Visibility Graphs and Oriented Matroids (Extended Abstract)

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Abstract. This paper describes a new set of necessary conditions for a given graph to be the visibility graph of a simple polygon. For every graph satisfying these conditions we show that a uniform rank 3 oriented matroid can be constructed in polynomial time, which if affinely co- ordinatizable would yield a simple polygon whose visibility graph is isomorphic to the given graph. This will in turn offer the first characterization of this class of graphs.

1 Introduction

Visibility graphs are fundamental structures in computational geometry. They find applications in areas such as graphics [13, 21] and robotics [16], yet very little is known about their combinatorial structure. This paper addresses the question of characterizing internal visibility graphs of simple plane polygons, henceforth simply called visibility graphs. Two vertices of a simple polygon P, are called visible, if the open line segment between them is either a boundary edge of P, or is completely contained in the interior of the polygon. Note that in this setting, two vertices are considered to be invisible if the open line segment between them passes through a third vertex of the polygon. The **visibility graph** of a polygon is the graph whose vertices correspond to the vertices of the polygon and edges correspond to visible pairs of vertices in the polygon. From the computational standpoint, the complexity of the recognition problem for visibility graphs is only known to be in $PSPACE$ [9]. It is not known to be in NP nor it is known to be NP-complete.

Visibility graphs do not lie in any of the well known classes of graphs such as planar graphs, perfect graphs *etc.* [9, 11]. The first set of necessary conditions for a graph to be a visibility graph were obtained by Ghosh [11]. However, it was

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shown by Everett [9] that these conditions were not sufficient. Further necessary conditions were developed by Coullard and Lubiw [8], but they also showed that they are not sufficient. Abello, Hua and Pisupati [2] have strengthened these results by showing that the proposed conditions are not sufficient, even for triconnected graphs, and in the case of the conditions of [8], even for planar graphs. O'Rourke [19] has an excellent review of current status of this research. In this paper, we develop stronger necessary conditions for a graph to be a visibility graph.

In order to show that a given set of conditions on a graph, are sufficient for the graph to be a visibility graph, one must demonstrate that every graph satisfying the conditions can be realized as the visibility graph of a simple polygon in the plane. However, this **reconstruction problem** appears to be quite difficult in the general case. In this paper, we solve a combinatorial version of the reconstruction problem for general visibility graphs. We prove new necessary conditions for visibility graphs and show that these conditions are sufficient to construct a uniform oriented matroid of rank 3 corresponding to each graph in this class. These oriented matroids are combinatorial representations of simple polygons realizing the graphs, in the sense that any affine realization of the oriented matroids yields a simple polygon whose visibility graph is isomorphic to the given graph. It would be sufficient to show that each of these oriented matroids is affinely realizable, in order to obtain a characterization of visibility graphs of simple polygons. The main results of the paper are summarized below.

1. A class of graphs called **Quasi-Persistent** graphs is defined and it is shown that visibility graphs are properly contained in this class.
2. Several new necessary conditions are proven for a given Quasi-Persistent graph to be a visibility graph. These conditions strengthen Ghosh's necessary conditions for visibility graphs.
3. For each Quasi-Persistent graph satisfying these necessary conditions, a uniform oriented matroid of rank 3 is constructed (in polynomial time) such that any affine realization of the oriented matroid yields a simple polygon whose visibility graph is isomorphic to the given graph.

Because of space restrictions, we just give the main ideas involved in the proofs. The details may be found in [4] and [14].

2 Definitions

It is clear that every visibility graph is Hamiltonian and we therefore restrict our attention to Hamiltonian graphs. We further assume that the graphs considered are undirected, loopless and do not have multiple edges.

Let $G = (V, E)$ be a Hamiltonian graph with a prescribed Hamiltonian cycle H. The vertices of G are labelled along H from 0 to $n-1$. The vertex labelled i is denoted v_i. v_{i-1} and v_{i+1} respectively, denote the predecessor and successor of v_i on H. All subscript arithmetic is modulo n. It will be convenient to think of G as being embedded in the plane so that H forms a simple closed curve. In this setting, a traversal of H from v_i to v_j in the order $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ may be thought of as a counterclockwise traversal of H, and the traversal that goes from v_i to v_j in the order $v_i, v_{i-1}, \ldots, v_{j+1}, v_j$ will correspond to clockwise traversals. In this paper, unless specified otherwise, ***traversals of H are implicitly assumed to be in counterclockwise order.***

For any two vertices v_i and v_j, the ordered set $\{v_i, v_{i+1}, \ldots, v_{j-1}, v_j\}$ of vertices encountered in traversing H from v_i to v_j, is called the chain from v_i to v_j and is denoted chain$[v_i, v_j]$. This set of vertices constitutes a simple path in G. The chain from v_{i+1} to v_{j-1} is denoted chain(v_i, v_j). We also use chain$[v_i, v_j)$ and chain$(v_i, v_j]$ in the obvious manner. We emphasize that chain(v_i, v_k) and chain(v_k, v_i) are always disjoint sets. We say that $v_i < v_j < v_k$ if v_j lies on chain(v_i, v_k).

Two vertices v_i and v_j of G are said to be **invisible** if $v_iv_j \in \overline{E}$. For an invisible pair v_i, v_k, a vertex v_j is called an **inner blocking vertex** [11], relative to H, if v_j lies on chain(v_i, v_k) and $v_xv_y \in \overline{E}$ for all v_x on chain$[v_i, v_j)$ and v_y on chain$(v_j, v_k]$. Similarly, a vertex v_j is called an **outer blocking vertex** relative to H for the invisible pair v_iv_k if v_j lies on chain(v_k, v_i) and $v_xv_y \in \overline{E}$ for all v_x on chain$(v_j, v_i]$ and v_y on chain$[v_k, v_j)$. In general, v_j is called a **blocking vertex** for the invisible pair v_iv_k if it is either an inner or an outer blocking vertex for this pair.

A simple path $P = u_0u_1 \ldots u_r$ is called an **ordered path** relative to H, if the vertices in P are encountered in the order $u_0, u_1, \ldots, u_r$ when $H$is traversed from u_0. Similarly, a simple cycle $C = u_0u_1 \ldots u_ru_0$ is called an **ordered cycle** relative to H if the vertices in C are encountered in the order $u_0, u_1, \ldots u_r, u_0$ (or its reverse) when H is traversed from u_0. Two pairs v_iv_j and v_kv_l, are said to be **separable** [11] with respect to a vertex v_p if both v_i and v_j are encountered before v_k and v_l (or vice versa), when H is traversed from v_p. In this case, we say that v_iv_j and v_kv_l are v_p-separable; otherwise we say that they are v_p-inseparable. Note that two pairs v_iv_j and v_kv_l are separable with respect to v_p when the two pairs do not interlace on the boundary, (*i.e* $v_i < v_j < v_k < v_l$) and v_p lies on chain(v_j, v_k) or on chain(v_k, v_i)).

We now introduce a new class of graphs called **Quasi-Persistent** graphs, and show that the visibility graphs of all simple polygons are contained in this class. This class is a natural generalization of *persistent* graphs, a class originally introduced by Abello and Egecioglu [1].A graph G with Hamiltonian cycle H, is said to be **Quasi-Persistent** (or q-persistent) relative to H, if for every triple of vertices $v_i < v_p < v_q$, such that v_iv_p and $v_iv_q \in E$, and $v_iv_j \in \overline{E}$, for all v_j in chain(v_p, v_q), the following conditions hold

1. v_p is adjacent to v_q.
2. For every v_j in chain(v_p, v_q), at least one of the vertices v_p or v_q is a blocking vertex for v_iv_j .

The graph in figure 1 is a q-persistent graph. For a pair v_iv_j of non-consecutive vertices, let v_p be the first vertex adjacent to v_i, that is encountered on a *clockwise* traversal of H starting from v_{j-1}. v_p is called the first neighbour of v_i

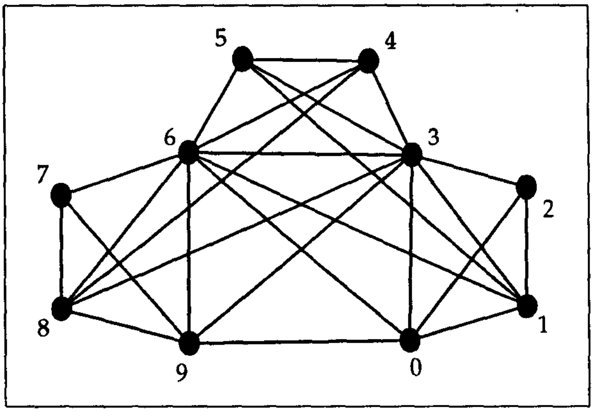

Fig. 1. A q-persistent graph.

before v_j and is denoted as $\mathrm{pn}(v_i v_j)$. Similarly, the first vertex v_q adjacent to v_i, encountered on a counter-clockwise traversal of H from v_{j+1}, is called the first neighbour of v_i after v_j and is denoted $\mathrm{sn}(v_i v_j)$. Since G is Hamiltonian, $\mathrm{pn}(v_i v_j)$ and $\mathrm{sn}(v_i v_j)$ exist for every invisible pair $v_i v_j$, *and they are distinct.* Also, note that the definition is not symmetric, *i.e*, it is not necessary that $\mathrm{pn}(v_i v_j)$ and $\mathrm{sn}(v_i v_j)$ be the same as $\mathrm{pn}(v_j v_i)$ and $\mathrm{sn}(v_j v_i)$ respectively. The q-persistence conditions imply that for any invisible pair $v_i v_j$, the vertices $\mathrm{pn}(v_i v_j)$ and $\mathrm{sn}(v_i v_j)$ are adjacent in G, and at least one of them is a blocking vertex for $v_i v_j$.

Ghosh [11] gave the first set of necessary conditions for a given graph to be a visibility graph. These conditions which we will henceforth call **Ghosh's conditions** are summarized below.

Proposition 1. *[Ghosh]If a graph G is the visibility graph of a simple polygon then*

1. *G has Hamiltonian cycle H.*
2. *Every ordered cycle relative to H of length ≥ 4 has a chord.*
3. *Every invisible pair in G has a blocking vertex relative to H.*
4. *If two invisible pairs are separable with respect to a vertex v_p, then v_p cannot be the only blocking vertex for both the invisible pairs.*

Our q-persistent graphs satisfy the first and third conditions of proposition 1 by definition. In fact, the second q-persistence condition(ordered chordality) appears, at first glance, to be much stronger than Ghosh's third condition. However, it can be shown that the class of q-persistent graphs is *equivalent* to the class of graphs that satisfies the first *three* of Ghosh's conditions. We summarize this as theorem 2 below.

Theorem 2. *A graph G with Hamiltonian cycle H is q-persistent relative to H if and only if every ordered cycle of length ≥ 4 has a chord and every invisible pair has a blocking vertex (relative to H).*

Thus, q-persistent graphs are not a fundamentally new class of graphs. The main advantage of the above formulation is that the simpler structure of the definition makes it easier to analyze and prove properties of the resulting class. It is interesting to note the relationship between the two q-persistence conditions and Ghoshs conditions 2 and 3. The first q-persistence condition is a "weaker" version of ordered chordality, in the sense that the graphs that are Hamiltonian and ordered chordal are *properly* contained in the class of (Hamiltonian) graphs satisfying the first q-persistence condition. On the other hand the second q-persistence condition is a stronger version of Ghosh's condition 3 since Hamiltonian graphs that satisfy the second q-persistence condition are *properly* contained in the class of Hamiltonian graphs satisfying Ghosh's condition 3. However, when both pairs of conditions are considered together, the classes become equivalent!

Since visibility graphs satisfy all four of Ghosh's conditions, it is evident from theorem 2 that visibility graphs are properly contained in the class of q-persistent graphs. The following section develops additional necessary conditions for a q-persistent graph to be a visibility graph and shows that these conditions are strictly stronger than Ghosh's conditions.

3 New Necessary Conditions for Visibility Graphs

We assume throughout this section, that P is a simple polygon in the Euclidean plane and that a q-persistent graph G is its visibility graph. Arbitrary points of the plane are denoted as p_x, p_y *etc.* For two points p_x and p_y, the ray from p_x in the direction of p_y will be denoted r_{xy}. For a vertex v_i of G the corresponding vertex of P is denoted v_i^*. We will also use r_{ij} to denote the ray from v_i^* in the direction of v_j^*. A polygon P whose visibility graph is G, is called a **realization** of the graph G. A given q-persistent graph that is a visibility graph, can have many different realizations.

Suppose $W = w_0, \ldots, w_k$ is the sequence of neighbors of a vertex v_i in G, obtained in traversing H, with $w_0 = v_{i+1}$ and $w_k = v_{i-1}$. In a polygon Prealizing G, $\angle w_k^* v_i^* w_{j-1}^* < \angle w_k^* v_i^* w_j^*$ for $1 \le j \le k-1$ (see [9], pg. 18 for a proof of this fact). The second q-persistence condition can now be interpreted geometrically. let $v_i < v_p < v_q$ be a triple of vertices in G such that $v_i v_p, v_i v_q \in E$ and $v_i v_j \in \overline{E}$ for all v_j on chain(v_p, v_q). For the corresponding triple of points v_i^*, v_p^*, and v_q^* in a realization P of G, there exists a *unique* segment $v_k^* v_{k+1}^*$ on the boundary of P, such that, v_k and v_{k+1} lie on chain$[v_p, v_q]$, and for *any* ray r_{ix} such that $\angle v_{i-1}^* v_i^* v_p^* < \angle v_{i-1}^* v_i^* p_x < \angle v_{i-1}^* v_i^* v_q^*$, the first segment on the boundary of P that is intersected by ray r_{ix} is $v_k^* v_{k+1}^*$. The fact to be emphasized here is that the segment so obtained does not depend on the specific ray r_{ix}, but only on the triple of points involved (see figure 2).

For any vertex v_j on chain$(v_p, v_k]$, the vertex v_p is a blocking vertex (in G) for $v_i v_j$ and for any vertex v_j on chain$[v_{k+1}, v_q)$, the vertex v_q is a blocking vertex for $v_i v_j$. The edge $v_k^* v_{k+1}^*$ is called the **split segment** for

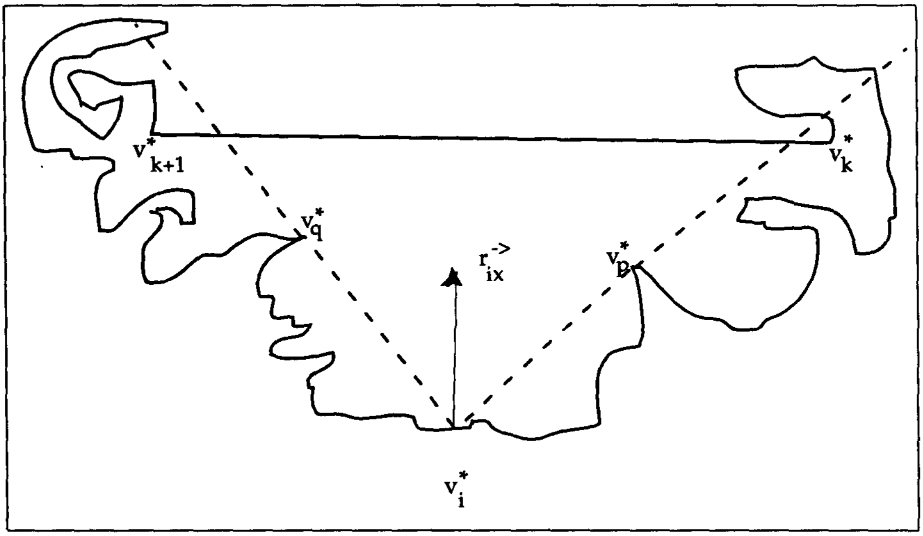

Fig. 2. Geometric interpretation of q-persistence

the triple of points $v_i^* v_p^* v_q^*$. The corresponding edge in the graph is called the **split edge**. Intuitively, the split edge determines, which one of the points v_p^* and v_q^* is involved in "physically" blocking a given pair $v_i v_j$ on chain(v_p, v_q) in a given polygon whose visibility graph is G. In general, the split edge is not determined by the visibility graph alone. Different polygons with the same underlying visibility graph may have different split edges for the same triple of vertices in G.

The q-persistence conditions stipulate that for any invisible pair $v_i v_j \in \overline{E}$ of a q-persistent graph G, at least one of the vertices pn$(v_i v_j)$ or sn$(v_i v_j)$ must be a blocking vertex for the invisible pair. However, according to the discussion above, in any fixed realization of G, at most one of these vertices "physically" blocks the corresponding invisible pair of points in the realization. This motivates the following definitions.

A vertex v_p is called a **primary blocking vertex** for an invisible pair $v_i v_j$ if v_p is a blocking vertex for $v_i v_j$ and $v_i v_p \in E$. By the definition of blocking vertices, the only possible choices for the primary blocking vertices for $v_i v_j$ are pn$(v_i v_j)$ and sn$(v_i v_j)$. Therefore, if either of the vertices pn$(v_i v_j)$ or sn$(v_i v_j)$ is a blocking vertex for the invisible pair $v_i v_j$ in a q-persistent graph G, then it is called a primary blocking vertex for $v_i v_j$. The q-persistence conditions imply that every invisible pair has at least one primary blocking vertex. Also, the primary blocking vertices of the pair $v_i v_j$ are not necessarily the same as those for $v_j v_i$.

A blocking vertex assignment [3] for a q-persistent graph G, is a function

[3] Everett, in [9], also defines a similar notion, but the requirement that vertices in the image of the function be *primary* blocking vertices makes the definition given here strictly stronger than the one in [9]

$\beta: \overline{E} \rightarrow V$ such that, for all $v_i v_j \in \overline{E}$, $\beta(v_i v_j)$ is a primary blocking vertex for $v_i v_j$. Any q-persistent graph has at least one blocking vertex assignment. If G is a visibility graph, then every fixed realization, P of a given q-persistent graph, determines a particular blocking vertex assignment for G as follows. For a triple $v_i < v_p < v_q$ of vertices in G, such that $v_i v_p$, $v_i v_q \in E$ and $v_i v_j \in \overline{E}$ for all v_j on chain(v_p, v_q), let $v_k^* v_{k+1}^*$ be the split segment in P for the triple $v_i^* v_p^* v_q^*$. We set $\beta(v_i v_j) = v_p$ for all v_j on chain$(v_p, v_k]$. For all v_j on chain$[v_{k+1}, v_q)$ we set $\beta(v_i v_j) = v_q$. From the discussion in the last section, it follows that β is a blocking vertex assignment for G. This blocking vertex assignment is called a **canonical blocking vertex assignment** for G determined by the realization P.

We now consider the following problem: Given a q-persistent graph together with a blocking vertex assignment β, determine the conditions under which there exists a polygon P whose visibility graph is G, and such that the canonical assignment on G determined by P is β. Such conditions will clearly yield a characterization of visibility graphs. It turns out that blocking vertex assignments on q-persistent graphs must satisfy four additional necessary conditions in order to be canonical assignments.

A blocking vertex assignment is said to be **locally inseparable** if any two invisible pairs $v_i v_j$ and $v_k v_l$ such that $\beta(v_i v_j) = \beta(v_k v_l) = v_p$ are v_p-inseparable (see definition on page 3). The following is a necessary condition for a blocking vertex assignment to be a canonical blocking vertex assignment [4].

Necessary Condition 1: *If β is a canonical blocking vertex assignment for a q-persistent graph G, determined by a realization P, then β is locally-inseparable.*

In order to state the remaining necessary conditions we need to introduce the following definition Given an invisible pair $v_i v_k$ in G, an **occluding path** generated by β, between v_i and v_k, denoted path$_\beta(v_i, v_k)$ is a path $v_i u_0 \ldots u_r v_k$ in G, such that $u_0 = \beta(v_i v_k)$, $u_j v_k \in \overline{E}$, and $u_{j+1} = \beta(u_j, v_k)$ for $0 \leq j \leq r-1$. It is readily seen that a given blocking vertex assignment determines a unique occluding path between every invisible pair of vertices. It can also bee shown that this path is simple and that every internal vertex on this path is a blocking vertex for the invisible pair. For notational convenience, we identify path$_\beta(v_i, v_k)$ with its underlying set of vertices.

When a graph G is the visibility graph of a polygon P, the graph theoretical notion of occluding path corresponds to the geometric notion of shortest path under the geodesic metric. This fact is stated in the following proposition

Proposition 3. *Let β be the canonical blocking vertex assignment for a visibility graph G, determined by a fixed realization P. A vertex v_x lies on path$_\beta(v_i, v_j)$ if and only if v_x^* lies on the Euclidean shortest path in P between v_i^* and v_j^*.*

[4] Everett [9] conjectures a similar result. However, since our definition of blocking vertex assignment is stricter, Necessary Condition 1 is stronger.

The remaining necessary conditions arise as a result of this correspondence between occluding and Euclidean shortest paths . A blocking vertex assignment is called **path-symmetric** if for every invisible pair $v_i v_k$ such that $\text{path}_\beta(v_i, v_k) = v_i u_0 \ldots u_r v_k$, we have $\text{path}_\beta(v_k, v_i) = v_k u_r \ldots u_0 v_i$. We denote this as $\text{path}_\beta(v_k, v_i) = \text{path}^R_\beta(v_i, v_k)$ In other words, even though a blocking vertex assignment is not symmetric in the blocking vertices it assigns to invisible pairs $v_i v_k$ and $v_k v_i$, it must ensure the symmetry of the occluding paths generated under the assignment between every invisible pair of vertices. Since Euclidean shortest paths between two points inside a simple polygon are unique, it readily follows that canonical blocking vertex assignments must be path-symmetric; a fact which we summarize as necessary condition 2 below.

Necessary Condition 2: *If β is a canonical blocking vertex assignment for a q-persistent graph determined by a realization P, then β is path-symmetric.*

The two remaining necessary conditions reflect the constraints imposed on occluding paths, generated by canonical blocking vertex assignments, because of their correspondence with Euclidean shortest paths A blocking vertex assignment satisfying these two conditions will be called a **path-consistent** assignment.

Necessary Condition 3: *If β is a canonical blocking vertex assignment for a q-persistent graph determined by a realization P, and if $u_x \in path_\beta(v_i, u_y)$, and $u_y \in path_\beta(u_x, v_k)$ then $u_x, u_y \in path_\beta(v_i, v_k)$.*

Necessary Condition 4: *If β is a canonical blocking vertex assignment for a q-persistent graph determined by a realization P, and*

1. *If $v_p \in path_\beta(v_i, v_k)$ is an inner blocking vertex for $v_i v_k$, then for all v_x on chain$[v_i, v_p)$ and v_y on chain$(v_p, v_k]$, $v_p \in path_\beta(v_x, v_y)$*
2. *If $v_p \in path_\beta(v_i, v_k)$ is an outer blocking vertex for $v_i v_k$, then for all v_x on chain$(v_p, v_i]$ and v_y on chain$[v_k, v_p)$, $v_p \in path_\beta(v_x, v_y)$*

The proofs of necessary conditions 3 and 4 are based on the fact that Euclidean shortest paths satisfy the above combinatorial conditions. It is natural to ask whether all the above four conditions are independent of each other. It can be shown that in fact they are. Namely, for any subset of these conditions, there exist q-persistent graphs for which blocking vertex assignments can be constructed that satisfy only that subset and no the others. On the other hand to contrast these conditions with Ghosh's conditions, Everett has exhibited a graph that satisfies Ghosh's conditions and yet it is not a visibility graph. It can be shown that Everett's example is a q-persistent graph, that does not have a blocking vertex assignment that satisfies condition 1. The graph in figure 1 is a q-persistent graph that was shown not to be a visibility graph in [2]. It can be shown that this graph has one blocking vertex assignment that satisfies necessary condition 1, and another that satisfies conditions 2, 3 and 4, but no one that satisfies all conditions simultaneously. Thus these four necessary conditions are a non-trivial strengthening of Ghosh's conditions. The key question is whether they are sufficient. The next section provides a partial answer to this question.

4 Q-Persistent Graphs and Oriented Matroids

We consider the problem of determining, given as input a q-persistent graph with a blocking vertex assignment satisfying the conditions of the previous section, a combinatorial representation of a potential polygon whose visibility graph is isomorphic to the given graph. The main result of this section is that such a combinatorial reconstruction seems to be significantly easier than the actual reconstruction of the polygon.

Oriented matroids are a well studied combinatorial representation [6, 10, 15] for point configurations. In the following, we adopt the conventions of [6] and identify oriented matroids with their representations by **chirotopes**. An equivalence proof for this representation and the classical definition in terms of signed circuits of matroids may be found in [15]. We are concerned here, only with the definition of oriented matroids of rank 3. Let $\mathcal{T}_n$, $n \geq 3$ denote the set of increasing triples from the set $\{0, \ldots, n-1\}$(*i.e* 3-tuples (i, j, k) where $i < j < k$). A mapping $\chi: \mathcal{T}_n \rightarrow \{-1, +1, 0\}$ (alternatingly extended to the set of all ordered triples from $\{0, \ldots, n-1\}$) is called a **chirotope** if for all i, $0 \leq i \leq n-1$ and all 4-tuples $0 \leq j < k < l < m \leq n-1$ from $\{0, \ldots n-1\}$ the set

$$\left\{ \begin{array}{l} \chi(i,j,k).\chi(i,l,m), \\ -\chi(i,j,l).\chi(i,k,m), \\ \chi(i,j,m).\chi(i,k,l) \end{array} \right\}$$

either contains $\{-1, +1\}$ or equals $\{0\}$. The chirotope is called simplicial if its image is contained in the set $\{-1, +1\}$.

A chirotope is called **co-ordinatizable** if there exists an $n \times 3$ matrix M such that for any triple $(i, j, k) \in \mathcal{T}_n$, $\chi(i, j, k)$ agrees with the sign of the corresponding 3×3 subdeterminant of M. The chirotope associated with a point configuration assigns to each triple of points its orientation (given by the signed area). The fact that these subdeterminants obey the chirotope conditions above follows from the well known Grassman-Plucker identities (see [6]). Deciding if a given rank 3 oriented matroid is co-ordinatizable is known to be NP-hard [20]. It is also polynomially equivalent to the decision problem for the existential theory of the reals [17] and thus in $PSPACE$ [7].

We now establish the existence of a simplicial chirotope, corresponding to every q-persistent graph G with a blocking vertex assignment β that is path-symmetric, path consistent and locally-separable. Call such β a feasible blocking vertex assignment. The chirotope has the property that any of its coordinatizations defines a simple polygon whose visibility graph is isomorphic to the input graph and induces a canonical blocking vertex assignment on G which is exactly β. This chirotope, called the **Normal Chirotope** for the pair (G, β) can be constructed in polynomial time given G and β.

Given a q-persistent graph G with a feasible blocking vertex assignment β, we define a function $\chi_{G,\beta}: \mathcal{T}_n \rightarrow \{-1, +1\}$ (alternatingly extended to the set of all ordered triples in $\{0, \ldots, n-1\}$) where

$$\chi(i,j,k) = \begin{cases} -1 & \text{if there exists an occluding path generated by } \beta \text{ that contains the vertices } v_i, v_j \text{ and } v_k \\ +1 & \text{otherwise.} \end{cases}$$

The function χ can be constructed from G and β in $O(n^4)$ time. Moreover, it defines a simplicial chirotope. This constitutes the main result of this paper.

Theorem 4. If G is a q-persistent graph and β is a blocking vertex assignment that is path-symmetric, path-consistent and locally-separable then $\chi_{G,\beta}$ is a simplicial chirotope.

We now sketch a proof of the fact that if the normal chirotope $\chi_{G,\beta}$ is realizable, then the corresponding realization yields a polygon whose visibility graph is isomorphic to G and such that the canonical assignment induced on the graph by the polygon is precisely β. Suppose that a normal chirotope is affinely co-ordinatizable. We first note that points $v_0^*, \ldots, v_{n-1}^*$ in a plane realization of the chirotope, together with the segments $v_i^* v_{i+1}^*$ mod n constitute a simple polygon P. To see this notice that if β is feasible, it is impossible that $\chi(i, i+1, j) . \chi(i, i+1, j+1) = -1$ and $\chi(j, j+1, i) . \chi(j, j+1, i+1) = -1$ when $|i-j| > 1$. In the realization this implies that no two segments of the polygon intersect, ensuring simplicity. Also note that since the chirotope is simplicial, the resulting point configuration is always non-degenerate.

Now consider a triple $v_i v_p v_q$ in G such that v_i is adjacent to no vertex in chain(v_p, v_q) and adjacent to v_p and v_q. Let $v_k v_{k+1}$ be the split edge for the triple determined by β. Interpreting signs assigned to the ordered triples by β as orientations of the corresponding triples of points, we can show that

1. The interior of the triangle $v_i^* v_p^* v_q^*$ contains no points of P.
2. The points corresponding to chain$(v_p, v_k]$ and the points corresponding to chain$[v_{k+1}, v_q]$ lie on opposite halfspaces of the line containing $v_i^* v_p^*$. Similarly the points corresponding to chain$[v_p, v_k]$ and the points corresponding to chain$[v_{k+1}, v_q)$ lie on opposite halfspaces of the line containing $v_i^* v_q^*$.

We also note that by local-inseparability v_i cannot lie both on an occluding path from v_p to a vertex on chain(v_i, v_p) and also on a path from v_q to one on chain(v_q, v_i). This together with item 1 allow us to claim that v_i^*, v_p^*, and v_q^* are visible from each other. Item 2 allows us to claim that v_i^* is invisible from all the points corresponding to those on chain(v_p, v_q).

Therefore v_p^* and v_q^* are succesive neighbors of v_i^* and $v_k^* v_{k+1}^*$ is the split segment for this triple. A similar argument shows the converse case, that is when v_p^* and v_q^* are the succesive neighbors of a vertex v_i^*, then the corresponding three vertices are all adjacent to each other. Also, if the split segment determined by the realization is $v_k^* v_{k+1}^*$, then the corresponding split edge determined by β for this triple is $v_k v_{k+1}$. Repeating the argument for each triple in G shows that the co- ordinatization gives a simple polygon P whose visibility graph is G, and determines the canonical assignment β on G.

From the previous discussion it follows that any coordinatization of the chirotopes described here will turn theorem 4 into the first characterization of visibility graphs. We know at this point such a characterization holds for values of n up to 7 and for visibility graphs of 2-spiral polygons. We notice in closing that all the conditions stated in the hypothesis of theorem 4 are used in its proof.

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