Complete Inference Systems for Weak Bisimulation Equivalences in the π -Calculus

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Abstract

Proof systems for weak bisimulation equivalences in the π -calculus are presented, and their soundness and completeness are shown. The proofs of the completeness results rely on the notion of *symbolic bisimulation*. Two versions of π -calculus are investigated, one without and the other with the *mismatch* construction. For each version of the calculus proof systems for both *late* and *early* weak bisimulation equivalences are studied. Thus there are four proof systems in all. These proof systems are related in a natural way: the proof systems for early and late equivalences differ only in the inference rule for the input prefix, while the proof system for the version of π -calculus with mismatch is obtained by adding a single inference rule for the version without it.

1 Introduction

This paper aims to provide complete proof systems for weak bisimulation equivalences in the π -calculus. *Bisimulation equivalences* are among the most well-developed semantic theories for process calculi. The π -calculus inherits from *CCS* the invisible τ action, so the usual distinction between *strong* and *weak* bisimulations applies: the strong version gives the τ action the same weight as any other normal, or visible, actions, while the weak version ignores τ moves. As τ action is used to represent internal communication and as such is not observable, the weak equivalence is what used for practical applications. On the other hand, for message-passing process calculi, of which the π -calculus is a special instance, another distinction can be made, namely *late* and *early* bisimulations [MPW92, HL92], according to when input variables are instantiated. Thus for the π -calculus we have at least four bisimulation equivalences to study.

The original π -calculus paper [MPW92] only gave an axiomatisation for the *late* version of *strong* ground bisimulation. Subsequently, efforts have been made to formulate complete proof systems for other equivalences for this calculus: [PS93, BD94] for both early and late strong bisimulation congruences,

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[Hen91, BD92] for *testing* equivalence, and [San93] for (strong) *open bisimulation*. Axiomatisations for weak bisimulations are still missing, though it has been widely conjectured that they can be obtained by adding Milner's τ -laws to proof systems for strong equivalences.

In [Lin94b] we presented proof systems for strong bisimulation equivalences in the π -calculus. These proof systems consist of a set of inference rules together with some standard equations. The judgements of the inference systems are of the form

$$C \triangleright t = u$$

where t, u are π -calculus terms and C , called a *condition*, is a set of equality or inequality tests between names. It is important to note that C is *not* a construction in the π -calculus; it is a construction used in our meta language in order to reason about bisimulation equivalences for the calculus. To give a taste of the proof system, here is the inference rule for match:

$$\text{MATCH} \quad \frac{C \cup \{x = y\} \triangleright t = u \quad C \cup \{x \neq y\} \triangleright \mathbf{0} = u}{C \triangleright [x = y]t = u}$$

It involves a case analysis: if we can establish $t = u$ under the condition $x = y$, and $\mathbf{0} = u$ under the condition $x \neq y$, then we can conclude $[x = y]t = u$. Here it can be seen clearly how the inequalities *in the meta-language* help to characterise constructions *in the calculus*. For the extension of the π -calculus with the *mismatch* operator, only a single inference rule, which is dual to the above rule for match, has to be added to the proof systems.

In this paper we shall extend these results to weak bisimulation equivalences. It turns out that, as in *CCS*, the three τ -laws ([Mil89]) are sufficient for this purpose. We shall demonstrate how, by exploiting the notion of *maximally consistent extensions* of a condition, the standard techniques used to prove the completeness results in *CCS* can be re-cast for the π -calculus.

As in *CCS*, weak bisimulations in the π -calculus are defined in terms of double arrows that abstract away τ -moves. But care must be taken in dealing with late equivalence: in this case input moves should not be allowed to absorb the τ -moves *after* them, for otherwise the resulted relation would not be transitive. Also as in *CCS*, weak bisimulations are not preserved by the choice operator, so we have to work with observation equivalences.

The proofs of the completeness results for our inference systems rely on the notion of *symbolic bisimulations* [HL92, HL93]. In [MPW92] a general notion of bisimulation, $\sim^D$, called *distinction indexed bisimulation*, is introduced, where the index D is a set of inequations on names such that $t \sim^D u$ iff $t\sigma \sim u\sigma$ for every substitution σ that satisfies D . In the setting of the π -calculus, symbolic bisimulation is a mild generalisation of distinction indexed bisimulation, it is indexed by *conditions* consisting of name inequations *as well as* name equations. This generalisation makes it possible to give a direct definition of symbolic bisimulation (in terms of *symbolic transitional semantics*) instead of as substitution closure on top of ground bisimulation, which facilitates the proofs of the completeness results for the inference systems.

The main advantage of working with π -calculus over general message-passing calculi is that the language for the message domain is known: it is simply a set of *port names* upon which the only allowed operations are tests for equality and inequality. To define the notion of symbolic bisimulation for the π -calculus, we first develop a simple theory of *conditions* which are sets of equalities and inequalities over names. A *maximally consistent* condition is a *complete*, or *saturated*, condition, in the sense that adding to it anything not implied by it will result in inconsistency. This notion plays an important rôle in the definition of symbolic bisimulation and in the proofs of the completeness results.

The rest of this paper is organised as follows: The calculus and its semantics are introduced in the next section. The inference system is presented in Section 3, along with the completeness proof. Section 4 demonstrates how the theory developed for the late equivalence in the previous sections can be carried over to the early case. Section 5 discusses extensions to the calculus. The paper is concluded with Section 6, where related work is also discussed.

Due to space limitation, most of the proofs have been omitted. They can be found in the full version of this paper [Lin94a].

2 The π -Calculus And Bisimulations

2.1 Names and Conditions

The basic entities of the π -calculus are *names*, *i.e.* identifiers for communication channels. Let $\mathcal{N}$, ranged over by $x, y, z, \dots$ be a countably infinite set of names.

Conditions, ranged over by C, D , are finite sets of equality or inequality tests on names. We will write $n(C)$ for the set of names appearing in C . We say C is a condition on a name set V if $n(C) \subseteq V$. The empty condition will sometimes be denoted by *true*. *Matches*, ranged over by M, N, L , are special conditions which consist of only name equalities. Matches are part of the π -calculus, while conditions are only used in our meta-theory about the π -calculus, namely in the definition of symbolic bisimulations and in the formulation of inference rules.

Substitutions, ranged over by σ and δ , are mappings from $\mathcal{N}$ to $\mathcal{N}$, and $[\bar{y}/\bar{x}]$ is the substitution sending $\bar{x}$ to $\bar{y}$ and is identity otherwise. Substitutions are postfix operators, and have higher precedence than the operators in the language. If $\sigma = [\bar{y}/\bar{x}]$ then $n(\sigma) = \{\bar{y}\} \cup \{\bar{x}\}$. So for any $x \notin n(\sigma)$, $y\sigma = x$ *iff* $y = x$.

A substitution σ satisfies a condition C , written $\sigma \models C$, if $x\sigma = y\sigma$ for any $x = y \in C$ and $x\sigma \neq y\sigma$ for any $x \neq y \in C$. We write $C \Rightarrow D$ to mean that $\sigma \models C$ implies $\sigma \models D$ for any substitution σ .

The elements in a condition are treated as conjuncts. We avoid introducing disjunction into the theory of conditions. The major advantage is that the relation $C \Rightarrow D$ can be tested in linear time w.r.t the size of C and D :

Proposition 2.1 *The relation $C \Rightarrow D$ is linear time decidable.*

Allowing disjunction does not affect our theoretical results. The only consequence is that the above implication relation would in general take exponential time to decide.

Two substitutions σ and σ' are equal on V , written $\sigma =^V \sigma'$, if $x\sigma = x\sigma'$ for all $x \in V$. A condition C is *consistent* if there are no $x, y \in \mathcal{N}$ such that $C \Rightarrow x = y$ and $C \Rightarrow x \neq y$. C is *maximally consistent* on $V \subset \mathcal{N}$ if for any $x, y \in V$ either $C \Rightarrow x = y$ or $C \Rightarrow x \neq y$.

C' is a *maximally consistent extension* of C on V , written $C' \in MCE_V(C)$, if $C \subseteq C'$ and C' is maximally consistent on V . The set of maximally consistent extensions of a given condition on a finite set of names V is finite. We will abbreviate $MCE_V(\text{true})$ as MC_V .

Lemma 2.2 $\bigvee MCE_V(C) = C$

Lemma 2.2 shows that the set of all consistent extensions of a condition (on a given name set) constitutes a boolean *partition* ([HL92, HL93]) of the condition.

Proposition 2.3 *Suppose C is maximally consistent on V . If σ and σ' both satisfy C , then $\sigma =^V \sigma'\delta$ for some injective substitution δ .*

2.2 The Language And Operational Semantics

The language of π -calculus can be given by the following BNF grammar

$$\begin{aligned} t &::= \mathbf{0} \mid \alpha.t \mid t+t \mid [x=y]t \mid (x)t \mid t \mid t \mid A(y_1, \dots, y_n) \\ \alpha &::= \tau \mid a(x) \mid \bar{a}x \end{aligned}$$

We refer to [MPW92] for intended meanings of these operators.

In $a(x).t$ and $(x)t$ x is a bound name with scope t . We use $bn(t)$ and $fn(t)$ for the set of bound and free names in t , respectively. Bound names induce the notion of α -equivalence as usual. In the sequel we will not distinguish between α -equivalent terms, using $\equiv$ for both syntactical equality and α -equivalence. In the π -calculus no clear distinction is made between constants and variables, and we will call “processes” those terms whose free names are not to be instantiated.

The operational semantics of the language is reported in Figure 1, where a transition is of the form $t \xrightarrow{\alpha} u$ with α ranging over four kinds of actions: τ , $a(x)$, $\bar{a}x$ and $\bar{a}(x)$. We have omitted the symmetric rules for sum and par. Transitions are defined upto α -equivalence, i.e. α -equivalent terms are deemed to have the same transitions. Bound names for actions are defined by: $bn(a(x)) = bn(\bar{a}(x)) = \{x\}$ and $bn(\tau) = bn(\bar{a}x) = \emptyset$.

We write $\alpha =^C \beta$ to mean

$$\begin{aligned} \text{if } \alpha &\equiv \tau \text{ then } \beta \equiv \tau \\ \text{if } \alpha &\equiv \bar{a}x \text{ then } \beta \equiv \bar{b}y \text{ and } C \Rightarrow a = b, C \Rightarrow x = y \\ \text{if } \alpha &\equiv \bar{a}(x) \text{ then } \beta \equiv \bar{b}(x) \text{ and } C \Rightarrow a = b \\ \text{if } \alpha &\equiv a(x) \text{ then } \beta \equiv b(x) \text{ and } C \Rightarrow a = b \end{aligned}$$

The transitions defined in Figure 1 are *concrete* in the sense that they will always be fired regardless the contexts in which terms are placed. In our work

$$\begin{array}{ll}
\text{pre} \quad \frac{}{\alpha.t \xrightarrow{\alpha} t} & \text{match} \quad \frac{t \xrightarrow{\alpha} t'}{[x = x]t \xrightarrow{\alpha} t'} \\
\text{sum} \quad \frac{t \xrightarrow{\alpha} t'}{t + u \xrightarrow{\alpha} t'} & \text{par} \quad \frac{t \xrightarrow{\alpha} t' \quad bn(\alpha) \cap fn(u) = \emptyset}{t \mid u \xrightarrow{\alpha} t' \mid u} \\
\text{res} \quad \frac{t \xrightarrow{\alpha} t'}{(x)t \xrightarrow{\alpha} (x)t'} \quad x \notin n(\alpha) & \text{com} \quad \frac{t \xrightarrow{a(x)} t' \quad u \xrightarrow{\bar{a}y} u'}{t \mid u \xrightarrow{\tau} t'[y/x] \mid u'} \\
\text{open} \quad \frac{t \xrightarrow{\bar{a}x} t'}{(x)t \xrightarrow{\bar{a}(x)} t'} \quad a \neq x & \text{close} \quad \frac{t \xrightarrow{a(x)} t' \quad u \xrightarrow{\bar{a}(x)} u'}{t \mid u \xrightarrow{\tau} (x)(t' \mid u')} \\
\text{id} \quad \frac{t[y_1, \dots, y_n/x_1, \dots, x_n] \xrightarrow{\alpha} t'}{A(y_1, \dots, y_n) \xrightarrow{\alpha} t'} & A(x_1, \dots, x_n) \stackrel{\text{def}}{=} t
\end{array}$$

Figure 1: π -Calculus Transitional Semantics

on general message-passing processes [HL92, HL93] we use a more abstract form of transitions called *symbolic transitions*. A symbolic transition takes the form $t \xrightarrow{b, \alpha} u$, where b is a boolean condition. Intuitively b represents the context under which action α can actually be fired from t . In the setting of the π -calculus b will be a set of *matches*, i.e. equality tests on names. This kind of transition has also been used in the work of open bisimulation by Sangiorgi [San93].

The symbolic transitional semantics of the π -calculus is given in Figure 2. For notational convenience we write MN for the union of M and N . Again the symmetric rules for Sum and Par have been omitted.

2.3 Weak Bisimulations

Strong bisimulations for the π -calculus and their proof systems have been studied in [Lin94b], so here we only consider weak equivalences. For the bulk of this paper we will concentrate on *late* bisimulation, and we will sketch in a later section how these results can be carried over to *early* bisimulation in a systematic manner. So until Section 4, the word “bisimulation” means “late bisimulation”.

As in *CCS*, weak bisimulation is defined in terms of double arrows which ignore τ -moves. But late bisimulation insists that an input move from one process be matched by a *single* input move from the other. As a consequence, in defining late double arrows input moves should *not* be allowed to absorb τ moves *after* them.

The (concrete) late double arrow relation is generated by the following rules

- $p \xRightarrow{\varepsilon}_l p$.
- $p \xrightarrow{\alpha}_l q$ implies $p \xRightarrow{\alpha}_l q$.

$$\begin{array}{ll}
\text{Pre} \frac{}{\alpha.t \xrightarrow{\text{true}, \alpha} t} & \text{Match} \frac{t \xrightarrow{M, \alpha} t'}{[x = y]t \xrightarrow{ML, \alpha} t'} \quad L = \begin{cases} x = y & \text{if } x \neq y \\ \emptyset & \text{otherwise} \end{cases} \\
\text{Sum} \frac{t \xrightarrow{M, \alpha} t'}{t + u \xrightarrow{M, \alpha} t'} & \text{Par} \frac{t \xrightarrow{M, \alpha} t'}{t \mid u \xrightarrow{M, \alpha} t' \mid u} \quad bn(\alpha) \cap fn(u) = \emptyset \\
\text{Res} \frac{t \xrightarrow{M, \alpha} t'}{(x)t \xrightarrow{M, \alpha} (x)t'} \quad x \notin n(M, \alpha) & \text{Open} \frac{t \xrightarrow{M, \bar{a}x} t'}{(x)t \xrightarrow{M, \bar{a}(x)} t'} \quad x \notin n(M, a) \\
\\
\text{Com} \frac{t \xrightarrow{M, a(x)} t' \quad u \xrightarrow{N, \bar{b}y} u'}{t \mid u \xrightarrow{MNL, \tau} t'[y/x] \mid u'} \quad L = \begin{cases} a = b & \text{if } a \neq b \\ \emptyset & \text{otherwise} \end{cases} \\
\text{Close} \frac{t \xrightarrow{M, a(x)} t' \quad u \xrightarrow{N, \bar{b}(x)} u'}{t \mid u \xrightarrow{MNL, \tau} (x)(t' \mid u')} \quad L = \begin{cases} a = b & \text{if } a \neq b \\ \emptyset & \text{otherwise} \end{cases} \\
\text{Id} \frac{t[y_1, \dots, y_n/x_1, \dots, x_n] \xrightarrow{M, \alpha} t'}{A(y_1, \dots, y_n) \xrightarrow{M, \alpha} t'} \quad A(x_1, \dots, x_n) \stackrel{\text{def}}{=} t
\end{array}$$

Figure 2: π -Calculus Symbolic Transitional Semantics

- $p \xrightarrow{\tau}_l \xRightarrow{\alpha}_l q$ implies $p \xRightarrow{\alpha}_l q$.
- If α does not have the form $a(x)$ then $p \xRightarrow{\alpha}_l \xrightarrow{\tau}_l q$ implies $p \xRightarrow{\alpha}_l q$.

Let $\hat{\tau}_l$ be $\xRightarrow{\epsilon}_l$, and $\hat{\alpha}_l$ be $\xRightarrow{\alpha}_l$ for $\alpha \neq \tau$.

Definition 2.4 A symmetric relation R is a late weak ground bisimulation if $(p, q) \in R$ implies:

- whenever $p \xrightarrow{a(x)} p'$ with $x \notin fn(p, q)$ then $q \xRightarrow{a(x)}_l q''$ for some q'' s.t for any y there is $q''[y/x] \xRightarrow{\tau}_l q'$ and $(p'[y/x], q') \in R$.
- whenever $p \xrightarrow{\alpha} p'$ for any other action α with $bn(\alpha) \cap fn(p, q) = \emptyset$ then $q \xRightarrow{\hat{\alpha}}_l q'$ for some q' and $(p', q') \in R$.

Write $p \dot{\approx}_l q$ if there exists a late weak ground bisimulation R s.t. $(p, q) \in R$.

t is late weak bisimilar to u , written $t \approx_l u$, if $t\sigma \approx_l u\sigma$ for every substitution σ . $\square$

Similar to CCS , $\approx_l$ is not preserved by $+$. So we have to work with the modified relation:

Definition 2.5 p and q are late ground observation equivalent, written $p \dot{\simeq}_l q$, if

- whenever $p \xrightarrow{a(x)} p'$ with $x \notin fn(p, q)$ then $q \xrightarrow{a(x)}_l q''$ for some q'' s.t for any y there is $q''[y/x] \xrightarrow{\hat{\tau}}_l q'$ and $p'[y/x] \approx_l q'$.
- whenever $p \xrightarrow{\alpha} p'$ for any other action α with $bn(\alpha) \cap fn(p, q) = \emptyset$ then $q \xrightarrow{\alpha}_l q'$ for some q' and $p' \approx_l q'$.

And similarly for u .

t is late observation congruent to u , written $t \simeq_l u$, if $t\sigma \simeq_l u\sigma$ for every substitution σ . $\square$

The late symbolic double arrows are defined as the least relations satisfying the following rules:

- $t \xRightarrow{true, \varepsilon}_L t$.
- $t \xrightarrow{M, \alpha}_L u$ implies $t \xRightarrow{M, \alpha}_L u$.
- $t \xrightarrow{M, \tau}_L \xrightarrow{N, \alpha}_L u$ implies $t \xRightarrow{MN, \alpha}_L u$.
- If α does not have the form $a(x)$ then $t \xrightarrow{M, \alpha}_L \xrightarrow{N, \tau}_L u$ implies $t \xRightarrow{MN, \alpha}_L u$.

Let $\xRightarrow{M, \hat{\tau}}_L$ be $\xRightarrow{M, \varepsilon}_L$, and $\xRightarrow{M, \hat{\alpha}}_L$ be $\xRightarrow{M, \alpha}_L$ for $\alpha \neq \tau$.

Definition 2.6 A condition indexed family of symmetric relations $\mathcal{S} = \{S^C\}$ is a late weak symbolic bisimulation if $(t, u) \in S^C$ implies whenever $t \xrightarrow{M, \alpha} t'$ with $bn(\alpha) \cap fn(t, u) = \emptyset$, then for each $D \in MCE_{fn(t, u)}(C \cup M)$ there is a $u \xRightarrow{N, \beta}_L u'$ such that $D \Rightarrow N$, $\alpha =^D \beta$, and

- If $\alpha \equiv a(x)$ then for each $D' \in MCE_{fn(t, u) \cup \{x\}}(D)$ there is $u' \xRightarrow{N', \hat{\tau}}_L u''$ s.t. $D' \Rightarrow N'$ and $(t', u'') \in S^{D'}$.
- If $\alpha \equiv \bar{a}(x)$ then $(t', u') \in S^{D \cup \{x \neq y \mid y \in fn(\alpha, t', \beta, u')\}}$.
- Otherwise $(t', u') \in S^D$.

Let $\approx_L$ be the largest late weak symbolic bisimulation. $\square$

Proposition 2.7 $t \approx_L^C u$ iff $t\sigma \approx_l u\sigma$ for every $\sigma \models C$.

Late symbolic observation equivalence is defined in terms of $\approx_L$ as usual:

Definition 2.8 Two terms t, u are *late symbolic observation equivalent* over C , written $t \simeq_L^C u$, if whenever $t \xrightarrow{M, \alpha} t'$ with $bn(\alpha) \cap fn(t, u) = \emptyset$, then for each $D \in MCE_{fn(t, u)}(C \cup M)$ there is a $u \xRightarrow{N, \beta}_L u'$ such that $D \Rightarrow N$, $\alpha =^D \beta$, and

- If $\alpha \equiv a(x)$ then for each $D' \in MCE_{fn(t, u) \cup \{x\}}(D)$ there is $u' \xRightarrow{M', \hat{\tau}}_L u''$ s.t. $D' \Rightarrow M'$ and $t' \approx_L^{D'} u''$.

ALPHA	$\frac{}{true \triangleright t = u} \quad t \equiv u$	
AXIOM	$\frac{}{true \triangleright t = u} \quad t = u \text{ an axiom instance}$	
CHOICE	$\frac{C \triangleright t_i = u_i}{C \triangleright t_1 + t_2 = u_1 + u_2}$	TAU $\frac{C \triangleright t = u}{C \triangleright \tau.t = \tau.u}$
L-INPUT	$\frac{C \triangleright t = u}{C \triangleright a(x).t = \bar{b}(x).u} \quad C \Rightarrow a = b, x \notin n(C)$	
OUTPUT	$\frac{C \triangleright t = u}{C \triangleright \bar{a}x.t = by.u} \quad C \Rightarrow a = b, C \Rightarrow x = y$	
MATCH	$\frac{C \cup \{x = y\} \triangleright t = u \quad C \cup \{x \neq y\} \triangleright 0 = u}{C \triangleright [x = y]t = u}$	
RES	$\frac{C \cup \{x \neq y \mid y \in fn((x)t, (x)u)\} \triangleright t = u}{C \triangleright (x)t = (x)u} \quad x \notin n(C)$	
CUT	$\frac{C \cup \{x = y\} \triangleright t = u \quad C \cup \{x \neq y\} \triangleright t = u}{C \triangleright t = u}$	
CONSEQ	$\frac{C \triangleright t = u}{C' \triangleright t = u} \quad C' \Rightarrow C$	ABSURD $\frac{}{false \triangleright t = u}$

Figure 3: The Inference Rules for Late Symbolic Bisimulation

- If $\alpha \equiv \bar{a}(x)$ then $t' \approx_L^{D \cup \{x \neq y \mid y \in fn(\alpha.t', \beta.u')\}} u'$.
- Otherwise $t' \approx_L^D u'$.

And similarly for u . □

Proposition 2.9 $t \simeq_L^C u$ iff $t\sigma \simeq_l u\sigma$ for every $\sigma \models C$.

3 The Inference System

In this section we will concentrate on a sublanguage without the parallel composition operator $|$. Later we will see that this operator can be axiomatised easily.

The inference system for late symbolic bisimulation is reported in Figure 3. The judgements are *conditional equations* of the form

$$C \triangleright t = u$$

where $t = u$ is an equation in the π -calculus and C is a condition.

S1	$X + \mathbf{0} = X$	R1	$(x)\mathbf{0} = \mathbf{0}$
S2	$X + X = X$	R2	$(x)\alpha.X = \alpha.(x)X \quad \text{if } x \notin n(\alpha)$
S3	$X + Y = Y + X$	R3	$(x)\alpha.X = \mathbf{0} \quad \text{if } x \text{ is the port of } \alpha$
S4	$(X + Y) + Z = X + (Y + Z)$	R4	$(x)(y)X = (y)(x)X$
		R5	$(x)(X + Y) = (x)X + (x)Y$
T1	$\alpha.\tau.X = \alpha.X$	T2	$X + \tau.X = \tau.X$
T3	$\alpha.(X + \tau.Y) + \alpha.Y = \alpha.(X + \tau.Y)$		

Figure 4: The Axioms for Choice, Restriction and τ

In order to derive interesting equations from this inference system we need some standard axioms. These include the axioms for the choice operator (S1 – S4), the axioms for the restriction operator (R1 – R5), and the τ -laws (T1 – T3) in Figure 4. Let $\mathcal{A}_s = \{S1, S2, S3, S4, R1, R2, R3, R4, R5\}$ and $\mathcal{A}_w = \mathcal{A}_s \cup \{T1, T2, T3\}$.

We write $\vdash C \triangleright t = u$ and $\vdash_{wL} C \triangleright t = u$ to mean that $C \triangleright t = u$ can be derived from $\mathcal{A}_s$ and $\mathcal{A}_w$, respectively, using the above inference rules. It is immediate from the definition that $\vdash C \triangleright t = u$ implies $\vdash_{wL} C \triangleright t = u$. Some properties of these proof systems are listed below.

Proposition 3.1 1. If $C \Rightarrow M$ and $\vdash C \triangleright t = u$ then $\vdash C \triangleright Mt = u$.
 2. If $C \cup M \Rightarrow \text{false}$ then $\vdash C \triangleright Mt = \mathbf{0}$.
 3. $\vdash t = t + Mt$.

Proposition 3.2 If $\vdash D \triangleright t = u$ for each $D \in MCE_V(C)$ then $\vdash C \triangleright t = u$.

Proposition 3.3 1. $\vdash (x)[x = x]t = (x)t$.
 2. $\vdash (x)[x = y]t = \mathbf{0}$.
 3. $\vdash (x)[y = z]t = [y = z](x)t$.

The soundness and completeness of $\vdash$ w.r.t $\sim_L$ (hence $\sim_l$) have been proved in [Lin94b]. The rest of this section is devoted to the proof of completeness result for $\vdash_{wL}$.

If $a \neq x$ then we abbreviate $(x)\bar{a}x.t$ as $\bar{a}(x).t$. $\bar{a}(x)$ is a derived action called *bound output*. Using RES the following two equations can be easily derived from T1 and T3:

$$\bar{a}(x).\tau.X = \bar{a}(x).X$$

$$\bar{a}(x).(X + \tau.Y) + \bar{a}(x).Y = \bar{a}(x).(X + \tau.Y)$$

So from now on we will allow the action α in T1 and T3 to range over bound output as well.

We also have the following derived inference rule for bound output:

Proposition 3.4 Suppose $C \Rightarrow a = b$, $x \notin n(C)$. If $\vdash C \cup \{x \neq y \mid y \in fn(\bar{a}(x).t, \bar{b}(x).u)\} \triangleright t = u$ then $\vdash C \triangleright \bar{a}(x).t = \bar{b}(x).u$.

A term is restriction-free if, using the above abbreviation, it does not explicitly contain any occurrences of the restriction operator.

A restriction-free term is a *standard form* if it has the form $\sum_i M_i \alpha_i . t_i$ and each t_i is a standard form. A standard form $\sum_i M_i \alpha_i . t_i$ is a *full standard form* if

- $t \xRightarrow{M, \alpha}_L t'$, where $bn(\alpha) \cap n(M) = \emptyset$, implies $t \xRightarrow{M, \alpha} t'$;
- Each t_i is in full standard form.

The following two crucial lemmas hold:

Lemma 3.5 (*Absorption*) If $t \xRightarrow{M, \alpha}_L t'$ with $bn(\alpha) \cap n(M) = \emptyset$ then $\vdash_{wL} t = t + M\alpha.t'$.

Lemma 3.6 If $t \xRightarrow{M, \hat{\tau}}_L t'$, $C \Rightarrow M$ and $\vdash_{wL} C \triangleright \tau.t' = \tau.u$, then $\vdash_{wL} C \triangleright \tau.t = \tau.t + \tau.u$.

Let the *height* of a term t , $|t|$, be defined as the maximum depth of action nesting in t .

Lemma 3.7 For any term t there is a full standard form t' such that $\vdash_{wL} t = t'$ and $|t'| \leq |t|$.

The following proposition relates observation equivalence to weak bisimulation. It will play an important rôle in the proof of the completeness theorem.

Proposition 3.8 Suppose $D \in MC_{fn(t,u)}$. Then $t \approx_L^D u$ if and only if either $t \simeq_L^D u$, or $\tau.t \simeq_L^D u$, or $t \simeq_L^D \tau.u$.

Using Lemmas 3.7, 3.6 and 3.8 we can prove the main theorem of this paper:

Theorem 3.9 (*Completeness of $\vdash_{wL}$*) If $t \simeq_L^C u$ then $\vdash_{wL} C \triangleright t = u$.

Proof: By Lemma 3.7 we may assume t, u are standard forms: $t \equiv \sum_i M_i \alpha_i . t_i$, $u \equiv \sum_j N_j \beta_j . u_j$. We may further assume that bound actions in α_i, β_j use the same bound name $z \notin fn(t, u, C)$. The proof is by induction on the joint height of t and u . By Proposition 3.2 we need only to show $\vdash_{wL} D \triangleright t = u$ for each $D \in MCE_{fn(t,u,C)}(C)$, and for this it is sufficient to show $\vdash_{wL} D \triangleright t = t + N_j \beta_j . u_j$ for each j .

If $D \not\Rightarrow N_j$ then, since D is maximally consistent, $D \cup N_j \Rightarrow false$. So by Proposition 3.1, $\vdash_{wL} D \triangleright N_j \beta_j . u_j = 0$. Hence $\vdash_{wL} D \triangleright t = t + N_j \beta_j . u_j$ by S1.

Now assume $D \Rightarrow N_j$ (then $D \in MCE_{fn(t,u,C)}(C \cup N_j)$), and consider four cases according to the type of β_j .

- $\beta_j \equiv \tau$. Then $u \xrightarrow{N_j, \tau} u_j$. Since $t \simeq_L^C u$ and t is a full normal form, $t \xrightarrow{M_i, \tau} t_i$ for some i s.t. $D \Rightarrow M_i$ and $t_i \simeq_L^D u_j$. By Proposition 3.8 we have either $t_i \simeq_L^D u_j$, or $\tau.t_i \simeq_L^D u_j$, or $t_i \simeq_L^D \tau.u_j$. In the first case we can apply induction to get $\vdash_{wL} D \triangleright t_i = u_j$, and by TAU, $\vdash_{wL} D \triangleright \tau.t_i = \tau.u_j$. In the second case, by Lemma 3.7 there is a full normal form t' with height no greater than t s.t. $\vdash_{wL} \tau.t_i = t'$. By the soundness of $\vdash_{wL}$, $t' \simeq_L^{true} \tau.t_i$, hence $t' \simeq_L^D u_j$. By induction $\vdash_{wL} D \triangleright t' = u_j$, so $\vdash_{wL} D \triangleright \tau.t_i = u_j$. Apply TAU and T1 we get $\vdash_{wL} D \triangleright \tau.t_i = \tau.u_j$. In the third case we can similarly get $\vdash_{wL} D \triangleright \tau.t_i = \tau.u_j$. Since $D \Rightarrow M_i$ and $D \Rightarrow N_j$, by Proposition 3.1 $\vdash_{wL} D \triangleright M_i \tau.t_i = N_j \tau.u_j$. Finally by S2, $\vdash_{wL} D \triangleright t = t + N_j \tau.u_j$.
- $\beta_j \equiv \bar{b}(z)$. Then $u \xrightarrow{N_j, \bar{b}(z)} u_j$, so $t \xrightarrow{M_i, \bar{a}_i(z)} t_i \xrightarrow{M', \bar{\tau}} t'$ for some i s.t. $D \Rightarrow M_i M'$, $D \Rightarrow a_i = b$ and $t' \simeq_L^{D \cup \{z \neq y \mid y \in fn(\bar{a}_i(z).t_i, \bar{b}(z).u_j)\}} u_j$. Note that $D \cup \{z \neq y \mid y \in fn(\bar{a}_i(z).t_i, \bar{b}(z).u_j)\} \in MC_{fn(t, u_j)}$, so by the same argument as in the τ case, $\vdash_{wL} D \cup \{z \neq y \mid y \in fn(\bar{a}_i(z).t_i, \bar{b}(z).u_j)\} \triangleright \tau.t' = \tau.u_j$. By Lemma 3.6, $\vdash_{wL} D' \triangleright \tau.t_i = \tau.t_i + \tau.u_j$. By Proposition 3.4, $\vdash_{wL} D \triangleright \bar{a}_i(z).\tau.t_i = \bar{a}_i(z).(\tau.t_i + \tau.u_j)$. Hence

$$\begin{aligned}
\vdash_{wL} D \triangleright \bar{a}_i(z).t_i &\stackrel{T1}{=} \bar{a}_i(z).\tau.t_i \\
&= \bar{a}_i(z).(\tau.t_i + \tau.u_j) \\
&\stackrel{T3}{=} \bar{a}_i(z).(\tau.t_i + \tau.u_j) + \bar{a}_i(z).u_j \\
&= \bar{a}_i(z).t_i + \bar{a}_i(z).u_j \\
&= \bar{a}_i(z).t_i + \bar{b}(z).u_j
\end{aligned}$$

Since $D \Rightarrow M_i$ and $D \Rightarrow N_j$, by Proposition 3.1, $\vdash_{wL} D \triangleright M_i \bar{a}_i(z).t_i = M_i \bar{a}_i(z).t_i + N_j \bar{b}(z).u_j$. Hence $\vdash_{wL} D \triangleright t = t + N_j \bar{b}(z).u_j$ by S2.

- The other two cases are similar. □

4 The Early Case

Early weak (concrete) bisimulation and congruence have been discussed in [Wal94], here we only give the definitions for thier symbolic counterpart.

To define early symbolic double arrow, $\Rightarrow_E$, we simply remove the “non-input action” restriction to the last rule in the definition of late double arrow in Section 2.3, with the other rules unchanged.

Definition 4.1 A condition indexed family of symmetric relations $\mathcal{S} = \{S^C\}$ is an early weak symbolic bisimulation if $(t, u) \in S^C$ implies

whenever $t \xrightarrow{M, \alpha} t'$ with $bn(\alpha) \cap fn(t, u) = \emptyset$, then for each $D \in MCE_V(C \cup M)$ there is a $u \xrightarrow{N, \beta}_E u'$ such that $D \Rightarrow N$, $\alpha =^D \beta$,

and $(t', u') \in S^{D'}$, where

$$V = \begin{cases} fn(t, u) \cup \{x\} & \text{if } \alpha \equiv a(x) \\ fn(t, u) & \text{otherwise} \end{cases}$$

$$D' = \begin{cases} D \cup \{x \neq y \mid y \in fn(\alpha.t', \beta.u')\} & \text{if } \alpha \equiv \bar{a}(x) \\ D & \text{otherwise} \end{cases}$$

Let $\approx_E$ be the largest early weak symbolic bisimulation. $\square$

Early symbolic observation equivalence $\simeq_E$ can then be defined in terms of $\approx_E$ as usual.

Proposition 4.2 $t \simeq_E^C u$ iff $t\sigma \simeq_e u\sigma$ for every $\sigma \models C$.

The proof systems for early weak equivalence can be obtained by replacing the L-INPUT rule in Figure 3 with the following one

$$\text{E-INPUT} \quad \frac{C \triangleright \sum_{i \in I} \tau.t_i = \sum_{j \in J} \tau.u_j \quad C \Rightarrow a_i = b_j, i \in I, j \in J}{C \triangleright \sum_{i \in I} a_i(x).t_i = \sum_{j \in J} b_j(x).u_j} \quad x \notin n(C)$$

Let us write $\vdash_{wE} C \triangleright t = u$ to mean $C \triangleright t = u$ can be derived from this new proof system. It is easy to see that, in the presence of the TAU rule, L-INPUT can be derived from E-INPUT, but not the other way round. So $\vdash_{wE}$ is strictly stronger than $\vdash_{wL}$.

All the lemmas in the previous section hold for $\vdash_{wE}$ as well. And we have the following counterpart of Theorem 3.9:

Theorem 4.3 (Completeness of $\vdash_{wE}$) If $t \simeq_E^C u$ then $\vdash_{wE} C \triangleright t = u$.

5 Other Operators

5.1 Parallel Composition

To deal with parallel composition all we need is a suitable form of expansion law which is presented in Figure 5. With this law it is standard that any term containing parallel operator can be reduced to a term without it, hence the normal form lemma still holds, as does the completeness theorem.

5.2 Mismatch

To include mismatch, *i.e.* testing inequality between names, into the language we first extend the operational semantics by including the following two rules: “mismatch” and “Mismatch” in Figure 1 and Figure 2, respectively (now M ranges over conditions)

$$\text{mismatch} \quad \frac{t \xrightarrow{\alpha} t'}{[x \neq y]t \xrightarrow{\alpha} t'} \quad x \neq y$$

$$\text{Mismatch} \quad \frac{t \xrightarrow{M, \alpha} t'}{[x \neq y]t \xrightarrow{M[x \neq y], \alpha} t'}$$

Let $t \equiv \sum_i M_i \alpha_i . t_i$ and $u \equiv \sum_j N_j \beta_j . u_j$ with $bn(\alpha_i) \cap fn(u) = bn(\beta_j) \cap fn(t) = \emptyset$. Then

$$t \mid u = \sum_i M_i \alpha_i . (t_i \mid u) + \sum_j N_j \beta_j . (t \mid u_j) + \sum_{\alpha_i \text{ opp } \beta_j} M_i N_j [a_i = b_j] \tau . v_{ij}$$

where $\alpha_i \text{ opp } \beta_j$ and v_{ij} are defined as follows

1. $\alpha_i \equiv a(x), \beta_j \equiv \bar{b}y$; then $v_{ij} \equiv t_i[y/x] \mid u_j$;
 2. The converse of the above clause;
 3. $\alpha_i \equiv a(x), \beta_j \equiv \bar{b}(y)$; then $v_{ij} \equiv (z)(t_i[z/x] \mid u_j[z/y])$ with $z \notin fn(t, u)$;
 4. The converse of the above clause.
-

Figure 5: The Expansion Law

The inference rule for mismatch is dual to that for match:

$$\text{MISMATCH} \quad \frac{C \cup \{x \neq y\} \triangleright t = u \quad C \cup \{x = y\} \triangleright 0 = u}{C \triangleright [x \neq y]t = u}$$

With this rule Proposition 3.1 can be generalised to allow M to be an arbitrary condition, not just a match.

Now, in a normal form $\sum_i M_i \alpha_i . t_i$, M_i may contain inequality tests as well as equality tests. This does not affect the proofs of the normal form lemma and the completeness theorem.

6 Conclusions And Related Work

We have presented complete proof systems for late and early weak observation equivalences for the π -calculus. The inference system for the early equivalence is obtained by adding the E-INPUT rule to the inference system for the late equivalence. To enrich the language with the *mismatch* construction we only need to extend the inference systems with a single rule, namely MISMATCH. In each case the proof system for weak equivalence is obtained by adding the standard τ -laws to the one for the strong equivalence. Thus in all we have eight proof systems. The relationship between them are pictured in Figure 6, where arrows stand for “extension” and $\pi^\neq$ is the version of π -calculus enriched with mismatch.

Our results verify a conjecture posed in [MPW92, PS93, BD94]: the three τ -laws are sufficient for the transition from strong to weak bisimulation equivalences for the π -calculus, as in the case of *CCS*.

The current work inherits the main ideas from our previous work on general message-passing process calculi [HL92, HL93]. As the process calculi considered

Figure 6: The Relationship between Proof Systems

there take the languages for data and boolean expressions as parameters, the completeness results in those papers are relative to data reasoning. Here, by exploiting the specific features presented in the π -calculus, there is no need to refer to an “oracle” for the data domain and the completeness results are no longer relative to reasoning about data.

Parrow and Sangiorgi ([PS93]) provided complete equational theories for both late and early strong congruences for name-passing calculi, *i.e.* extensions of the π -calculus with the mismatch construction. Mismatch plays an essential rôle in at least two places: the definition of normal form and the axiom for early equivalence. The equational characterisation of the restriction operator also relies on mismatch.

In [BD94] Moreale and De Nicola proposed “symbolic” semantics and proof systems very similar to [Lin94b]. Again they work with the extension of the π -calculus with mismatch instead of the π -calculus proper, and only strong equivalences were treated.

[Liu94] gives a symbolic version of distinction-indexed bisimulation for a sub-language of the π -calculus (without the *match* operator) and shows that this notion of bisimulation captures the conventional bisimulation equivalences. But no proof system is considered there.

In [San93] another notion of bisimulation, called *open bisimulation*, for the π -calculus is proposed, along with an equational axiomatisation. A “symbolic” characterisation of open bisimulation is also given. To deal with the restriction operator, distinctions have to be exploited as indices in the definition of open bisimulation as well as in the axiomatisation. Open bisimulation has been further studied in [Wal94].

We are not aware any other published work on axiomatising weak bisimulation equivalences in the π -calculus.

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