Patching Proofs for Reuse (Extended Abstract)

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1 Introduction

We investigate the application of machine learning paradigms [2, 4, 3] in automated reasoning for improving a theorem prover by reusing previously computed *proofs* [7]. Assume that we have already computed a proof P of a conjecture

$$\varphi := \big(\forall u\ \mathsf{plus}(\mathsf{sum}(x), \mathsf{sum}(u)) \equiv \mathsf{sum}(\mathsf{append}(x, u))\big) \\ \rightarrow \mathsf{plus}(\mathsf{sum}(\mathsf{add}(n, x)), \mathsf{sum}(y)) \equiv \mathsf{sum}(\mathsf{append}(\mathsf{add}(n, x), y))$$

from a set of axioms AX. The schematic conjecture $\Phi := \mathrm{H} \rightarrow \mathrm{C}\ :=$

$$\big(\forall u\ F(G(x), G(u)) \equiv G(H(x, u))\big) \rightarrow F(G(D(n, x)), G(y)) \equiv G(H(D(n, x), y))$$

is obtained from φ via the generalization $\{\mathsf{plus} \mapsto F, \mathsf{sum} \mapsto G, \mathsf{append} \mapsto H, \mathsf{add} \mapsto D\}$ of function *symbols* $\mathsf{plus}, \mathsf{sum}, ...$ to function *variables* $F, G, ...$ In the same way a *schematic catch*, i.e. a set of schematic axioms $AX' = \{(1), (2), (3)\}$ is obtained from AX where e.g. (1) stems from the axiom $\mathsf{sum}(\mathsf{add}(n, x)) \equiv \mathsf{plus}(n, \mathsf{sum}(x))$. The generalization of P finally yields a schematic proof P' of Φ in which the schematic conclusion C is modified in a backward chaining style:

$$G(D(n, x)) \equiv F(n, G(x)) \quad (1)$$
$$H(D(n, x), y) \equiv D(n, H(x, y)) \quad (2)$$
$$F(F(x, y), z) \equiv F(x, F(y, z)) \quad (3)$$

$F(G(D(n, x)), G(y))$	$\equiv$	$G(H(D(n, x), y))$	C
$F(\underline{F(n, G(x))}, G(y))$	$\equiv$	$G(H(D(n, x), y))$	Replace (1)
$F(F(n, G(x)), G(y))$	$\equiv$	$G(\underline{D(n, H(x, y))})$	Replace (2)
$F(F(n, G(x)), G(y))$	$\equiv$	$\underline{F(n, G(H(x, y)))}$	Replace (1)
$F(F(n, G(x)), G(y))$	$\equiv$	$F(n, \underline{F(G(x), G(y))})$	Replace (H)
$\underline{F(n, F(G(x), G(y)))}$	$\equiv$	$F(n, F(G(x), G(y)))$	Replace (3)
	$\underline{\mathrm{TRUE}}$		Reflexivity

The key idea of our reuse procedure is to *instantiate* such a schematic proof with a second-order substitution π obtained by matching Φ with a new conjecture ψ which is (formally) similar to φ, i.e. $\psi = \pi(\Phi)$. As long as the matcher π only replaces function variables with function symbols, the instantiated schematic proof $\pi(P')$ is a proof of ψ from the axioms $\pi(AX')$ because the structure of P' is preserved. However, the success of the method is limited by such a restriction. Therefore function variables are also replaced using general second-order substitutions[1] like $\pi := \{F/w_2,\ G/\mathsf{minus}(w_1, w_1),\ H/\mathsf{plus}(w_1, w_2),\ D/\mathsf{succ}(w_2)\}$ obtained by matching Φ with the new conjecture $\psi := \pi(\Phi) = \pi(\mathrm{H} \rightarrow \mathrm{C}) =$

[1] A second-order substitution replaces a n-ary function variable V with a (first-order)

$$\begin{aligned}&(\forall u\ \mathsf{minus}(u,u) \equiv \mathsf{minus}(\mathsf{plus}(x,u),\mathsf{plus}(x,u)))\\ &\to \mathsf{minus}(y,y) \equiv \mathsf{minus}(\mathsf{plus}(\mathsf{succ}(x),y),\mathsf{plus}(\mathsf{succ}(x),y)).\end{aligned}$$

AX' is instantiated yielding the set of axioms $\pi(AX') = \{\pi(1), \pi(2), \pi(3)\}$:

$$\begin{aligned}\mathsf{minus}(\mathsf{succ}(x),\mathsf{succ}(x)) &\equiv \mathsf{minus}(x,x) && \pi(1)\\ \mathsf{plus}(\mathsf{succ}(x),y) &\equiv \mathsf{succ}(\mathsf{plus}(x,y)) && \pi(2)\\ z &\equiv z && \pi(3)\end{aligned}$$

If the proof P shall be reused for proving ψ from the set of axioms $\pi(AX')$ by instantiating the schematic proof P' with π, we obtain $\pi(P')$ as

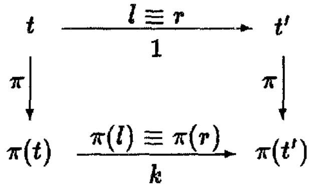
$\mathsf{minus}(y,y) \equiv \mathsf{minus}(\mathsf{plus}(\mathsf{succ}(x),y),\mathsf{plus}(\mathsf{succ}(x),y))$		π(C)
$\mathsf{minus}(y,y) \equiv \mathsf{minus}(\mathsf{plus}(\mathsf{succ}(x),y),\mathsf{plus}(\mathsf{succ}(x),y))$		Replace ($\pi(1)$)
$\mathsf{minus}(y,y) \equiv \mathsf{minus}(\underline{\mathsf{succ}(\mathsf{plus}(x,y))},\underline{\mathsf{succ}(\mathsf{plus}(x,y))})$		Replace ($\pi(2)$)
$\mathsf{minus}(y,y) \equiv \underline{\mathsf{minus}(\mathsf{plus}(x,y),\mathsf{plus}(x,y))}$		Replace ($\pi(1)$)
$\mathsf{minus}(y,y) \equiv \underline{\mathsf{minus}(y,y)}$		Replace (π(H))
$\underline{\mathsf{minus}(y,y)} \equiv \mathsf{minus}(y,y)$		Replace ($\pi(3)$)
TRUE		Reflexivity

But $\pi(P')$ is *not* a proof: Although each statement is implied by the statement in the line below, the *justifications* of the inference steps are not valid. E.g. the first *replace*($\pi(1)$)-step is illegal because the *position* of the replacement (the former first argument of F) does not exist in π(C). Also the *replace*($\pi(2)$)-step is illegal, as it actually consists of *two* replacements which have to be performed separately at different positions. Finally, the *replace*($\pi(3)$)-step is redundant and should be omitted. Thus $\pi(P')$ has to be *patched* for obtaining a proof of ψ.

Such a machine-found proof can be processed subsequently, e.g. by translating it into natural language to obtain a proof similar to those found in mathematical textbooks [5]. Furthermore proofs can be worked up for planning or synthesis tasks if plans or programs should be extracted form proofs [1]. These applications require a *specific* proof, i.e. it is not enough to know that *some* proof exists.

2 An Algorithm for Patching Proofs

We first illustrate the patching of a single replacement step: Let t be a schematic term (containing function variables) which can be modified by *one* replacement step with a certain schematic equation $l \equiv r$ at a certain position p (i.e. $t|_p = l$) yielding another schematic term $t' = t[p \leftarrow r]$ as the result. The function call *patch_positions*(t, p, π) yields for an arbitrary second-order substitution π a list of positions $[p_1, ..., p_k]$ such that the instance $\pi(t)$ can be modified by a (possibly empty) *sequence* of k replacement steps with the instantiated equation $\pi(l) \equiv \pi(r)$ at the positions $p_1, ..., p_k$ such that the instance $\pi(t')$ is obtained.



term where special *argument variables* $w_1, \ldots, w_n$ serve as the formal parameters of V. For instance π replaces the binary function variable D with the function symbol succ, where the first argument w_1 of D is ignored.

function *patch_positions* $(t,\ p,\ \pi)$: *list of positions in* $\pi(t)$
if $p = \epsilon$ **then return** $[\epsilon]$
else let $p =: ip'$; $t =: X(t_1, ..., t_n)$; $[p_1, ..., p_k] := patch\_positions(t_i, p', \pi)$
  if $X \in \mathrm{dom}(\pi)$ **then** $s := \pi(X)$; $[q_1, ..., q_m] := \{q \in \mathrm{Pos}(s) \mid s|_q = w_i\}$
    return $[q_1p_1, ..., q_1p_k, ..., q_mp_1, ..., q_mp_k]$
  else return $[ip_1, ..., ip_k]$ **fi fi**

Theorem 1. [6] *Let* t, l, r *be schematic terms,* p *a position in* t *and* π *a second-order substitution. If* $t|_p = l$ *then the call* $patch\_positions(t, p, \pi)$ *terminates yielding a list of positions* $[p_1, ..., p_k]$ *in* $\pi(t)$ *such that for* $i, j \in \{1, ..., k\}$
1) if $i \neq j$ *then there is no* $p \in \mathbb{N}^*$ *such that* $p_i = p_jp$ *or* $p_j = p_ip$,
2) $\pi(t)|_{p_j} = \pi(l)$ *and* $\pi(t)[p_1, ..., p_k \leftarrow \pi(r)] = \pi(t[p \leftarrow r])$.

The goal of a (schematic) proof is a so-called *sequent* $H \rightarrow C$ with a conjunction H of hypotheses each of which is of the form $\forall u^*\ t_1 \equiv t_2$ and a conclusion C of the form $s_1 \equiv s_2$. A *proof* of $H \rightarrow C$ (from a set of axioms AX) is a list $[S_0, j_1, S_1, j_2, \ldots, S_n]$ of sequents S_i (with $S_0 = H \rightarrow C$) and *justifications* j_i, where the latter contain the information how the next sequent is derived. A proof is constructed by applying the following inference rules,[2] where σ is a first-order substitution, p is a position in C and $m \in \{\text{"}AX\text{"}, \text{"}H\text{"}\}$:

Reflexivity $$\frac{}{[H \rightarrow t \equiv t]}$$

Replacement $$\frac{[H \rightarrow C[p \leftarrow \sigma(r)] \mid L]}{[H \rightarrow C[p \leftarrow \sigma(l)],\ \langle p, \sigma, u^*, l, r, m\rangle,\ H \rightarrow C[p \leftarrow \sigma(r)] \mid L]}$$

if either $\forall u^*\ l \equiv r\ \in AX$ and $m = \text{"}AX\text{"}$
or $\forall u^*\ l \equiv r\ \in H$, $\mathrm{dom}(\sigma) \subseteq u^*$ and $m = \text{"}H\text{"}$.

function *patch_proof* $(P',\ \pi)$: *proof*
if $P' = [H \rightarrow C]$ **then return** $[\pi(H) \rightarrow \pi(C)]$
else let $P' =: [H \rightarrow C,\ \langle p, \sigma, u^*, l, r, m\rangle,\ H \rightarrow C' \mid L]$
  $P_\pi := patch\_proof([H \rightarrow C' \mid L],\ \pi)$
  if $\pi(C) \neq \pi(C')$ **then** $[p_1, ..., p_k] := patch\_positions(C, p, \pi)$
    $\sigma_\pi := \{v/\pi(\sigma(v)) \mid v \in \mathrm{dom}(\sigma)\}$; $C_k := \pi(C')$
    for $j := k$ **downto** 1 **do** $C_{j-1} := C_j[p_j \leftarrow \sigma_\pi(\pi(l))]$
      $P_\pi := [\pi(H) \rightarrow C_{j-1},\ \langle p_j, \sigma_\pi, u^*, \pi(l), \pi(r), m\rangle \mid P_\pi]$ **od fi**
  return P_π **fi**

In a replacement step an *instance* $\sigma(l) \equiv \sigma(r)$ of an equation $l \equiv r$ is applied, but in the patched proof only (instances of) the equation $\pi(l) \equiv \pi(r)$ are available. However, we can use the first-order substitution $\sigma_\pi := \{v/\pi(\sigma(v)) \mid v \in \mathrm{dom}(\sigma)\}$ in *patch_proof* because $\pi(\sigma(u)) = \sigma_\pi(\pi(u))$ holds for each (schematic) term u.

Now we can compute $P_\pi := patch\_proof(P', \pi)$ to obtain a patched proof for

[2] Proofs can be extended to deal with arbitrary formulas instead of equations only if we define further inference rules. Then H may also contain additional conditions.

the conjecture $\psi = \pi(\mathrm{H}) \rightarrow \pi(\mathrm{C})$ from Section 1:

$$\begin{array}{lll}
\mathsf{minus}(y,y) \equiv \mathsf{minus}(\mathsf{plus}(\mathsf{succ}(x),y),\mathsf{plus}(\mathsf{succ}(x),y)) & & \pi(\mathrm{C}) \\
\mathsf{minus}(y,y) \equiv \mathsf{minus}(\underline{\mathsf{succ}(\mathsf{plus}(x,y))},\mathsf{plus}(\mathsf{succ}(x),y)) & & \text{Replace } (\pi(2)) \\
\mathsf{minus}(y,y) \equiv \mathsf{minus}(\mathsf{succ}(\mathsf{plus}(x,y)),\underline{\mathsf{succ}(\mathsf{plus}(x,y))}) & & \text{Replace } (\pi(2)) \\
\mathsf{minus}(y,y) \equiv \underline{\mathsf{minus}(\mathsf{plus}(x,y),\mathsf{plus}(x,y))} & & \text{Replace } (\pi(1)) \\
\mathsf{minus}(y,y) \equiv \underline{\mathsf{minus}(y,y)} & & \text{Replace } (\pi(\mathrm{H})) \\
\underline{\mathbf{TRUE}} & & \text{Reflexivity}
\end{array}$$

Compared to the schematic proof P from Section 1, the first *replace*(1)-step is eliminated while the *replace*(2)-step is doubled. The test $\pi(C) \neq \pi(C')$ in *patch_proof* is merely an optimization to avoid redundant steps like *replace*($\pi(3)$), cf. Section 1.

Theorem 2. [6] *Let P' be a proof of the sequent $H \rightarrow C$ from the set of axioms AX. Then for each second-order substitution π, the call patch_proof(P', π) terminates and yields a proof P_π of $\pi(H) \rightarrow \pi(C)$ from $\pi(AX)$.*

Summing up, we have presented an algorithm that constructs a proof for the instantiated conjecture from a schematic proof of a schematic conjecture and a second-order substitution. This allows us to exploit the full flexibility of second-order instantiations for the reuse procedure developed in [7]. Thus more conjectures are (formally) similar than by just instantiating function variables with function symbols, i.e. the applicability of a schematic catch is increased. Furthermore the obtained proofs may be more flexible, i.e. the reusability of a schematic catch is increased.

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