Fine Hierarchy of Regular ω -Languages

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Abstract

By applying descriptive set theory we get several facts on the fine structure of regular ω -languages considered by K.Wagner. We present quite different, shorter proofs for main his results and get new results. Our description of the fine structure is new, very clear and automata-free. We prove also a closure property of the fine structure under Boolean operations. Our results demonstrate deep interconnections between descriptive set theory and theory of ω -languages.

1 Introduction

Regular ω -languages were introduced by J.R.Büchi in the sixties and studied by many people including B.A.Trakhtenbrot, R.McNaughton and M.O.Rabin. The subject quickly developed into a rich topic with several deep applications. For the history and references on the subject see e.g. [Th90,St87,Wa79]. We assume acquaintance with some basic concepts, notation and results in this field, all of them may be found in the cited papers.

One of branches of the discussed topic deals with the classifications of regular ω -languages by means of topology, hierarchies and reducibilities. A series of papers culminated with the paper [Wa79] giving in a sense the finest possible classification. Its formulation needs some notions.

Let X be a finite nontrivial alphabet (for simplicity assume that $X = \{x|x < k\}$ for a natural number $k > 1$). Let X^* and X^ω denote respectively the sets of all words and of all ω -words (i.e. sequences $\nu : \omega \rightarrow X$) over X . For $B, C \subseteq X^\omega$, B is said to be *DA-reducible* to C (in symbols $B \leq_{DA} C$), if $B = f^{-1}(C)$ for some function $f : X^\omega \rightarrow X^\omega$ computed by a deterministic asynchronous finite transducer (this relation was studied in [Wa79]). The relation $\leq_{DA}$ is clearly a preordering, and the join $B \oplus C$ gives as usual the supremum of B and C under this relation. By $\equiv_{DA}$ we denote the induced equivalence relation which gives rise to the structure of *DA-degrees*. Our notation and terminology on

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reducibilities are standard and applicable to several modifications of the introduced notions which will be studied further on. Let ω^ω be the supremum of ordinals ω^k , $k < \omega$. One of the main results from [Wa79] may be now reformulated as follows.

1.1. Theorem ([Wa79]). *There are regular ω -languages A_α ($\alpha < \omega^\omega$) such that $A_\alpha \not\leq_{DA} \bar{A}_\alpha$ and $A_\alpha \oplus \bar{A}_\alpha <_{DA} A_\beta$ for all $\alpha < \beta < \omega^\omega$, and any regular ω -language is DA-equivalent to one of $A_\alpha, A_\alpha \oplus \bar{A}_\alpha$ ($\alpha < \omega^\omega$).*

Note that we denote ordinals by $\alpha, \beta, \gamma, \dots$ while ω -words—by $\mu, \nu, \xi, \dots$

An immediate corollary of Theorem 1.1 is the following assertion in which $\mathcal{R}$ denotes the class of all regular ω -languages. For a class $\mathcal{A}$ of ω -languages (i.e. of subsets of X^ω), the structure $(\mathcal{A}; \leq_{DA})$ is said to be *almost well-ordered*, if it is well founded and for all $B, C \in \mathcal{A}$ either $B \leq_{DA} C$ or $\bar{C} \leq_{DA} B$. To any such a structure an ordinal may be related in the usual way.

1.2. Corollary ([Wa79]). *The structure $(\mathcal{R}; \leq_{DA})$ is almost well-ordered with the corresponding ordinal ω^ω .*

The next fact follows from the constructiveness of the proof in [Wa79].

1.3. Corollary ([Wa79]). (i) *There is an algorithm computing from (a Muller automaton for) a regular ω -language B the (notation of the) language from 1.1 DA-equivalent to B .*

(ii) *The DA-reducibility on $\mathcal{R}$ coincides with the Wadge reducibility (i.e. with the reducibility $\leq_{CA}$ by functions continuous in the Cantor topology on the set X^ω).*

Our paper presents an alternative proof of the results stated above which are shorter than the quite complicated proofs from [Wa79]. The reason why it is possible is the fact that we use some nontrivial facts on Wadge degrees, on determinateness of games from [BL69] and on the so called fine hierarchy introduced by me in [Se83] in the context of recursion theory and then applied to some other fields in several subsequent publications. Our paper contains also alternative proofs of some other results from [Wa79] as well as several new results, e.g. an interesting closure property of the hierarchy in 1.1 under the Boolean operations.

Our proof of 1.1 (as well as of other facts from [Wa79]) gives additional information as compared with the proof of K.Wagner, namely:

(i) it gives the estimation of sets from 1.1 in the structure of Wadge degrees of Borel sets (which is known to be almost well-ordered under $\leq_{CA}$);

(ii) it gives a very clear algebraic description of the hierarchy from 1.1;

(iii) our description of the hierarchy in 1.1 is automata-free (as far as we see, the paper [Wa79] does not provide an automata-free description).

So our revision of [Wa79] is not just a simplification. It uses different methodology and is an automata-free supplement to [Wa79].

Though our paper implies most of the results from [Wa79], it by no means covers the paper [Wa79] completely. That paper contains e.g. sophisticated constructions on automata which are interesting in their own right.

The rest of the paper is organized as follows: in Sections 2—5 we formulate some auxiliary facts most of which are known, and in Section 6 we prove our main results.

2 Wadge Hierarchy of Borel Sets

Here we summarize some notation and well-known facts on Borel hierarchy and Wadge reducibility which are useful further on.

Let $\mathcal{B}$ denote the class of Borel subsets of X^ω , i.e. the least class containing the open sets and closed under complementation and countable union. These sets can be classified in a hierarchy the lowest levels of which are as follows: G and F are the classes of open and closed sets, respectively; G_δ (F_σ) is the class of countable intersections (unions) of open (resp. closed) sets; $G_{\delta\sigma}$ ($F_{\sigma\delta}$) is the class of countable unions (intersections) of G_δ - (resp. of F_σ -) sets, and so on.

In notation of hierarchy theory, $\Sigma_0^0 = G \cap F$ is the class of clopen sets, $\Sigma_1^0 = G$, $\Sigma_2^0 = F_\sigma$, $\Sigma_3^0 = G_{\delta\sigma}$, $\Sigma_4^0 = F_{\sigma\delta\sigma}$ and so on, $\Pi_n^0 = \{\bar{B} \mid B \in \Sigma_n^0\}$ is the dual class for Σ_n^0 , and $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$. The sequence $\{\Sigma_n^0\}_{n < \omega}$ is known as *the finite Borel hierarchy*. It may be in a natural way extended on all countable ordinals; the resulting sequence, called *the Borel hierarchy*, exhausts the class $\mathcal{B}$.

Recall that a class $\mathcal{A}$ of sets is said to have *the reduction property*, if for all $B_0, B_1 \in \mathcal{A}$ there are disjoint $B'_0, B'_1 \in \mathcal{A}$ such that $B'_i \subseteq B_i$ and $B_0 \cup B_1 = B'_0 \cup B'_1$. For $B, C \subseteq X^\omega$, let $B \oplus C = \{0 \cdot \mu, i \cdot \nu \mid 0 < i < k, \mu \in B, \nu \in C\}$ (recall that we assume $X = \{x \mid x < k\}, 1 < k < \omega$, and use some almost standard notation on words and ω -words). A set B is called *selfdual*, if $B \leq_{CA} \bar{B}$. A set B is called Σ_n^0 -*complete*, if $B \in \Sigma_n^0$ and any Σ_n^0 -set is CA -reducible to B ; similarly one can define completeness for other classes of sets and other reducibilities. Let us state some well-known properties of the introduced notions. Proofs may be found e.g. in [Mo80].

2.1. Properties. (i) For any n , the class Σ_n^0 contains $\emptyset, X^\omega$, is closed under $\cup, \cap$, is closed downwards under $\leq_{CA}$ and has the reduction property.

(ii) Any level Σ_{n+1}^0 has a universal set which is Σ_{n+1}^0 -complete and not selfdual.

(iii) For any n , $\Sigma_n^0 \cup \Pi_n^0 \subseteq \Delta_{n+1}^0$, and for $n > 0$ the inclusion is strict.

(iv) For all $B, C \subseteq X^\omega$, $B \oplus C$ is the supremum of B and C under $\leq_{CA}$.

(v) For any $B \subseteq X^\omega$, the set $B \oplus \bar{B}$ is selfdual.

Now let us formulate some more sophisticated properties of Wadge reducibility and of one of its modifications defined as follows. A set B is called *Lipschitz reducible* to a set C (in symbols $B \leq_{CS} C$), if $B = f^{-1}(C)$ for a synchronous continuous function $f : X^\omega \rightarrow X^\omega$ (i.e. a function satisfying $f(\xi)(n) = \varphi(\xi[n+1])$, where $\varphi : X^* \rightarrow X$ and $\xi[n+1]$ is the initial segment of ξ of length $n+1$). Note that for uniformity we use notation for Wadge and Lipschitz reducibilities from [Wa79] which is different from notation in descriptive set theory. Of course,

$\leq_{CS}$ implies $\leq_{CA}$. By $(B_0, B_1) \leq_{CA} (A_0, A_1)$ we denote the fact that for some continuous function f it holds $B_i = f^{-1}(C_i), i < 2$. Define the sets U_0, U_1 by $U_i = \bigcup_{n < \omega} 0^{2n+i} \cdot 1 \cdot X^\omega, i < 2$. For the proofs of the next assertions see e.g. [VW76].

2.2. Properties. (i) *For any $B \in \mathcal{B}$, $B \leq_{CS} \bar{B}$ iff $B \leq_{CA} \bar{B}$.*

(ii) *The structures $(\mathcal{B}; \leq_{CS})$ and $(\mathcal{B}; \leq_{CA})$ are almost well-ordered.*

(iii) *The sets U_0, U_1 are open and disjoint, and for all open disjoint sets V_0, V_1 it holds $(V_0, V_1) \leq_{CA} (U_0, U_1)$.*

3 Regular ω -Languages

Here we summarize some facts on regular sets most of which are known. Let $\xi = \langle \nu_0, \nu_1 \rangle$ be the code of the pair of ω -words defined by $\xi(2n+i) = \nu_i(n), i < 2$; in the same way one can code longer tuples of ω -words. Note that this coding provides a homeomorphism between $X^\omega \times X^\omega$ and X^ω . Along with the reducibility $\leq_{DA}$ from Introduction we consider the reducibility $\leq_{DS}$ by deterministic synchronous finite transducers which is defined in the obvious way and which was also studied in [Wa79]. Note that $\leq_{DS}$ implies $\leq_{DA}$, and these relations are effective versions of $\leq_{CS}$ and $\leq_{CA}$, respectively. Recall that $\mathcal{R}$ denotes the class of all regular ω -languages.

The assertions (i)–(vii) below are well-known (see [Th90, Wa79, SW74]) while the assertion (viii) is evident.

3.1. Properties. (i) *The class $\mathcal{R}$ is closed under Boolean operations.*

(ii) *Any regular set is a Boolean combination of regular F_σ -sets.*

(iii) *The classes of regular open and of regular F_σ -sets are closed under $\cup, \cap$ and are downward closed under $\leq_{DA}$.*

(iv) *There are regular sets in $G \setminus F$ and $F_\sigma \setminus G_\delta$.*

(v) *The relations $\leq_{DS}$ and $\leq_{CS}$ on $\mathcal{R}$ coincide.*

(vi) *The class of regular sets in $F_\sigma \cap G_\delta$ coincides with the class of Boolean combinations of regular open sets.*

(vii) *A regular set B is F_σ iff $B = \{\xi : I_A(\xi) \subseteq T\}$ for a deterministic finite automaton $\mathcal{A} = [X, Z, f, z_0]$ and a set of states $T \subseteq Z$, where $I_A(\xi)$ is the set of states which occur infinitely often when $\mathcal{A}$ runs along ξ starting from the initial state $z_0 \in Z$.*

(viii) *The "projections" $\langle \nu_0, \dots, \nu_k \rangle \mapsto \nu_i$ are computable by deterministic asynchronous finite transducers.*

Note that the proofs of these facts were given before [Wa79] and do not use that paper (the assertion (v) was indeed proved in [Wa79] but it follows also from [BL69], see Remark 3 at the end of [Wa79]).

The next assertion seems formally new but it follows from the known proofs and constructions.

3.2. Proposition. (i) *The classes of regular open and of regular F_σ -sets have the reduction property.*

(ii) Every two disjoint regular G_δ -sets A_0 and A_1 are separable by a Boolean combination of regular open sets (i.e. $A_0 \subseteq F \subseteq \bar{A}_1$ for a Boolean combination F of regular open sets).

Proof. (i) Let $B_i, i < 2$, be regular open sets. By a result of L. Staiger (see [Wa79]), $B_i = W_i \cdot X^\omega$ for some regular sets of words $W_i \subseteq X^*$. Let

$$W'_0 = \{a \in W_0 \mid \forall b \sqsubset a (b \notin W_1)\}, \quad W'_1 = \{a \in W_1 \mid \forall b \sqsubset a (b \notin W_0)\}$$

and $B'_i = W'_i \cdot X^\omega$. Then the sets B'_0, B'_1 clearly reduce B_0, B_1 and are open. By considering the product of the automata recognizing the sets W_0 and W_1 one easily sees that the sets W'_0, W'_1 are regular, hence B'_0, B'_1 are regular too.

For the case of regular F_σ -sets $B_0, B_1 \subseteq X^\omega$, choose an automaton $\mathcal{A} = [X, Z, f, z_0]$ such that there are sets $T_0, T_1 \subseteq Z$ satisfying $B_i = \{\xi : I_{\mathcal{A}}(\xi) \subseteq T_i\}$. Then $B_0 \cap B_1 = \{\xi : I_{\mathcal{A}}(\xi) \subseteq T_0 \cap T_1\}$.

Let $\mathcal{A}' = [X, Z \times \{0, 1\}, f', (z_0, 1)]$ where

$$f'((z, k), x) = \begin{cases} (f(z, x), 2 - k) & , \text{ if } k = 0 \wedge f(z, x) \notin T_0, \\ (f(z, x), 2 - k) & , \text{ if } k = 1 \wedge f(z, x) \notin T_1, \\ (f(z, x), k) & , \text{ otherwise.} \end{cases}$$

By cases one easily checks that the sets $B'_i = \{\xi : I_{\mathcal{A}'}(\xi) \subseteq (T_i \times \{i\})\}, i \leq 1$, reduce the sets B_0, B_1 .

(ii) The sets $B_i = \bar{A}_i$ are F_σ , so they are reduced by some regular F_σ -sets $B'_i, i \leq 1$. We have $B'_0 \cap B'_1 = \emptyset$ and $B'_0 \cup B'_1 = X^\omega$ (because $B_0 \cup B_1 = X^\omega$). So the sets B'_i are in $F_\sigma \cap G_\delta$. By 3.1.(vi), both these sets are Boolean combinations of regular open sets, hence the set $F = B'_1$ has the desired property. This completes the proof.

4 Fine Hierarchy

Here we summarize some facts on the abstract version of the fine hierarchy considered in a series of my papers. The proofs are elementary but some of them are not at all trivial.

Let $(B; \cup, \cap, \bar{}, 0, 1)$ be a Boolean algebra. By a *base* (in B) we mean any sequence $\{L_n\}_{n < \omega}$ of sublattices of $(B; \cup, \cap, \bar{}, 0, 1)$ satisfying $L_n \cup \bar{L}_n \subseteq L_{n+1}$, where $\bar{L}_n = \{\bar{a} \mid a \in L_n\}$. Note that by Stone Representation Theorem we may think of the L_n 's as of classes of sets.

4.1. Definition. (i) A base L is *interpolable*, if for all $n < \omega$ any two disjoint elements $a, b \in \bar{L}_{n+1}$ are separable by a Boolean combination c of elements of L_n (i.e. $a \subseteq c \subseteq \bar{b}$).

(ii) A base L is *reducible*, if any L_n has the reduction property.

We need an operation *Bisep* on subsets of B defined by

$$\text{Bisep}(X, Y_0, Y_1, Y_2) = \{x_0 y_0 \cup x_1 y_1 \cup \bar{x}_0 \bar{x}_1 y_2 \mid x_i \in X, y_j \in Y_j, x_0 x_1 y_0 = x_0 x_1 y_1\},$$

where x_0y_0 stand for $x_0 \cap y_0$. This operation generalizes the following operation from [Lo83] used for a description of some levels of the Wadge hierarchy:

$$\text{bise}p(X, Y_0, Y_1, Y_2) = \{x_0y_0 \cup x_1y_1 \cup \bar{x}_0\bar{x}_1y_2 \mid x_i \in X, y_j \in Y_j, x_0x_1 = \emptyset\}.$$

Now we define a notion which will be the main technical instrument in further considerations. It uses the ordinal $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$.

4.2. Definition. By the fine hierarchy over L we mean the sequence $\{S_\alpha\}_{\alpha < \varepsilon_0}$, where $S_\alpha = S_\alpha^0$ and the classes S_α^n ($n < \omega$) are defined by induction on α : $S_0^n = \{0\}$; $S_{\omega^\gamma}^n = S_{\omega^\gamma}^{n+1}$ for $\gamma > 0$; $S_{\beta+1}^n = \text{Bise}p(L_n, S_\beta^n, \check{S}_\beta^n, S_0^n)$ for all $\beta < \varepsilon_0$, and $S_{\beta+\omega^\gamma}^n = \text{Bise}p(L_n, S_\beta^n, \check{S}_\beta^n, S_{\omega^\gamma}^n)$ for $\gamma > 0$ and β of the form $\beta = \omega^\gamma \cdot \beta_1 > 0$.

Definition uses some ordinal arithmetic as described e.g. in [KM67]. To see that this definition is correct note that every nonzero ordinal $\alpha < \varepsilon_0$ is uniquely representable in the form $\alpha = \omega^{\gamma_0} + \dots + \omega^{\gamma_k}$ for a finite sequence $\gamma_0 \geq \dots \geq \gamma_k$ of ordinals $< \alpha$. Applying 4.2 we subsequently get $S_{\omega^{\gamma_0}}^n, S_{\omega^{\gamma_0}+\omega^{\gamma_1}}^n, \dots, S_\alpha^n$. The classes S_γ^n for $n > 0$ play a technical role, they are among the classes S_α .

Let us formulate some technical properties of the introduced notions. The proofs may be found in [Se89, Se91] and, in a systematized form, in [Se9?].

4.3. Properties. (i) If $\alpha < \beta < \varepsilon_0$ then $S_\alpha \subseteq S_\beta \cap \check{S}_\beta$.

(ii) For any $n < \omega$, $\{S_k^n\}_{k < \omega}$ is the difference hierarchy over L_n , i.e. $S_0^n = \{0\}$ and $S_{k+1}^n = \{\bar{a}u \mid a \in S_k^n, u \in L_n\}$.

(iii) Classes of the fine hierarchy over a reducible base L coincide with the corresponding classes obtained by using the operation $\text{bise}p$ in place of $\text{Bise}p$.

(iv) If L is interpolable then $S_\alpha \cap \check{S}_\alpha = \bigcup_{\beta < \alpha} (S_\beta \cup \check{S}_\beta)$ for all limit ordinals $\alpha < \varepsilon_0$.

(v) If L is reducible then, for all $\beta < \varepsilon_0$, $S_{\beta+1} \cap \check{S}_{\beta+1}$ is the class of elements $a_0u_0 \cup a_1u_1$, where $a_0, \bar{a}_1 \in S_\beta$, $u_0, u_1 \in L_0$ and $u_0u_1 = 0, u_0 \cup u_1 = 1$.

The fine hierarchy as defined above may seem at first glance ad hoc. In [Se9?] we have given an especially natural and clear description of it in terms of the Boolean operations. Let T be the set of terms in the language $\{\cup, \cap, -, 0, 1\}$ with variables v_k^n ($k, n < \omega$). Relate to any $t \in T$ the set $t(L)$ of all values of t when the variables v_k^n ($k < \omega$) range over L_n for all $n < \omega$. The next result from [Se94, Se9?] relates these classes to the levels of the fine hierarchy over L . The proof is elementary, though rather technical.

4.4. Theorem. For any reducible base L , $\{S_\alpha, \check{S}_\alpha \mid \alpha < \varepsilon_0\} = \{t(L) \mid t \in T\}$. There are algorithms computing from any ordinal $\alpha < \varepsilon_0$ a corresponding term $t \in T$ and vice versa.

5 Examples of Fine Hierarchy

In our publications one could find several examples of the fine hierarchy with applications. Here we consider two concrete examples of the fine hierarchy

relevant to the topic of this paper. By 2.1, $L = \{\Sigma_{n+1}^0\}_{n < \omega}$ is a reducible base. Let $\{\mathcal{S}_\alpha\}$ be the fine hierarchy over this base. The hierarchy $\{\mathcal{S}_\alpha\}$ is clearly a fragment of the Wadge hierarchy of Borel sets; in [Se9?] we proved that the hierarchy $\{\mathcal{S}_\alpha\}$ is just the finite version of the Wadge hierarchy of Borel sets.

Let B_n be a Σ_{n+1}^0 -complete set for any $n < \omega$. Relate to the sequence $\{B_n\}$ the sets $A_\alpha^n (n < \omega)$ by induction on $\alpha < \varepsilon_0$ as follows: $A_0^n = \emptyset$, $A_{\omega^\gamma}^n = A_\gamma^{n+1}$, $A_{\beta+1}^n = \{(\mu, \nu, \xi) | (\mu \in U_0 \wedge \nu \in A_\beta^n) \vee (\mu \in U_1 \wedge \xi \notin A_\beta^n)\}$, and $A_{\beta+\omega^\gamma}^n = \{(\mu, \nu, \xi, \rho) | (\mu \in U_0 \wedge \nu \in A_\beta^n) \vee (\mu \in U_1 \wedge \xi \notin A_\beta^n) \vee (\mu \notin (U_0 \cup U_1) \wedge \rho \in C_{\omega^\gamma}^n)\}$, where the ordinals satisfy the same conditions as in Definition 4.2 and U_0, U_1 are the sets from 2.2.

Let us state some properties of the introduced objects.

5.1. Proposition. (i) *For any $\alpha < \varepsilon_0$, the class $\mathcal{S}_\alpha$ is downward closed under $\leq_{CA}$ and has a universal set.*

(ii) *For any $\alpha < \varepsilon_0$, the set A_α^0 is $\mathcal{S}_\alpha$ -complete with respect to $\leq_{CA}$ and is not selfdual.*

Proof. By induction on α one easily checks the assertion (i) for $\mathcal{S}_\alpha^n (n < \omega)$, so (i) is true. As is well-known from descriptive set theory, (i) implies that the universal set is not selfdual.

The evident induction shows that $A_\alpha^n \in \mathcal{S}_\alpha^n$ for all n, α , so it remains to reduce any $D \in \mathcal{S}_\alpha^n$ to A_α^n . This is also by induction. The cases $\alpha = 0, \omega^\gamma$ are trivial. The remaining cases are considered similarly to one another, so consider only the case $\alpha = \beta + 1$. By 4.3.(iii), $D = D_0 V_0 \cup D_1 V_1$ for some $D_0, \bar{D}_1 \in \mathcal{S}_\beta^n$ and disjoint open sets V_0, V_1 . By 2.2 and induction hypothesis, $(V_0, V_1) \leq_{CA} (U_0, U_1)$, $D_0 \leq_{CA} A_\beta^n$ and $\bar{D}_1 \leq_{CA} A_\beta^n$. Let f, g_0 and g_1 be continuous functions witnessing these reductions. Then the function $\mu \mapsto \langle f(\mu), g_0(\mu), g_1(\mu) \rangle$ CA -reduces D to A_α^n . The set universal in $\mathcal{S}_\alpha^n$ is clearly $\mathcal{S}_\alpha^n$ -complete, so it is CA -equivalent to A_α^n , hence A_α^n is not selfdual. This completes the proof.

Now consider our second example. Let $\mathcal{L}_0 = \mathcal{R} \cap G$, $\mathcal{L}_1 = \mathcal{R} \cap F_\sigma$ and $\mathcal{L}_{n+2} = \mathcal{R}$. By 3.1, $\mathcal{L} = \{\mathcal{L}_n\}_{n < \omega}$ is a reducible base, so we can construct the fine hierarchy $\{\mathcal{R}_\alpha\}$ over $\mathcal{L}$. Note that $\mathcal{R}_\alpha = \mathcal{R}$ for $\alpha \geq \omega^\omega$, hence the classes $\mathcal{R}_\alpha$ are interesting only for $\alpha < \omega^\omega$. By 3.1.(ii) and 4.3.(ii), $\cup_{\alpha < \omega^\omega} \mathcal{R}_\alpha = \mathcal{R}$. It is clear also that $\mathcal{R}_\alpha \subseteq \mathcal{S}_\alpha$ for all $\alpha < \varepsilon_0$.

Let B_0, B_1 be regular sets respectively from $G \setminus F$ and $F_\sigma \setminus G_\delta$, see Properties 3.1. Let $A_\alpha = A_\alpha^0, \alpha < \omega^\omega$, be sets constructed from B_0, B_1 as described above (note that the construction for $\alpha < \omega^\omega$ needs only the sets B_0, B_1). By 5.1, any A_α is $\mathcal{S}_\alpha$ -complete and not selfdual. Repeating the proof of 5.1 we get the following "effective" version of this assertion.

5.2. Proposition. (i) *For any $\alpha < \omega^\omega$, the class $\mathcal{R}_\alpha$ is downward closed under $\leq_{DA}$.*

(ii) *For any $\alpha < \omega^\omega$, the set A_α is $\mathcal{R}_\alpha$ -complete with respect to $\leq_{DA}$ and $A_\alpha \not\leq_{DA} \bar{A}_\alpha$.*

The introduced fine hierarchy of regular sets has several other interesting properties. We consider them in the next Section.

6 Main Results

Now we are able to prove our main results.

Proof of 1.1. By 5.2, the sets in 1.1 (defined as in 5.2) are ordered as stated in 1.1. It remains to show that any $E \in \mathcal{R}$ is DA -equivalent to one of $A_\alpha, \bar{A}_\alpha, A_\alpha \oplus \bar{A}_\alpha (\alpha < \omega^\omega)$. By 3.1.(ii) and 4.3.(ii), $E \in \mathcal{R}_\alpha \cup \bar{\mathcal{R}}_\alpha$ for some $\alpha < \omega^\omega$. Choose the least such α . By 4.3.(i), one of the following alternatives holds:

$$E \in \mathcal{R}_\alpha \setminus \bar{\mathcal{R}}_\alpha, E \in \bar{\mathcal{R}}_\alpha \setminus \mathcal{R}_\alpha, E \in (\mathcal{R}_\alpha \cap \bar{\mathcal{R}}_\alpha) \setminus \cup_{\beta < \alpha} (\mathcal{R}_\beta \cup \bar{\mathcal{R}}_\beta). \quad (1)$$

By 3.2.(ii), the base L from the end of Section 5 is interpolable. By 4.3.(iv), in the third alternative the ordinal α must be a successor, say $\alpha = \beta + 1$. It suffices to show that for the alternatives in (1) we respectively have $E \equiv_{DA} A_\alpha$, $E \equiv_{DA} \bar{A}_\alpha$ and $E \equiv_{DA} A_\beta \oplus \bar{A}_\beta$.

Assume first that $E \in \mathcal{R}_\alpha \setminus \bar{\mathcal{R}}_\alpha$. By 5.2, $E \leq_{DA} A_\alpha$, so it remains to show $A_\alpha \leq_{DA} E$. Suppose the contrary, then $A_\alpha \not\leq_{DS} E$ and (by 3.1.(v)) $A_\alpha \not\leq_{CS} E$. By 2.2.(ii), $\bar{E} \leq_{CS} A_\alpha$, hence $\bar{E} \leq_{DS} A_\alpha$, $\bar{E} \leq_{DA} A_\alpha$ and (by 5.2) $E \in \bar{\mathcal{R}}_\alpha$. A contradiction.

The second alternative is dual to the first one, so it remains to consider the third alternative. By the argument from preceding paragraph, $A_\beta, \bar{A}_\beta \leq_{DA} E$, so $A_\beta \oplus \bar{A}_\beta \leq_{DA} E$ and it remains to show that $E \leq_{DA} A_\beta \oplus \bar{A}_\beta$. By 4.3.(v), $E = E_0 V_0 \cup E_1 V_1$ for some $E_0, \bar{E}_1 \in \mathcal{R}_\beta$ and $V_0, V_1 \in L_0$ with $V_0 V_1 = \emptyset$, $V_0 \cup V_1 = X^\omega$. Then $V_0 = \bar{V}_1$ and V_0 is clopen, so $V_0 = W \cdot X^\omega$ for a finite set $W \subseteq X^*$. By 5.2, there are deterministic asynchronous finite automata $\mathcal{A}_0, \mathcal{A}_1$ reducing respectively E_0 to A_β and E_1 to $\bar{A}_\beta$. It is an easy exercise to construct a deterministic asynchronous finite automaton reducing E to $A_\beta \oplus \bar{A}_\beta$. This completes the proof.

The proof shows that the hierarchy $\{\mathcal{R}_\alpha\}_{\alpha < \omega^\omega}$ coincides with the hierarchy from the end of [Wa79] for DA -reducibility. This gives very clear, purely algebraic and automata-free description of this hierarchy.

Proof of 1.3. (i) By 3.1.(v) and 2.2.(ii), for all regular ω -languages B, C , one of the conditions $B \leq_{DS} C$, $\bar{C} \leq_{DS} B$ holds. By a result in [BL69], there is an algorithm computing which one of the conditions holds (because the condition $B \leq_{DS} C$ may be written as a sequential finite-state condition, see Remark 3 at the end of [Wa79]). This algorithm together with the proof of 1.1 yields a desired algorithm.

(ii) By 1.1, it suffices to show that the relations $\leq_{CA}$ and $\leq_{DA}$ coincide on the class $\{A_\alpha, \bar{A}_\alpha, A_\alpha \oplus \bar{A}_\alpha | \alpha < \omega\}$. But this follows from 5.1 and 5.2 completing the proof.

The next result states close relationship between hierarchies from Section 5. This is an interesting fact because its analogs for other examples of the fine hierarchy usually fail.

6.1. Theorem. *For any $\alpha < \omega^\omega$, $\mathcal{R}_\alpha = \mathcal{R} \cap \mathcal{S}_\alpha$.*

Proof. The inclusion from left to right is trivial, so consider the converse inclusion. By 1.1, it suffices to show that if one of the sets $A_\beta, \bar{A}_\beta, A_\beta \oplus \bar{A}_\beta$ ($\beta < \omega$) is in $\mathcal{S}_\alpha$ then it is in $\mathcal{R}_\alpha$. But this follows from 1.3.(ii), 5.1 and 5.2 completing the proof.

Theorem 1.1 describes the structure $(\mathcal{R}; \leq_{DA})$. In [Wa79] there is also a similar description of the structure $(\mathcal{R}; \leq_{DS})$. We can give an equivalent shorter description. For all $\alpha < \omega^\omega$ and $k < \omega$, let $B_\alpha^k = \{0^{k+1} \cdot A_\alpha, a \cdot \bar{A}_\alpha \mid a \in X^{k+1}, a \neq 0^{k+1}\}$, where X^{k+1} is the set of words from X^* of length $k+1$.

6.2. Theorem. (i) For all $\alpha < \beta < \omega^\omega$ and $k < \omega$, $A_\alpha \not\leq_{DS} \bar{A}_\alpha$ and $A_\alpha, \bar{A}_\alpha <_{DS} B_\alpha^k <_{DS} B_\alpha^{k+1} <_{DS} A_\beta, \bar{A}_\beta$.

(ii) Any $E \in \mathcal{R}$ is DS -equivalent to one of $A_\alpha, \bar{A}_\alpha, B_\alpha^k$ ($\alpha < \omega^\omega, k < \omega$).

Proof. First we check that for any $F \in \mathcal{R}$, $F \leq_{DA} A_\alpha$ iff $F \leq_{DS} A_\alpha$. Nontrivial is only the implication from left to right. Suppose it is false, so $F \leq_{DA} A_\alpha$ and $F \not\leq_{DS} A_\alpha$. By 3.1.(v) and 2.2.(ii), $\bar{A}_\alpha \leq_{DS} F$, hence $\bar{A}_\alpha \leq_{DA} A_\alpha$. A contradiction. Note that the same argument proves also the equivalence $A_\alpha \leq_{DA} F$ iff $A_\alpha \leq_{DS} F$, as well as the analogs of the both equivalences with $\bar{A}_\alpha$ in place of A_α .

Now we prove (i). The condition $A_\alpha \not\leq_{DS} \bar{A}_\alpha$ follows from 5.2. For any $k < \omega$, $A_\alpha \oplus \bar{A}_\alpha \equiv_{DA} B_\alpha^k$, hence, by the preceding paragraph, $A_\alpha, \bar{A}_\alpha <_{DS} B_\alpha^k <_{DS} A_\beta, \bar{A}_\beta$. It remains to prove that $B_\alpha^k <_{DS} B_\alpha^{k+1}$. The condition $B_\alpha^k \leq_{DS} B_\alpha^{k+1}$ is clear while the condition $B_\alpha^{k+1} \not\leq_{DS} B_\alpha^k$ follows from Lemma 1 in [BL69] (consider the finite-state condition $\{(\mu, \nu) \mid \mu \in B_\alpha^k \leftrightarrow \nu \in B_\alpha^{k+1}\}$).

The assertion (ii) is proved as in the proof of 1.1. For the first two alternatives in (1) we have respectively $E \equiv_{DS} A_\alpha$ and $E \equiv_{DS} \bar{A}_\alpha$, by the remarks from the beginning of this proof. For the third alternative, consider representations for E and V_0 from the proof of 1.1. Let $k < \omega$ satisfies $W \subseteq X^{k+1}$. The class $\{(a \cdot X^\omega) \cap E \mid a \in X^{k+1}\}$ coincides, modulo DS -equivalence, with $\{A_\beta, \bar{A}_\beta\}$ (otherwise $S \in \mathcal{R}_\beta \cup \bar{\mathcal{R}}_\beta$). From this one easily deduces that $E \equiv_{DS} B_\beta^k$. This completes the proof.

6.3. Corollary. The structure $(\mathcal{R}; \leq_{DS})$ is almost well-ordered with the corresponding ordinal ω^ω , but it is not isomorphic to the structure $(\mathcal{R}; \leq_{DA})$.

Note that the argument above is very similar to that in the proof of properties of the Lipschitz reducibility in [VW76].

A good deal of [Wa79] treats the so called coarse structure $\{\hat{C}_n^k, \hat{D}_n^k\}$. It turns out that the coarse structure is a fragment of the hierarchy $\{\mathcal{R}_\alpha\}$ having especially nice and easy description in terms of Boolean operations. For classes of sets $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} + \mathcal{B} = \{A \Delta B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$, where Δ is the symmetric difference. Define classes $\mathcal{P}_\alpha$ ($\alpha < \omega^2$) as follows: $\mathcal{P}_0 = \{\emptyset\}$, $\mathcal{P}_{k+1} = \mathcal{L}_0 + \dots + \mathcal{L}_0$ ($k+1 < \omega$ members in the sum), $\mathcal{P}_{\omega \cdot (k+1)} = \mathcal{L}_1 + \dots + \mathcal{L}_1$ ($k+1 < \omega$ members in the sum), and $\mathcal{P}_{\omega \cdot (k+1) + l+1} = \mathcal{P}_{\omega \cdot (k+1)} + \mathcal{P}_{l+1}$ for $k, l < \omega$, where $\mathcal{L}_0, \mathcal{L}_1$ are the classes from 5.2. The next result follows from [Wa79, Theorem 15], and results from [Se94].

6.4. Theorem. For all $n, k < \omega$, $\hat{C}_{2n+1}^{k+1} = \mathcal{P}_{\omega \cdot 2n+k}$, $\hat{D}_{2n+2}^{k+1} = \mathcal{P}_{\omega \cdot (2n+1)+k}$, $\mathcal{P}_k = \mathcal{R}_k$ and $\mathcal{P}_{\omega \cdot (n+1)+k} = \mathcal{R}_{\omega^{n+1} \cdot (k+1)}$.

We conclude the paper by the following corollary of 4.4 and 3.2.(i). This result gives probably the clearest possible description of the classes $\mathcal{R}_\alpha$ in terms of $\mathcal{L}_0, \mathcal{L}_1$. Let T_1 be the set of Boolean terms with variables $v_k^0, v_k^1 (k < \omega)$.

6.5. Theorem. *It holds $\{\mathcal{R}_\alpha, \tilde{\mathcal{R}}_\alpha | \alpha < \omega^\omega\} = \{t(\mathcal{L}) | t \in T_1\}$.*

The next corollary of 6.5 (for details see [Se9?]) states a natural closure property of the collection $\{\mathcal{R}_\alpha, \tilde{\mathcal{R}}_\alpha | \alpha < \omega^\omega\}$. Relate to any term $s = s(x_0, \dots, x_{n-1})$ in the language $\{\cup, \cap, \bar{}, 0, 1\}$ an n -ary operation F_s on classes of sets as follows:

$$F_s(A_0, \dots, A_{n-1}) = \{s(a_0, \dots, a_{n-1}) | a_i \in A_i\}.$$

E.g., for $s = x_0 \cap x_1$ we have $F_s(A_0, A_1) = \{a_0 \cap a_1 | a_i \in A_i\}$.

6.6. Corollary. *The collection $\{\mathcal{R}_\alpha, \tilde{\mathcal{R}}_\alpha | \alpha < \omega^\omega\}$ is closed under the operations F_s , and the structure formed by this collection and these operations is recursive.*

6.7. Remark. In [Wa79] conditions on Muller automaton were found equivalent to the condition that the corresponding regular ω -language belongs to a given level of the coarse structure. Recently, we found similar conditions for all levels of the fine hierarchy, as well as for the hierarchy generated by DS -reducibility. Proofs of these results should appear somewhere else.

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