Minimum-Width Grid Drawings of Plane Graphs (Extended Abstract)

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Abstract

Given a plane graph G, we wish to draw it in the plane, according to the given embedding, in such a way that the vertices of G are drawn as grid points, and the edges are drawn as straight-line segments between their endpoints. An additional objective is to minimize the size of the resulting grid. It is known that each plane graph can be drawn in such a way in a $(n-2) \times (n-2)$ grid (for $n \geq 3$), and that no grid smaller than $(2n/3-1) \times (2n/3-1)$ can be used for this purpose, if n is a multiple of 3. In fact, it can be shown that, for all $n \geq 3$, each dimension of the resulting grid needs to be at least $\lfloor 2(n-1)/3 \rfloor$, even if the other one is allowed to be infinite. In this paper we show that this bound is tight, by presenting a grid drawing algorithm that produces drawings of width $\lfloor 2(n-1)/3 \rfloor$. The height of the produced drawings is bounded by $4\lfloor 2(n-1)/3 \rfloor - 1$.

1 Introduction

The problem of automatic graph drawing has attracted recently a lot of attention, due to its numerous practical applications and challenging mathematical and algorithmic questions that arise in this area. Generally, given a graph G, the task is to produce an *æsthetic* drawing of G, one that accurately reflects the topological structure of G in a graphical form. Many versions of this problem have been considered, and there is a variety of techniques, algorithms, and software packages that are currently available. (See the survey in [BETT] for more information.)

For planar graphs, we typically require that vertices are represented by points in the plane, and edges are drawn as non-intersecting straight-line segments between their endpoints. Additionally, we are often given a *plane graph*, that is a planar graph with a given planar embedding, represented combinatorially by cyclic orderings of edges incident to all vertices. Then the drawing needs to be consistent with that given planar embedding, in the sense that for each vertex v, the given cyclic ordering of edges incident to v needs to be the same as their clockwise ordering in the drawing.

In this paper we deal with the following version: given a plane graph G, we want to map its vertices into integer grid points in such a way that the edges between them can be drawn as straight, non-intersecting line segments. The resulting drawing has to be consistent with the planar embedding of G. We call such mappings *grid drawings*.

It has been proven that each plane graph has a straight-line drawing [Fa48, Wa36, St51], which implies that it also has a grid drawing, since we can approximate real vertex coordinates by

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rational numbers, and then use appropriate scaling. The grids obtained by following this method are, unfortunately, of exponential size.

The question whether smaller, polynomial-size, grids can be used for this purpose was open until 1988, when de Fraisseix, Pach and Pollack [FPP88, FPP90] proved that each plane graph with n vertices can be embedded into a $(2n-4)\times(n-2)$ grid. (Throughout the paper we assume that $n \geq 3$.) We will refer to their method as the *shift method*. Their paper initiated intensive research in this area, and led to new results and implementations. Chrobak and Payne [CP89] gave a simple, linear-time implementation of the shift method. Schnyder [Sc90] presented a different technique, based on so called *barycentric representations*, that led to smaller grid drawings of size $(n-2)\times(n-2)$. He also pointed out later (personal communication) that there is a close relationship between the shift method and barycentric representations, and that the grid size in [CP89] can be reduced to $(n-2)\times(n-2)$ without affecting time complexity.

The obvious question is: what is the minimum size of grid drawings? In their paper, [FPP88] proved that, in the worst case, no grid smaller than $(2n/3-1)\times(2n/3-1)$ is possible for n-vertex plane graphs, if n is a multiple of 3. The simple argument they presented can be easily modifed to show that, for all $n \geq 3$, each dimension of the grid needs to be at least $\lfloor 2(n-1)/3 \rfloor$, even if the other one is unbounded.

In this paper we show that this bound is tight, by presenting an algorithm that embeds each n-vertex plane graph into a grid of width at most $\lfloor 2(n-1)/3 \rfloor$. The height of the resulting drawings is at most $4\lfloor 2(n-1)/3 \rfloor - 1 \leq 8n/3 - 3$.

2 Preliminaries

Let $G = (V, E)$ be an arbitrary maximal (triangulated) plane graph with n vertices, where $n \geq 3$, and $\pi = v_1, \ldots, v_n$ an ordering of V such that (v_1, v_2, v_n) is the external face of G. Define G_k to be the subgraph of G induced by $v_1, \ldots, v_k$ and C_k to be its external face. We say that π is a *canonical ordering of* G if the following conditions are satisfied for each $k = 3, \ldots, n$:

(col) Each G_k is 2-connected and internally triangulated (that is, all internal faces of G_k are triangles).

(co2) C_k contains (v_1, v_2).

(co3) If $k < n$, then v_{k+1} is in the exterior face of G_k, and all neighbors of v_{k+1} in G_k belong to C_k.

It is easy to see that Conditions (col) and (co3) imply that the neighbors of v_{k+1} must, in fact, be consecutive in C_k. The existence of canonical orderings was proven by de Fraisseix, Pach and Pollack in [FPP88] (See also [Ka93]).

Lemma 1 *Let G be a maximal plane graph, and (v_1, v_2) an edge on its external face. Then there exists a canonical ordering $\pi = v_1, v_2, \ldots, v_n$ of G, and π can be constructed in linear time.*

By an *ordered plane* graph (G, π) we will understand a plane graph G with a given canonical ordering $\pi = v_1, \ldots, v_n$. By the *contour of* G_k we mean its external face written as $C_k = (w_1 = v_1, w_2, \ldots, w_m = v_2)$.

Let $k \geq 3$, and let the neighbors of $v = v_{k+1}$ in G_k be $w_p, w_{p+1}, \ldots, w_q$. The *in-degree of* v, denoted $deg^-(v)$, is the number of neighbors of v in G_k, that is $deg^-(v) = q - p + 1$. For $i = p, \ldots, q$, we denote $ind_v(w_i) = i - p + 1$ and call it the *index of w_i with respect to v*. For $k = 1, 2, 3$, $deg^-(v_k)$ and ind_{v_k} are undefined.

By a and b we will usually denote, respectively, the number of vertices of in-degree 2, and the number of vertices of in-degree ≥ 3, among $v_4, \ldots, v_n$. If $n \geq 4$, clearly, $deg^-(v_k) \geq 2$ for each $k = 4, \ldots, n-1$, $deg^-(v_n) \geq 3$, and $n = a + b + 3$.

Let N denote the set of non-negative integers, and G a given plane graph with vertex set V. Let $P = (x, y) : V \to N \times N$ be a function that maps V into an integer grid, where $x(v)$ and $y(v)$ represent the x and y coordinates of a vertex v.

Given a mapping $P = (x, y)$, we say that P is a *grid drawing* of G, if it satisfies the following conditions: (gd1) If $u \neq v$, then $P(u) \neq P(v)$. (gd2) No two edges of G intersect in P. (gd3) For each vertex v, the clockwise ordering of the segments $[P(v), P(u)]$, where u is a neighbor of v, is identical to their cyclic ordering in the given planar embedding of G.

The *width* of a given drawing is defined as the distance between its leftmost and rightmost vertices, that is $\max_{u,v} |x(u) - x(v)|$. The *height* is defined similarly: $\max_{u,v} |y(u) - y(v)|$.

By a minor modification of the construction in [FPP88], we obtain the following theorem.

Theorem 1 *For each $n \geq 3$ there is an n-vertex plane graph H_n, such that each grid drawing of H_n has width at least $\lfloor 2(n-1)/3 \rfloor$.*

3 The Shift Method for Grid Drawings

Let (G, π) be a given ordered maximal plane graph, where $\pi = v_1, \ldots, v_n$ and $n \geq 3$. Our general strategy is similar to the methods from [FPP88, CP89]: we add vertices one at a time, in canonical order. At every time step, the contour C_k satisfies a certain invariant that involves restrictions on the slopes of contour edges. When adding a vertex v_{k+1} we determine its location in the grid and, if necessary, shift some parts of G_k to the right in order to preserve the invariant. The difficult part is to determine which internal vertices of G_k can be shifted to the right without violating planarity. We will describe such a method in this section.

We will maintain a set $U(v)$ for each vertex v. This set will contain vertices located "under" v that need to be shifted whenever v is shifted. Initially, $U(v_i) = \{v_i\}$ for $i = 1, 2, 3$. Suppose that $3 \leq k \leq n - 1$ and that we are about to add v_{k+1} to G_k. Let the contour of G_k be $C_k = (w_1 = v_1, w_2, \ldots, w_m = v_2)$ and that $w_p, \ldots, w_q$ are the neighbors of v_{k+1} in G_k. Then we set $U(v_{k+1}) := \{v_{k+1}\} \cup \bigcup_{i=p+1}^{q-1} U(w_i)$.

Shifting a contour vertex w_j is achieved by operation *shift*(w_j), that increases the x-coordinate of each $u \in \bigcup_{i=j}^{m} U(w_i)$ by 1.

Initially, v_1, v_2, v_3 are mapped into different grid points so that $x(v_2) > x(v_1) \geq 0$, and v_3 is located at a point above the line segment joining v_1, v_2 and satisfying $x(v_1) \leq x(v_3) \leq x(v_2)$.

Inductively, suppose that $3 \leq k \leq n - 1$, that G_k has already been embedded, and that we are about to add $v = v_{k+1}$. Let $C_k = (w_1, \ldots, w_m)$ be the contour of G_k, and $w_p, \ldots, w_q$ be the neighbors of v in G_k. Apply *shift*(w_i) to some of $w_1, \ldots, w_m$ (possibly none), so that afterwards there exists at least one point (x', y') such that (gsm1) $x(w_p) \le x' \le x(w_q)$, (gsm2) (x', y') is located *above* C_k in the following sense: the half line $\{(x', z) : z \ge y'\}$ does not intersect C_k, and (gsm3) all vertices $w_p, \ldots, w_q$ are visible from (x', y'). Pick an arbitrary such point (x', y') and set $(x, y)(v) = (x', y')$.

Lemma 2 *For all choices of shift operations and vertex coordinates, as long as (gsm1), (gsm2) and (gsm3) are satisfied, the Generic Shift Algorithm produces a correct grid drawing.*

4 Minimum-Width Grid Drawings

Let (G, π) be a given ordered, maximal plane graph, where $\pi = v_1, \ldots, v_n$. For a given $3 \le k \le n-1$, let $w_p, \ldots, w_q$ be the neighbors of $v = v_{k+1}$ in C_k. When we add v to G_k, its leftmost and rightmost edges $(w_p, v), (v, w_q)$ become contour edges. We call (w_p, v) a *forward edge* and (v, w_q) a *backward edge*. All vertices and edges that disappear from the contour when we add v are said to be *covered* by v.

A vertex $v \ne v_1, v_2, v_3$ of in-degree 2 is called *forward-oriented (backward-oriented)* if it covers a forward (backward) edge. Let a_f and a_b be the numbers of forward-oriented and backward-oriented vertices. Note that the values of a_f, a_b depend on the canonical ordering.

Assume now that $n \ge 4$. Each vertex $v \ne v_1, v_2$ will be classified as *stable* or *unstable*. Also, with each such vertex we will associate a sequence of vertices called its *domino chain*, $DC(v)$, and a vertex $dom(v)$ called the *dominator* of v.

These concepts are defined as follows: For $v = v_n$, we define $DC(v_n) = (v_n)$, $dom(v_n)$ is undefined, and v_n is stable.

Suppose now $2 \le k \le n-2$, $v = v_{k+1}$, and let u be the leftmost neighbor of v in G_k, that is $ind_v(u) = 1$ (for $k = 2$ we assume $u = v_1$). Let also z be the vertex that covers edge (u, v). Such z must exist because $v \ne v_n$. Then:

(dc1) If $ind_z(v) = 2$, then $DC(v) := (v)$, $dom(v) := z$ and v is unstable.

(dc2) If $ind_z(v) \ge 4$, then $DC(v) := (v)$, $dom(v) := z$ and v is stable.

(dc3) If $ind_z(v) = 3$ and $DC(z) = (z_1, \ldots, z_i, z)$, then $DC(v) := (z_1, \ldots, z_i, z, v)$ and $dom(v) := dom(z)$. Also, v is stable iff z is stable.

An unstable vertex of in-degree 2 is called a *room-shift vertex*.

Note that, for $n \ge 4$, $DC(v_3)$ contains only vertex v_3, and the dominator of v_3 is the vertex z that covers edge (v_1, v_3); thus $ind_z(v_3) = 2$.

Example: Consider the ordered graph in Fig 1. We have $DC(10) = (13, 11, 10)$, $dom(10) = 19$, $DC(8) = (13, 11, 10, 9, 8)$ and $dom(8) = 19$, Since $ind_{18}(17) = 4$, vertex 17 is stable. Since $ind_{17}(16) = 2$, vertices 16, 15 are unstable. Vertices 4, 5, 7, 8, 15 are room-shift vertices, and 14 has in-degree 2 but is not a room-shift vertex. ♠

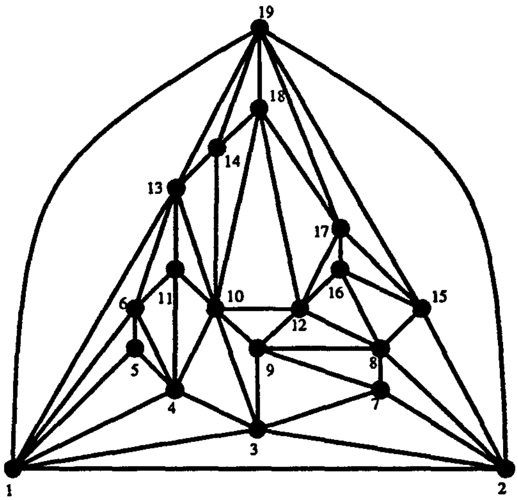

Figure 1: An example of a canonical ordering.

Algorithm $\mathcal{A}$: Given a maximal plane graph G, with $n \geq 3$ vertices, pick any edge (v_1, v_2) on the external face of G, and find its canonical ordering $\pi = v_1, \ldots, v_n$. If the number of forward-oriented vertices in π is greater than the number of backward-oriented vertices, then we modify π by swapping v_1 and v_2.

At this point, we are given an ordered maximal plane graph (G, π). If $n = 3$, we define $(x, y)(v_1) = (0, 0)$, $(x, y)(v_2) = (1, 0)$ and $(x, y)(v_3) = (0, 1)$, and the algorithm terminates.

Assume now that $n \geq 4$. We first embed v_1, v_2, v_3, as follows: $(x, y)(v_1) = (0, 0)$, $(x, y)(v_2) = (2, 0)$ and $(x, y)(v_3) = (1, 1)$.

After this initialization, we add vertices in order $v_4, \ldots, v_n$. Suppose $3 \leq k \leq n - 1$, and that we are about to add $v = v_{k+1}$. As usual, let $C_k = (w_1, \ldots, w_m)$ and $w_p, \ldots, w_q$ be the neighbors of v in G_k. If v is stable then $x(v) := x(w_p)$. Otherwise, $x(v) := x(w_p) + 1$ and, additionally, if $deg^-(v) = 2$ then we do $shift(w_q)$.

In both cases the $y(v)$ is chosen to be the smallest integer such that $(x', y') = (x(v), y(v))$ satisfies requirements (gsm2) and (gsm3).

If we swapped v_1, v_2 at the beginning of the algorithm, the clockwise ordering of edges incident to each vertex will be reversed with respect to the one in the given planar embedding. This can be easily modified, if desired, by using the left-right reflection: set $x_0 := x(v_2)$ and then $x(v_k) := x_0 - x(v_k)$ for all k.

Lemma 3 *Assume $n \geq 4$, and let $u, v \neq v_1, v_2$. Then* (a) *If $u \in DC(v)$ then $DC(u)$ is a prefix of $DC(v)$.* (b) *If $u \notin DC(v)$ and $v \notin DC(u)$ then:* (b1) *$DC(u) \cap DC(v) = \emptyset$.* (b2) *If u, v are unstable, then $dom(u) \neq dom(v)$.*

Theorem 2 *If G is a given maximal plane graph with $n \geq 3$ vertices, then Algorithm $\mathcal{A}$ produces a grid drawing of G of width $\lfloor 2(n-1)/3 \rfloor$ and height $n^2/4$.*

Proof: Correctness: Omitted.

Width estimate: If π is an arbitrary canonical ordering of G and π' is obtained by swapping v_1 with v_2, then each vertex of degree 2 is forward-oriented in π iff it is backward-oriented in π'. Thus in the canonical ordering π computed in $\mathcal{A}$ we have $a_f \leq a_b$.

Let ω be the width of the drawing constructed by algorithm $\mathcal{A}$, and denote by a^{rs} the number of room-shift vertices (other than v_1, v_2, v_3). Then $\omega = a^{rs} + 2$. Observe that a dominator cannot be a backward-oriented vertex of in-degree 2. By Lemma 3, dominators of room-shift vertices are distinct, and they are distinct from the dominator of v_3. This implies that $a^{rs} \leq a_f + b - 1$, and we get $\omega = a^{rs} + 2 \leq (a_f + b - 1) + 2 \leq a/2 + b + 1 = n - a/2 - 2$. On the other hand, $\omega \leq a + 2$. Therefore $\omega \leq \min(a+2, n - a/2 - 2) \leq 2(n-1)/3$, as required.

Height estimate: Let $-\gamma$ be the smallest slope among the edges in the current contour. After we add a room-shift vertex to the contour, the smallest contour slope is at least $-\gamma$. After we add a vertex which is not a room-shift vertex the smallest contour slope will be at least $-\gamma - 1$. Thus, if G has a^{rs} room-shift vertices, then the width of the drawing is exactly $a^{rs} + 2$ and the edge (v_n, v_2) has slope at least $-(n - 2 - a^{rs})$. Therefore the height of the drawing is at most $(a^{rs} + 2) \cdot (n - a^{rs} - 2) \leq n^2/4$. □

5 Reducing Height

As usual, let $C_k = (w_1 = v_1, w_2, \ldots, w_m = v_2)$ be the contour of G_k. Then each contour edge (w_i, w_{i+1}) belongs to one of the following four types: *vertical:* when $x(w_i) = x(w_{i+1})$, *horizontal:* when $y(w_i) = y(w_{i+1})$, *upward:* when $y(w_i) < y(w_{i+1})$, *downward:* when $y(w_i) > y(w_{i+1})$.

We define the *slack* between u and v by $slack(u, v) = y(v) + 4[x(v) - x(u)] - y(u)$. We will distinguish two types of shifts. Let v be a vertex to be installed. As in Algorithm $\mathcal{A}$, a *room-shift* occurs when v is a room-shift vertex. A *slope-shift* occurs when we shift the rightmost neighbor w_q of v in order to reduce the absolute value of the slope of edge (v, w_q). We will call such v *slope-shift vertices.* No two shifts will occur simultaneously. A *shift vertex* is either a room-shift or a slope-shift vertex.

A vertex v is *slack-preserving* if either (sp1) $deg^-(v) \geq 4$, or (sp2) $deg^-(v) = 3$ and v is stable. Also, v is *slack-reducing* if either (sr1) $deg^-(v) = 3$ and v is unstable, or (sr2) $deg^-(v) = 2$ and v is stable. Note that the two above concepts are not complementary, since room-shift vertices are neither slack-preserving nor slack-reducing.

Algorithm $\mathcal{B}$: The choice of the canonical ordering $\pi = v_1, \ldots, v_n$ and the initialization are exactly the same as in Algorithm $\mathcal{A}$.

Assume now that $n \geq 4$, and suppose that we are now about to add a vertex $v = v_{k+1}$, for $3 \leq k \leq n-1$. As usual, we denote $C_k = (w_1, \ldots, w_m)$, and $w_p, \ldots, w_q$ are the neighbors of v in G_k.

Let $x(v) := x(w_p)$ if v is stable, otherwise $x(v) := x(w_p) + 1$. Then we consider three cases.

Case 1: If v is slack-preserving, then $y(v) := y(w_q) + 4[x(w_q) - x(v)] - slack(w_{q-1}, w_q)$. Note that $slack(v, w_q) = slack(w_{q-1}, w_q)$.

Case 2: If v is slack-reducing, then $y(v) := y(w_q) + 4[x(w_q) - x(v)] - slack(w_{q-1}, w_q) + 1$. Note that $slack(v, w_q) = slack(w_{q-1}, w_q) - 1$. Then, if $slack(v, w_q) = 0$, we slope-shift w_q by executing $shift(w_q)$.

Case 3: If v is a room-shift vertex, then we room-shift w_q by executing $shift(w_q)$. If (w_p, w_q) is upward, then $y(v) := y(w_q)$. If (w_p, w_q) is horizontal, then $y(v) := y(w_p) + 1$. If (w_p, w_q) is downward, then $y(v) := y(w_p)$.

Lemma 4 *Algorithm B produces a correct grid drawing, and the height of this drawing is less than 4 times its width, that is* $y(v_n) < 4x(v_2)$.

Lemma 5 *If G is a n-vertex plane graph, then Algorithm B computes a grid drawing of G of width at most* $\lfloor 2(n-1)/3 \rfloor$.

Theorem 3 *Given a maximal n-vertex plane graph G, Algorithm B constructs a grid drawing of G into a* $\lfloor 2(n-1)/3 \rfloor \times 4\lfloor 2(n-1)/3 \rfloor - 1$ *grid.*

Theorem 4 *Algorithm B can be implemented in linear time.*

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