $|u^p|_z > |v^q|_z$.

Let $N = |u_2|_b = |u^p|_b - |v^q|_b$ with $b \in \text{alph}(u_2)$.

For every $k \in \mathbb{N}$ we have $v^{qk} u^{pk} \in f_\gamma(u^{pk} v^{qk})$ and there exist words $u_{1k}, u_{2k}, v_{1k}, v_{2k}$ such that: $u_{1k} u_{2k} \in f_\gamma(u^{kp})$, $v_{1k} v_{2k} \in f_\gamma(v^{kq})$, $\text{alph}(u_{2k}) \times \text{alph}(v_{1k}) \subseteq \gamma$, $v^{kq} \in f_\gamma(u_{1k} v_{1k})$ and $u^{kp} \in f_\gamma(u_{2k} v_{2k})$.

Clearly we have $|u_{2k} v_{1k}|_a = 0$ and $|u_{2k}|_b = k \cdot N$. Taking $k > 2|u|_b * \text{Card}(\text{alph}(u))$ we obtain a contradiction with Lemma 7 : we cannot have $u_{1k} u_{2k} \in f_\gamma(u^{pk})$ with $|u_{2k}|_b > 2|u|_b * \text{Card}(\text{alph}(u))$ and $|u_{2k}|_a = 0$ since u is strongly connected.

Thus $u_2 = \varepsilon$ and, by symmetry, $v_1 = \varepsilon$ and then we get $v^q \in f_\gamma(u^p)$ and $u^p \in f_\gamma(v^q)$ that is $vu \in f_{\gamma \cap \gamma^{-1}}(uv)$. So the inclusion $f_\gamma(vu) \subseteq f_\gamma(uv)$ is equivalent to the trace equation $f_{\gamma \cap \gamma^{-1}}(vu) = f_{\gamma \cap \gamma^{-1}}(uv)$. $\square$

3 From words into semi-traces – $\vec{\varphi} \equiv \tau$

This section is devoted to a particular case: we want the transduction to do all the work... Since rational transductions have a lot of good properties, they are sometimes powerless, so the class of semi-trace morphisms which are equivalent to rational transductions is poor.

For instance, consider the following identity morphism $\vec{\varphi} : \mathbf{M}(\{a, b\}, \{(a, b)\}) \mapsto \mathbf{M}(\{a, b\}, \{(a, b)\})$. The morphism $\vec{\varphi}$ is not a rational transduction:

$\vec{\varphi}((ab)^*) \cap b^* a^* = \{b^n a^n \mid n \in \mathbb{N}\}$ this is not a rational function.

The goal of this section is, in fact, to introduce a model of rational transduction which will be helpful for the general case. But first let us characterize semi-trace morphisms which are rational transductions.

Proposition 9. *A semi-trace morphism $\vec{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ is equivalent to a rational transduction iff*

- $\vec{\varphi}$ is strongly connected.
- $\forall (x, y) \in \theta$ there exist two integers α and β and a word $t \in B^*$ such that $\vec{\varphi}(x) = f_\gamma(t^\alpha)$ and $\vec{\varphi}(y) = f_\gamma(t^\beta)$.

Proof. If there exists a word w in A^* such that $\varphi(w)$ is not strongly connected for $\bar{\gamma}$ then from Corollary 3 the language $\vec{\varphi}(w^*)$ is not recognizable.

The second condition is necessary: since $\vec{\varphi}$ is a strongly connected semi-trace morphism we must have $\vec{\varphi}(xy) = \vec{\varphi}(yx)$, now if the second condition is not fulfilled (i.e. $\text{alph}(\varphi(x) \times \text{alph}(\varphi(y))) \subseteq \gamma$, $\varphi(x) \neq \varepsilon$ and $\varphi(y) \neq \varepsilon$) it is not difficult to prove that $\vec{\varphi}((xy)^*)$ is not a recognizable language.

For the converse we construct a rational transduction τ :

3.1 The transduction

We are going to use a bimorphism to define the transduction – assume that $A \cap B = \emptyset$ – the first idea is to define the following transduction $\tau = \Pi_B \circ (\cap L) \circ \Pi_A^{-1}$,

where $L = f_\lambda \left(\left(\bigcup_{a \in A} a\varphi(a) \right)^* \right)$ with $(A \cup B, \lambda) = (A \cup B, \gamma \cup (A \times B) \cup (B \times A))$; and Π_A and Π_B are the projections over the alphabets A and B .

But generally this is not a rational function: the language L is not recognizable since $a\varphi(a)$ is not a strongly connected word as soon as $\varphi(a)$ is not ε . Hence we have to add a symmetric dependency between a and $\varphi(a)$, a good solution seems to connect the letter a with the last letter of $\varphi(a)$ but then, in general, there exist words such that $\tau(w) \subset \bar{\varphi}(w)$. It is impossible to define a general rational transduction over the alphabet $(A \cup B)$ with a semi-commutation. For this purpose we use a new alphabet B_ψ and a finite substitution ψ which allows to distinguish the position of any letter of each element of $\bar{\varphi}(a)$; then for any a in A we connect the letter a with the first letter and the last letter of each word in $\psi(a)$.

Let B_ψ be the following alphabet

$$B_\psi = \bigcup_{a \in A} \{a_{n,l} \mid 0 < n \leq \text{Card}(\bar{\varphi}(a)) \text{ and } 0 < l < |\varphi(a)|\}$$

Let $\psi : A^* \mapsto 2^{B_\psi^*}$ be the following finite substitution:

$$\psi(a) = \begin{cases} \bigcup_{n=1}^{\text{Card}(\bar{\varphi}(a))} \{a_{n,1}a_{n,2} \dots a_{n,|\varphi(a)|}\} & \text{if } \varphi(a) \neq \varepsilon \\ \{\varepsilon\} & \text{if } \varphi(a) = \varepsilon \end{cases}$$

By definition, for each letter $a \in A$, there exists a bijection between $\psi(a)$ and $\bar{\varphi}(a)$: we associate $a_{n,1}a_{n,2} \dots a_{n,|\varphi(a)|}$ with the n^{th} element of $\bar{\varphi}(a)$, with respect to the lexicographic order.

The morphism $H_B : (A + B_\psi)^* \mapsto B^*$ erases letters of A and transforms a letter in B_ψ into a letter of B with respect to the bijection between $\psi(a)$ and $\bar{\varphi}(a)$:

$$H_B(x) = \begin{cases} \varepsilon & \text{if } x \in A \\ b & \text{if } x = a_{n,k} \text{ where } b \text{ is the } k^{\text{th}} \text{ letter of the } n^{\text{th}} \text{ element} \\ & \text{of } \bar{\varphi}(a) \text{ with respect to the lexicographic order.} \end{cases}$$

Let us define the morphism $\Pi_A : (A + B_\psi)^* \mapsto A^*$ which erases letters of B_ψ : $\Pi_A(x) = x$ if $x \in A$, ε otherwise.

The rational transduction works on the alphabet $(A + B_\psi)$:

$$\tau = H_B \circ (\cap L) \circ \Pi_A^{-1} \quad \text{with} \quad L = f_\lambda \left(\left(\bigcup_{a \in A} a\psi(a) \right)^* \right)$$

where f_λ is the semi-commutation function associated with the semi-commutation $(A \cup B_\psi, \lambda)$:

$$\begin{aligned} \lambda = & \{(a_{n,k}, b_{n',k'}) \in B_\psi \times B_\psi \mid (H_B(a_{n,k}), H_B(b_{n',k'})) \in \gamma\} \\ & - \{(a_{n,1}, b_{n',|\varphi(b)|}) \in B_\psi \times B_\psi\} \\ & + A \times B_\psi - \{(a, a_{n,|\varphi(a)|}) \in A \times B_\psi\} \\ & + B_\psi \times A - \{(a_{n,1}, a) \in B_\psi \times A\} \end{aligned}$$

Actually the semi-commutation $(A \cup B_\psi, \lambda)$

- simulates f_γ over B_ψ^* but any first letter of any element of $\psi(a)$ cannot commute to the right with the last letter of any element of $\psi(b)$ – this is important to simplify proofs and definitions but not necessary.
- forbids any commutation between letters of A – there is no need to change the input word.
- forbids any letter a in A to commute to the right with the last letter of any element of $\psi(a)$ and to the left with the first letter of any word in $\psi(a)$. Thus any word in $a\psi(a)$ is strongly connected w.r.t. $\bar{\lambda}$ and consequently the language L is recognizable.

3.2 The power of τ

First it is easy to see that the set $H_{B \circ f_\lambda \circ \psi}(u)$ is included in $\bar{\varphi}(u)$: remind that λ is very close to γ – the main difference is that the complete commutation of the images of any two letters is forbidden in λ :

Lemma 10. *Let $u \in A^*$ and $w \in B^*$. If $w \in \bar{\varphi}(u)$ and if the images of any two letters of u have not commuted completely in w w.r.t. u then $w \in H_{B \circ f_\lambda \circ \psi}(u)$.*

The next lemma shows that letters of A do not make troubles. Indeed, if we have

$$a_{r,s} \xrightarrow{\bar{\lambda}} a \xrightarrow{\bar{\lambda}} b \xrightarrow{\bar{\lambda}} b_{p,q} \quad \text{or} \quad a_{r,s} \xrightarrow{\bar{\lambda}} a \xrightarrow{\bar{\lambda}} a_{p,q}$$

then $a_{r,s}$ is the first letter of an element of $\psi(a)$ and $b_{p,q}$ (resp. $a_{p,q}$) is the last letter of an element of $\psi(b)$ (resp. $\psi(a)$). Thus we have

Lemma 11. *Let $u \in A^*$ and $w \in B^*$. If $w \in H_{B \circ f_\lambda \circ \psi}(u)$ then $w \in \tau(u)$.*

To establish Proposition 9 we prove that the rational transduction τ is equal to $\bar{\varphi}$: observe that from the necessary conditions we have $\text{alph}(\varphi(a)) \times \text{alph}(\varphi(b)) \not\subseteq \gamma$ for any non erased letters a and b in A : on one hand we know that for any $(a, b) \in \theta$ we have $\text{alph}(\varphi(a)) = \text{alph}(\varphi(b))$; on the other hand φ is strongly connected.

Therefore the images of two letters cannot totally commute with γ . Thus, due to Lemma 10, for any u we have $H_{B \circ f_\lambda \circ \psi}(u) = \bar{\varphi}(u)$, and thanks to Lemma 11, we conclude that $\tau = \bar{\varphi}$.

4 The general case – $\bar{\varphi} \equiv \tau \circ f_\theta$

In this part we apply first f_θ and then the transduction τ i.e. we first compute a set of words equivalent to a semi-trace and then we apply to this set a transduction. We will show $\bar{\varphi}$ is equivalent to $\tau \circ f_\theta$ iff the the transduction defined in the previous section fits.

Example 12. Let $A = \{a, b, c\}$, $B = \{x, y_1, y_2, y_3, z\}$, $\varphi(a) = x$, $\varphi(b) = y_1 y_2 y_3$, $\varphi(c) = z$ and the semi-commutations defined by the following non-commutation graphs:

we give two different transductions such that $\vec{\varphi} = \tau \circ f_{\theta}$.

First Solution: a transducer

Second solution: $\tau = \Pi_{B \circ (\cap f_{\lambda}((ax + by_1 y_2 y_3 + cz)^*))} \circ \Pi_A^{-1}$

with $\lambda = \{a\} \times \{y_1, y_3, z\} \cup \{b\} \times \{x, y_1, y_3, z\} \cup \{c\} \times \{x, y_1, y_2, y_3\} \cup \{x\} \times \{b, c, y_1, y_3, z\} \cup \{y_1\} \times \{a, b, c, x, z, y_3\} \cup \{y_2\} \times \{a, c, x\} \cup \{y_3\} \times \{a, b, c, x, y_1, z\} \cup \{z\} \times \{a, b, x, y_1, y_2, y_3\}$

In the previous section we saw that φ must be strongly connected. Of course this condition is still necessary. First observe that the equation $\vec{\varphi} = \tau \circ f_{\theta}$ concerns only semi-trace morphisms: $\vec{\varphi}$ must be as powerful as the function $\vec{\varphi} \circ f_{\theta}$.

Fact 13. Let (A, θ) and (B, γ) be two semi-commutations and $\varphi : A^* \mapsto B^*$ be a morphism. If $\vec{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ is not a semi-trace morphism then there does not exist a rational transduction τ such that $\vec{\varphi} = \tau \circ f_{\theta}$.

Proof. Suppose that $(a, b) \in \theta$. If $\vec{\varphi} = \tau \circ f_{\theta}$ then we have $\varphi(ba) \in \tau \circ f_{\theta}(ba)$. On the other hand we have $\tau \circ f_{\theta}(ba) \subseteq \tau \circ f_{\theta}(ab)$ hence $\vec{\varphi}(ba) \subseteq \vec{\varphi}(ab)$ therefore $\varphi(ba) \in \vec{\varphi}(ab)$. $\square$

Let us define the notion of local semi-trace morphisms: intuitively φ is local iff for every couple (u, v) , such that $v \in \vec{\varphi}(u)$, there exists a word in $f_{\theta}(u)$ near v such that f_{γ} does not need to commute the images of any two letters in u in order to reach v . More formally we have:

Definition 14. Let $\vec{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ be a semi-trace morphism. The morphism φ is local iff for any u in A^* and for each v in $\vec{\varphi}(u)$ there exists a word $w \in f_{\theta}(u)$ such that $v \in H_{B \circ f_{\lambda} \circ \psi}(w)$.

Clearly if φ is strongly connected and local then the transduction τ defined in the previous section is convenient: we have, from Lemma 10, $H_B \circ f_\lambda \circ \psi \circ f_\theta(m) = \bar{\varphi}(m)$ hence, due to Lemma 11, $\tau \circ f_\theta = \bar{\varphi}$.

What about the converse ? Let us give some examples of non-local semi-trace morphisms – the first one deals with partial commutations:

Example 15. Let (A, θ) with $A = \{a, b, c\}$ and $\theta = \{(a, c), (c, a)\}$, (B, γ) with $B = \{a, b_1, b_2, c\}$ and $\gamma = \{(a, c), (c, a), (a, b_1), (b_1, a), (b_2, c), (c, b_2)\}$.

Let the semi-trace morphism $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ with $\varphi(a) = a$ $\varphi(b) = b_1 b_2$ $\varphi(c) = c$. Notice that $\bar{\varphi}$ is connected. The word abc is rigid for θ : $f_\theta(abc) = \{abc\}$. Moreover $\varphi(a^n b c^n) = a^n b_1 b_2 c^n \xrightarrow[\gamma]{*} b_1 a^n c^n b_2 \xrightarrow[\gamma]{*} b_1 (ac)^n b_2$.

Observe that the language $L = \{b^p a^p b c^n b^n \mid n, p \in \mathbb{N}\}$ is θ -closed and context-free. But $\bar{\varphi}(L) \cap (b_1 b_2)^* b_1 (ac)^* b_2 (b_1 b_2)^* = \{(b_1 b_2)^n b_1 (ac)^n b_2 (b_1 b_2)^n \mid n \in \mathbb{N}\}$ is not context-free, therefore $\bar{\varphi}(L)$ is not context-free. Since any word in L is rigid, we must have $\tau(L) = \bar{\varphi}(L)$, this is impossible: τ must be a rational function.

As a matter of fact this example is very close to the general case for partial commutations:

Definition 16. A trace morphism $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ is rigid if for each word $w = x_1 \dots x_n$ in A^* rigid for θ there exists a subword $y_1 \dots y_k$ of $\varphi(w)$ which is rigid for γ and such that $y_1 \in \text{alph}(\varphi(x_1))$ and $y_k \in \text{alph}(\varphi(x_n))$.

Theorem 17. Let $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ be a trace morphism. There exists a rational transduction τ such that $\bar{\varphi} = \tau \circ f_\theta$ iff φ is connected and rigid.

The condition is sufficient since if φ is rigid then it is local. The proof of the necessary condition is only a generalization of Example 15.

For semi-trace morphisms the situation is more complex: let us start with an example.

Example 18. Let (A, θ) with $A = \{a, b, c\}$, $\theta = \{(a, c), (a, b)\}$, (B, γ) with $B = \{a, b_1, b_2, c\}$ and $\gamma = \{(a, c), (a, b_1), (b_1, a), (a, b_2), (b_2, c)\}$.

The semi-trace morphism $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ is defined by $\varphi(a) = a$ $\varphi(b) = b_1 b_2$ $\varphi(c) = c$. Notice that φ is strongly connected. Unlike the previous example the word abc is not rigid for θ : $f_\theta(abc) = \{abc, bac, bca\}$.

Nevertheless we have $\varphi(abc) = ab_1 b_2 c \xrightarrow[\gamma]{2} b_1 a c b_2 \xrightarrow[\gamma]{} b_1 c a b_2$.

But $b_1 c a b_2$ belongs neither to $\bar{\varphi}(bac)$, nor to $\bar{\varphi}(bca)$: the couple (b_2, a) does not belong to γ . Thus, as in the previous example, we have for all integers p, q : $b_1 c^q a^p b_2 \in \tau(a^p b c^q)$. The function $(\cap b_1 c^* a^* b_2) \circ \tau$, defined by the graph $\{(a^p b c^q, b_1 c^q a^p b_2) \mid p, q \in \mathbb{N}\}$, is not a rational subset of $A^* \times B^*$. There does not exist a rational transduction τ such that $\bar{\varphi} = \tau \circ f_\theta$.

This is the characterization of local semi-trace morphisms:

Proposition 19. *Let $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ be a semi-trace morphism. The morphism φ is local iff there does not exist a word $w = x_0 \dots x_n$ in A^* which does not contain two occurrences of the same letter and a word v in $\bar{\varphi}(w)$ such that:*

1. $\varphi(x_0) \neq \varepsilon$, $\varphi(x_n) \neq \varepsilon$ and $\text{alph}(\varphi(x_0)) \cap \text{alph}(\varphi(x_n)) = \emptyset$.
2. for any index i in $[0, n-1]$ the couple (x_i, x_{i+1}) belongs to $\bar{\theta} \cup \bar{\theta}^{-1}$ and moreover if $(x_i, x_{i+1}) \in \theta$ then there exist two letters $\alpha \in \text{alph}(\varphi(x_i))$ and $\beta \in \text{alph}(\varphi(x_{i+1}))$ such that:
 - $\Pi_\alpha(\varphi(w)) = \Pi_\alpha(\varphi(x_i))$.
 - $\Pi_\beta(\varphi(w)) = \Pi_\beta(\varphi(x_{i+1}))$.
 - $\Pi_{\alpha\beta}(v) \notin f_{\gamma \circ \Pi_{\alpha\beta} \circ \varphi}(x_{i+1} x_i)$.
3. $v = v_1 \varphi(x_n) \varphi(x_0) v_2$ with
 $\text{alph}(\varphi(x_0)) \times \text{alph}(v_1) \subseteq \gamma$ and $\text{alph}(v_2) \times \text{alph}(\varphi(x_n)) \subseteq \gamma$.

Observe that if a morphism is strongly connected on letters and local then it is strongly connected. With this characterization it is easy to show that being local is a necessary condition – the proof of the following lemma is based on a generalization of Example 18:

Lemma 20. *Let $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ be a semi-trace morphism. If φ is not local then there does not exist a rational transduction τ such that $\bar{\varphi} = \tau \circ f_\theta$.*

Now we are able to conclude this section:

Theorem 21. *Let $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ be a semi-trace morphism. There exists a rational transduction τ such that $\bar{\varphi} = \tau \circ f_\theta$ iff φ is strongly connected and local.*

5 Context-free trace morphisms

First let us define the notion of context-free trace morphisms:

Definition 22. *A trace morphism $\bar{\varphi} : \mathbf{M}(A, \theta) \mapsto \mathbf{M}(B, \gamma)$ is a context-free trace morphism if for any θ -closed context-free language L the language $\bar{\varphi}(L)$ is context-free.*

We have seen that a trace morphism is recognizable iff it is connected, this condition is not necessary for context-free trace morphisms: consider the trace morphism $\bar{\varphi} : \mathbf{M}(\{a\}, \emptyset) \mapsto \mathbf{M}(\{a, b\}, \{(a, b), (b, a)\})$ with $\varphi(a) = ab$.

It is well known that on a one letter alphabet the family of recognizable languages and context-free languages are equal. For any recognizable language L we have $\bar{\varphi}(L) = (L \sqcup b^*) \cap D_1^*$, where $D_1^* = \{m \in (a+b)^* \mid |m|_a = |m|_b\}$ is the Dyck language, a famous context-free language. Since $L \sqcup b^*$ is recognizable, $\bar{\varphi}(L)$ is context-free, therefore $\bar{\varphi}$ is a context-free trace morphism.

The following proposition characterizes context-free trace morphisms when the alphabet B contains only two letters. Then we generalize this result for non connected morphisms.

Proposition 23 [Lat79]. *Let $(B, \gamma) = (\{a, b\}, \{(a, b), (b, a)\})$ be a partial commutation. If the language $L \subseteq B^*$ is context-free then $f_\gamma(L)$ is context-free.*

Thanks to rational transductions we have:

Lemma 24. *A trace morphism $\bar{\varphi} : \mathbb{M}(A, \theta) \mapsto \mathbb{M}(B, \gamma)$ is a context-free trace morphism if there exist two words t_1 and t_2 connected w.r.t. $\bar{\gamma}$ such that for each $x \in A$ we have $\bar{\varphi}(x) \subseteq f_\gamma(t_1^* t_2^*)$.*

Proof. Let $A = \{a_1, \dots, a_n\}$; for all a_i we have $\bar{\varphi}(a_i) = f_\gamma(t_1^{\alpha_i} t_2^{\beta_i})$.

Let $X = \{x, y\}$, and the morphism $\psi : A^* \mapsto X^*$ defined by $\psi(a_i) = x^{\alpha_i} y^{\beta_i}$ for all $a_i \in A$.

Let (X, λ) be the total commutation. Let $\vec{\delta} : \mathbb{M}(X, \lambda) \mapsto \mathbb{M}(B, \gamma)$ be the trace morphism defined by $\vec{\delta}(x) = t_1$ and $\vec{\delta}(y) = t_2$.

Clearly we have for all word u in A^* $\bar{\varphi}(u) = \vec{\delta} \circ \psi(u)$; the trace morphism $\vec{\delta}$ is connected and rigid. Thus there exists a rational transduction τ such that $f_{\gamma \circ \delta} = \tau \circ f_\lambda$. From Proposition 23, for all context-free language $L \subseteq A^*$, $f_{\lambda \circ \psi}(L)$ is context-free thus the language $\tau \circ f_{\lambda \circ \psi}(L) = \bar{\varphi}(L)$ is context-free too. $\square$

When the morphism $\bar{\varphi}$ is not connected this condition becomes necessary:

Lemma 25. *A non-connected trace morphism $\bar{\varphi} : \mathbb{M}(A, \theta) \mapsto \mathbb{M}(B, \gamma)$ is a context-free trace morphism iff there exist two words t_1 and t_2 connected for $\bar{\gamma}$ such that for each $x \in A$ we have $\bar{\varphi}(x) \subseteq f_\gamma(t_1^* t_2^*)$.*

If a connected trace morphism $\bar{\varphi}$ is rigid, then there exists a rational transduction τ such that $\bar{\varphi} = \tau \circ f_\theta$ and therefore $\bar{\varphi}$ is a context-free morphism. The converse is true: Example 15 is closed to the general case.

Lemma 26. *A connected trace morphism $\bar{\varphi} : \mathbb{M}(A, \theta) \mapsto \mathbb{M}(B, \gamma)$ is a context-free trace morphism iff it is rigid.*

Thanks to the three previous lemmas we obtain

Proposition 27. *A trace morphism $\bar{\varphi} : \mathbb{M}(A, \theta) \mapsto \mathbb{M}(B, \gamma)$ is a context-free trace morphism iff*

- *Either there exist two words t_1 and t_2 connected for $\bar{\gamma}$ such that for each $x \in A$ we have $\bar{\varphi}(x) \subseteq f_\gamma(t_1^* t_2^*)$.*
- *Or φ is connected and rigid.*

We have seen that trace morphisms which are equivalent to some rational transduction form a proper subclass of recognizable trace morphisms. Since a context-free trace morphism may be not connected, the class of recognizable trace morphisms and the class of context-free trace morphisms are incomparable. At the first glance this is quite surprising, but observe that, unlike the family of recognizable trace languages, the family of context-free trace languages is not closed under concatenation.

Notice that we are not able to characterize context-free semi-trace morphisms: as usually the introduction of non-symmetric dependences modifies considerably the problem.

6 Deciding whether $f_\gamma(u^*)$ is context-free

Clerbout and Roos [CR90] have proved that, given a semi-commutation (A, θ) and word u , the language $f_\theta(u^*)$ is context-free iff the graph $(\text{alph}(u), \bar{\theta})$ contains at most two strongly connected components. Thanks to rational transductions we are able to give a new proof of this characterization.

Proposition 28 [CR90]. *Let (B, γ) be a semi-commutation and u be a word in B^* . The language $f_\gamma(u^*)$ is context-free iff the graph $(\text{alph}(u), \bar{\gamma})$ contains at most two strongly connected components.*

This condition is clearly necessary: for instance consider $\gamma = \{(a, b), (c, b), (c, a)\}$ we have $f_\gamma((abc)^*) \cap b^*a^*c^* = \{b^n a^n c^n \mid n \in \mathbb{N}\}$.

Here we focus on the converse. Of course if $(\text{alph}(u), \bar{\gamma})$ strongly connected we know that (Corollary 3) the language $f_\gamma(u^*)$ is recognizable. We have to deal with the case “ $(\text{alph}(u), \bar{\gamma})$ contains two strongly connected components”.

We may assume that u does not contain twice a same letter (otherwise we numbered the different occurrences of the letters of u). We will denote m and w the projections of u over the two components.

6.1 The case $f_\gamma(u) = f_\gamma(mw)$

Here we assume that $f_\gamma(u) = f_\gamma(mw)$ i.e. $u \xrightarrow[\gamma \cap \gamma^{-1}]{*} mw$.

Let $A = \{a, b\}$ and let $\varphi : A^* \mapsto B^*$ be the morphism defined by

$$\varphi(a) = m \quad \text{and} \quad \varphi(b) = w$$

Let (A, θ) be semi-commutation defined by:

$$(a, b) \in \theta \text{ if } \text{alph}(m) \times \text{alph}(w) \subseteq \gamma$$

$$(b, a) \in \theta \text{ if } \text{alph}(w) \times \text{alph}(m) \subseteq \gamma$$

Remark that $f_\gamma(u^*) = f_\gamma((mw)^*) = f_{\gamma \circ \varphi}((ab)^*) = \bar{\varphi}((ab)^*)$.

Note that the morphism φ is strongly connected and, due to the definition of θ , it is a semi-trace morphism and local. Thus there exists a rational transduction τ such that $\bar{\varphi} = \tau \circ f_\theta$. So $\bar{\varphi}((ab)^*)$ is context-free iff $f_\theta((ab)^*)$ is context-free, this is obvious:

$$f_\theta((ab)^*) = \begin{cases} D_1^*(a, b) & \text{if } (a, b) \in \theta \cap \theta^{-1} \\ D_1'^*(a, b) & \text{if } (b, a) \in \theta \cap \bar{\theta}^{-1} \\ (D_1'^*(b, a) + ab)^* & \text{if } (a, b) \in \theta \cap \bar{\theta}^{-1} \end{cases}$$

Thus when $f_\gamma(u) = f_\gamma(mw)$, $f_\gamma(u^*)$ is context-free.

6.2 The case $f_\gamma(u) \neq f_\gamma(mw)$

Here we assume that we have neither $u \xrightarrow[\gamma \cap \gamma^{-1}]{*} mw$, nor $u \xrightarrow[\gamma \cap \gamma^{-1}]{*} wm$. Note that it is impossible to use the same construction as above.

Set $u \xrightarrow[\gamma]{*} mw$ with $wm \xrightarrow[\gamma]{+} mw$.

Let B_{12} be the alphabet $B_1 \cup B_2$ with $B_i = \{x_i \mid x \in B\}$.

We denote by $H_B : B_{12}^* \mapsto B^*$ the morphism defined by $H_B(x_i) = x, \forall x \in B_{12}$.

Let (B_{12}, λ) be the following semi-commutation:

$$\lambda = \{(x_i, y_j) \in B_{12} \times B_{12} \mid (x, y) \in \gamma\} \cup \{(x_i, y_i) \in \text{alph}(m_i) \times \text{alph}(w_i) \mid \Pi_{xy}(u) = yx\}$$

Remark that $f_\gamma(u) = H_B \circ f_\lambda(m_1 w_1) = H_B \circ f_\lambda(m_2 w_2)$, moreover we have

$$f_\gamma(u^*) = H_B \circ f_\lambda((m_1 w_1 m_2 w_2)^*(\varepsilon + m_1 w_1))$$

Let A_{12} be the alphabet $\{a_1, a_2, b_1, b_2\}$ and $\varphi : A_{12}^* \mapsto B_{12}^*$ be the morphism defined by $\varphi(a_i) = m_i$ and $\varphi(b_i) = w_i$, where $m_i \in B_i^*$, $H_B(m_i) = m$ and $w_i \in B_i^*$, $H_B(w_i) = w$. We have

$$f_\gamma(u^*) = H_B \circ f_\lambda((m_1 w_1 m_2 w_2)^*(\varepsilon + m_1 w_1)) = H_B \circ f_\lambda \circ \varphi((a_1 b_1 a_2 b_2)^*(\varepsilon + a_1 b_1)) .$$

Let $\theta_{12} = \{(b_1, a_1), (b_1, a_2), (b_2, a_1), (b_2, a_2)\}$.

It is not difficult to see that $\varphi : \mathbb{M}(A_{12}, \theta_{12}) \mapsto \mathbb{M}(B_{12}, \lambda)$ is a semi-trace morphism, strongly connected and local: therefore there exists a rational transduction τ such that $f_\lambda \circ \varphi = \bar{\varphi} = \tau \circ f_{\theta_{12}}$. Thus $H_B \circ \bar{\varphi} = H_B \circ \tau \circ f_{\theta_{12}}$ and we get

$$f_\gamma(u^*) = H_B \circ \bar{\varphi}((a_1 b_1 a_2 b_2)^*(\varepsilon + a_1 b_1)) = H_B \circ \tau \circ f_{\theta_{12}}((a_1 b_1 a_2 b_2)^*(\varepsilon + a_1 b_1))$$

To conclude we have to check that $f_{\theta_{12}}((a_1 b_1 a_2 b_2)^*(\varepsilon + a_1 b_1))$ is context-free:

Let $(A, \theta) = (\{a, b\}, \{(b, a)\})$ and $\text{alt} : A^* \mapsto 2^{A_{12}^*}$ be the following rational transduction:

$$\text{alt}(vx) = \text{alt}(v)x_i \text{ with } \begin{cases} i = 1 & \text{if } |v|_x = 2p \\ i = 2 & \text{if } |v|_x = 2p + 1 \end{cases} \quad \forall v \in A^* \forall x \in A .$$

Clearly $\text{alt} \circ f_\theta((ab)^*) = f_{\theta_{12}}((a_1 b_1 a_2 b_2)^*(\varepsilon + a_1 b_1))$. As $f_\theta((ab)^*) = D_1'^*(a, b)$ is context-free, the language $f_{\theta_{12}}((a_1 b_1 a_2 b_2)^*(\varepsilon + a_1 b_1))$ is context-free too.

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