Using Abstractions for the Verification of Linear Hybrid Systems *

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1 Introduction

Hybrid systems are dynamical systems consisting of interacting discrete and continuous components [NSY91, MMP91]. They are used to model the combined behavior of embedded real-time systems and their physical environments. Recently, there have been attempts to develop verification methods for hybrid systems by working in two complementary directions:

- The first direction concerns the identification of subclasses of hybrid systems for which there exist decidability results and effective verification methods for various classes of properties. The main decidability results concern very restricted classes of hybrid systems like timed automata [AD90, ACD90, Alu91, NSY91] and integration graphs [KPSY93].
- The second direction concerns the elaboration of a general verification methodology for classes of systems for which tractable semi-decision methods are applicable. One interesting class is linear hybrid systems [ACHH93, NOSY93], i.e., systems where conditions on discrete transitions and evolution laws of continuous variables are linear constraints.

This paper deals with the verification of linear hybrid systems by bringing results in the two mentioned directions:

- It identifies a class of linear hybrid systems which are shown to be bisimilar to timed automata for which the satisfaction of TCTL properties can be decided by model-checking.
- It studies transformations of general linear hybrid systems inducing abstractions which are timed automata. The properties expressed in VTCTL, a fragment of TCTL, are preserved on these systems.

The paper is organized as follows:

Section 2 presents the model of Linear Hybrid Systems (LHS) where continuous variables evolve according to linear functions with slopes varying within an interval. We define two sub-classes of LHS: Constant Slope Hybrid Systems (CSHS) [KPSY93], where the slopes of the evolution functions are constant and K -Timed Graphs (K -TG) that are CSHS where for all variables the slope is equal to a constant K .

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In Sect. 3 we review the syntax and the semantics of the logic TCTL used to describe system properties.

In Sect. 4 we give a simple transformation that maps the sub-class of CSHS with non-zero slopes into equivalent timed graphs in the sense that the underlying models are bisimilar. This implies in particular, that the initial and the transformed system satisfy the same TCTL formulas modulo a linear time transformation. These results are used to show that satisfaction of TCTL is decidable for transformable CSHS.

In Sect. 5 we show that the results of Sect. 4 cannot be extended to general LHS. We propose an extension of the transformation of Sect. 4 which applied to a LHS yields a timed graph which is an abstraction of it. Then we prove that the validity of $\forall$ TCTL formulas is preserved for timed graphs that are abstractions of LHS.

2 Linear Hybrid Systems

2.1 Syntax

Let X be a set of *variables* ranging over the set $\mathbb{R}$ of real numbers. A *valuation* X_0 of X assigns to each variable $x \in X$ a real number $x_0 \in \mathbb{R}$. A *linear predicate* over X is a boolean combination of linear inequalities with integer coefficients over X .

A *linear hybrid system* (LHS) H is a structure $\langle S, X, E, A, B, \phi \rangle$ where:

- S is a finite set of *locations*. In a graphical representation of the system they are drawn as the nodes of the graph.
- X is a finite set of real-valued *variables*. These variables change *continuously* at locations and *discretely* via transitions.
- E is a finite set of *edges*. Each edge $e \in E$ is a tuple $\langle s, a, \psi, v, s' \rangle$, where $s \in S$ is the *source*, $s' \in S$ is the *target*, a is the *label*, ψ is a linear predicate called the *guard*, and v is an *assignment* represented as a set $\{x := v_x \mid x \in Y \subseteq X\}$ where v_x is an interval with end-points in $\mathbb{Z} \cup \{-\infty, +\infty\}$. In the graphical representation, each $e \in E$ is drawn as an edge from the source location to the target location with the label (a, ψ, v) .
- A and B associate to each location $s \in S$ the functions A_s and B_s from X to $\mathbb{Z}$ such that $A_s \leq B_s$. The rate of change of the variable x at the location s belongs to the closed interval $[A_s(x), B_s(x)]$.
- ϕ associates with each location $s \in S$ a linear predicate ϕ_s over X called the *invariant* at s .

A *Constant Slope Hybrid System* (CSHS) [KPSY93] is a linear hybrid system such that $A = B$. That is, any variable x changes continuously at a constant rate $A_s(x)$ at s . We denote a CSHS as a tuple $H = \langle S, X, E, A, \phi \rangle$ rather than $H = \langle S, X, E, A, A, \phi \rangle$.

A *K-Timed Graph* (K-TG) is a constant slope hybrid system where all variables are nonnegative and change *uniformly* at the same rate $K > 0$, that is $A_s(x) = K$. Guards are boolean combinations of inequalities of the form

$l \prec x \prec u$ where $\prec \in \{<, \leq\}$ and $l \in \mathbb{N}$ and $u \in \mathbb{N} \cup \{+\infty\}$. We denote a K -TG as a tuple $H = \langle S, X, E, \mathbf{K}, \phi \rangle$.

A *1-Timed Graph* differs from a timed graph [ACD90, Alu91, NSY91] in the sense that assignments are intervals rather than resets to zero. 1-Timed Graphs are an extension of timed graphs for which all the decidability and model-checking results concerning timed graphs can also be applied. For this reason 1-timed graphs are hereafter simply called timed graphs.

Notation: If A and B are tuples of scalars and c is a scalar, we write:

- $A(i)$ to represent the i -th component of A .
- $\frac{c}{A}$ to represent the tuple with i -th component $\frac{c}{A(i)}$.
- $A \prec B$ if $A(i) \prec B(i)$ for any component and $\prec \in \{<, \leq\}$.

2.2 Semantics

Let $H = \langle S, X, E, A, B, \phi \rangle$ be a linear hybrid system. A *state* of H is a pair (s, X_0) such that $\phi_s(X_0)$. We denote by Q the set of states of H . For $q = (s, X_0) \in Q$, $q(x)$ is the value x_0 of x at the valuation X_0 .

H changes its state either by a *discrete* and *instantaneous* move through an edge, or by a *continuous* transformation of its variables while time elapses. During the continuous transformations at location s we require that the associated invariant ϕ_s continuously holds. The semantics of H is a labeled transition system $\langle Q, \rightarrow \rangle$ where the transition relation $\rightarrow$ between states is defined by the following rules:

- An edge $\langle s_0, a, \psi, v, s_1 \rangle \in E$ is *enabled* at a state (s_0, X_0) if X_0 satisfies the guard ψ . Whenever an edge is enabled at a state it may be taken, and the resulting state is a pair (s_1, X_1) where X_1 is such that for every variable $x \in X$, $x_1 \in v_x$ if $x := v_x \in v$, and $x_1 = x_0$ otherwise. We denote by $v(X_0)$ the set of such valuations.

$$\frac{\langle s_0, a, \psi, v, s_1 \rangle \in E, \psi(X_0), X_1 \in v(X_0)}{(s_0, X_0) \xrightarrow{a} (s_1, X_1)}$$

- Time can progress by t at (s, X_0) , if there is a valuation X_1 reached from X_0 for a rate $\Lambda \in [A_s, B_s]$ and all the intermediate states satisfy the invariant associated to s .

$$\frac{X_1 = X_0 + \Lambda t, \Lambda \in [A_s, B_s], \forall 0 \leq t' \leq t. \phi_s(X_0 + \Lambda t')}{(s, X_0) \xrightarrow{t} (s, X_1)}$$

We introduce the following definitions:

A *trajectory* σ is an infinite sequence $q_0 \xrightarrow{t_0} q_1 \xrightarrow{t_1} \dots$ where for all $i \geq 0$ $q_i \xrightarrow{t_i} q_{i+1}$ is such that $q_i \xrightarrow{t_i} q_{i+1}$, or there exists a label a such that $q_i \xrightarrow{a} q_{i+1}$ and $t_i = 0$. A *position* π of σ is a pair (i, t) with $0 \leq t \leq t_i$. We define a total order $\preceq$ on the positions, as $(i, t) \preceq (j, t')$ iff $i < j$ or $i = j \wedge t < t'$. Let $q_i = (s_i, X_{0i})$ and $q_{i+1} = (s_i, X_{0i+1})$ where $X_{0i+1} = X_{0i} + \Lambda_i \cdot t_i$, with $\Lambda_i \in [A_{s_i}, B_{s_i}]$. The *state at position* (i, t) of σ is $\sigma(i, t) = (s_i, X_{0i} + \Lambda_i \cdot t)$. The *time elapsed at position*

(i, t) is $\tau_\sigma(i, t) = t + \sum_{j < i} t_j$, that is, the time elapsed since the beginning of the trajectory. A trajectory σ *diverges* if for all $t \in \mathbb{R}^+$ there exists a position π such that $\tau_\sigma(\pi) > t$. The set of all divergent trajectories is represented by Σ , and the set of all divergent trajectories starting at q by $\Sigma(q)$.

Remark. A different semantics for LHS is proposed in [AHH93] for which the results developed in the paper are also valid.

Example 1. The temperature of a tank grows linearly with time at rates belonging to the interval $[4, 6]$. For safety reasons the temperature must be kept within a minimum of 79 and a maximum of 90 degrees. A controller cools the tank by moving two independent rods. The rates of cooling belong to the intervals $[-4, -1]$ and $[-3, -2]$, respectively. Furthermore, each rod can be used again only if a given time has elapsed since the end of its previous movement. These minimum times are 7 and 5 minutes, respectively. If the temperature cannot be decreased because there is no available rod, a complete shutdown is required. A variant of this example is given in [JLHM91, NOSY93]. Figure 1 shows a linear hybrid system for the controller.

We use three variables: θ measures the temperature of the tank and x_1 and x_2 count the time elapsed since the last use of the respective rod. Initially, the temperature takes its minimum value and both rods can be used.

The equations at locations specify the evolution laws and the predicate specifies the invariant for the location.

Fig. 1. Temperature control system

3 The Logic TCTL

The real-time temporal logic TCTL is a real-time extension of CTL [CES86]. Many versions of TCTL have been defined [ACD90, Alu91, HNSY92]. We adopt here the one defined in [ACD90] using the temporal operators $\exists\mathcal{U}$ and $\forall\mathcal{U}$ of CTL with subscripts that limit their scope in time.

Intuitively, $f_1 \exists\mathcal{U}_{\#n} f_2$ means that there exists a trajectory such that f_2 holds at some state at a time t satisfying $t \# n$ and f_1 continuously holds at all intermediate states. $f_1 \forall\mathcal{U}_{\#n} f_2$ means that for every trajectory the above property holds. For example, we can write $\text{true} \exists\mathcal{U}_{<5} p$ to say that along some trajectory, p becomes true before time 5.

3.1 Syntax

Let P be a finite set of atomic propositions. The formulas of TCTL are defined by the following grammar:

$$f ::= p \mid l < x < u \mid \neg f \mid f_1 \vee f_2 \mid f_1 \exists\mathcal{U}_{\#n} f_2 \mid f_1 \forall\mathcal{U}_{\#n} f_2$$

where $p \in P$, $x \in X$, $l, u \in \mathbb{Z} \cup \{-\infty, +\infty\}$, $< \in \{<, \leq\}$, $n \in \mathbb{N}$ and $\# \in \{<, \leq, >, \geq, =\}$.

We use standard abbreviations such as $\forall\Diamond_{\#n} f$ for $\text{true} \forall\mathcal{U}_{\#n} f$, $\exists\Diamond_{\#n} f$ for $\text{true} \exists\mathcal{U}_{\#n} f$, $\Box_{\#n} f$ for $\neg\forall\Diamond_{\#n} \neg f$ and $\Diamond_{\#n} f$ for $\neg\Box_{\#n} \neg f$. In Sect. 5 we use the abbreviation $f_1 \forall\mathcal{W}_{\#n} f_2$ for the formula $\neg(\neg f_1 \exists\mathcal{U}_{\#n} \neg f_2)$. The unrestricted temporal operators of CTL correspond to TCTL operators subscripted by ≥ 0 .

3.2 Semantics

TCTL formulas are interpreted over the divergent trajectories generated by the model of a linear hybrid system. Let H be a LHS, $\langle Q, \rightarrow \rangle$ its model, and M an interpretation of atomic propositions over the set of states, i.e. $M(p) \subseteq Q$ for all $p \in P$.

The satisfaction relation *the state $q \in Q$ satisfies the formula f under the interpretation M* , denoted $q \models_M f$, is inductively defined as follows:

- $q \models_M p$ iff $q \in M(p)$;
- $q \models_M l < x < u$ iff $l < q(x) < u$;
- $q \models_M \neg f$ iff $q \not\models_M f$;
- $q \models_M f_1 \vee f_2$ iff $q \models_M f_1$ or $q \models_M f_2$;
- $q \models_M f_1 \exists\mathcal{U}_{\#n} f_2$ iff for some sequence $\sigma \in \Sigma(q)$ there exists a position π such that $\sigma(\pi) \models_M f_2$ and $\tau_\sigma(\pi) \# n$, and for all $\pi' \leq \pi$, $\sigma(\pi') \models_M f_1 \vee f_2 \wedge \tau_\sigma(\pi') \# n$;
- $q \models_M f_1 \forall\mathcal{U}_{\#n} f_2$ iff for all sequences $\sigma \in \Sigma(q)$ there exists a position π such that $\sigma(\pi) \models_M f_2$ and $\tau_\sigma(\pi) \# n$, and for all $\pi' \leq \pi$, $\sigma(\pi') \models_M f_1 \vee f_2 \wedge \tau_\sigma(\pi') \# n$;

H satisfies f under M , denoted $H \models_M f$, if all the states of H satisfy f .

Example 2. We express in TCTL three properties of Example 1:

- The location *Shutdown* is not reachable:

$$(Heating \wedge \theta = 79 \wedge x_1 = 7 \wedge x_2 = 5) \Rightarrow \forall \Box \neg Shutdown \quad (1)$$

- Rod 1 (Rod 2) can be used again only if at least 7 minutes (5 minutes) have elapsed since the end of its previous movement:

$$(Heating \wedge \theta = 79 \wedge x_1 = 0) \Rightarrow \neg \exists \Diamond_{<7} Cooling_1 \quad (2)$$

$$(Heating \wedge \theta = 79 \wedge x_2 = 0) \Rightarrow \neg \exists \Diamond_{<5} Cooling_2 \quad (3)$$

4 From Constant Slope Hybrid Systems to Timed Graphs

We define the class of Non-Zero Constant Slope Hybrid System, denoted $CSHS_{\neq 0}$, such that:

- All the rates are different from zero.
- The invariant and guards are boolean combinations of inequalities of the form $l < x < u$ with $< \in \{<, \leq\}$.
- For all edges $e = \langle s, a, \psi, v, s' \rangle \in E$ and for all variables $x \in X$, if the rates of change of x are different at s and s' , then $x := v_x$ is in v .
- At any location s any variable with positive (negative) slope has a finite lower (upper) bound in the invariant ϕ_s .

In this section we show how a given $CSHS_{\neq 0}$ can be transformed into an equivalent TG. To this end, we use two transformations: the K -transformation and the $\frac{1}{K}$ -transformation, whose successive application to a $CSHS_{\neq 0}$ gives a K -timed graph and a timed graph. The key idea is to apply linear transformations of variables which are proper to each location. For the K -transformation the aim is to obtain the same evolution laws for all variables and locations by multiplying by appropriate constants. Consequently, conditions and assignments are modified. The $\frac{1}{K}$ -transformation consists in changing the time scale. Both transformations preserve bisimilarity of the underlying models.

4.1 Translating $CSHS_{\neq 0}$ into K -TG

The K -transformation. Given $H = \langle S, X, E, A, \phi \rangle$ in $CSHS_{\neq 0}$, we define the set of functions $\kappa = \{\kappa_s \mid s \in S\}$ as follows:

$$\kappa_s(X) = \frac{K}{A_s} X + C_s$$

where:

- $K = \text{lcm}\{\text{abs}(A_s(x)) \mid s \in S, x \in X\}$, that is, K is the least common multiple of the absolute values of the rates appearing in H , and
- C_s is a tuple such that $C_s(x)$ is the maximum of the absolute values of all the constants appearing in assignments to x in edges with target location s or in conditions on x in guards of edges with source location s , multiplied by $\text{abs}(\frac{K}{A_s(x)})$.

For given $H = \langle S, X, E, A, \phi \rangle$ in $\text{CSHS}_{\neq 0}$, we construct a K -timed graph $H_K = \langle S, X', E', K, \phi' \rangle$ such that:

- X' is in bijection with X .
- For each $e = \langle s, a, \psi, v, s' \rangle \in E$ we construct the edge $e' = \langle s, a, \psi', v', s' \rangle \in E'$ where ψ' is obtained replacing each condition of the form $l \prec x \prec u$ appearing in ψ by a condition $l' = \frac{K}{A_s(x)}l + C_s(x) \prec x' \prec \frac{K}{A_s(x)}u + C_s(x) = u'$ when the rate $A_s(x)$ is positive, or $u' \prec x' \prec l'$ otherwise. The assignment v' is obtained replacing each $x := [\alpha, \beta] \in v$ by $x' := [\frac{K}{A_s(x)}\alpha + C_s(x), \frac{K}{A_s(x)}\beta + C_s(x)] = [\alpha', \beta']$ when the rate $A_s(x)$ is positive, or $x' := [\beta', \alpha']$ otherwise.
- ϕ'_s is obtained from ϕ_s exactly as ψ' is obtained from ψ .

Remark. When the slope of x is positive (negative) and l or α (u or β) are $-\infty$ (∞) we take the conjunction with the invariant ϕ_s before applying the transformation. This operation does not change the semantics of the system.

Notice that H_K is constructed from H so as the relation $X' = \kappa_s(X)$ holds at each location s .

Property Preservation by K -transformation. We want to prove that H and H_K are bisimilar. We recall here the definition of bisimulation.

Definition 1. Let $T_1 = \langle Q_1, \rightarrow_1 \rangle$ and $T_2 = \langle Q_2, \rightarrow_2 \rangle$ be two labeled transition systems over a set $\mathcal{L}$ of labels and a relation $\rho \subseteq Q_1 \times Q_2$.

The relation ρ is a *bisimulation* between T_1 and T_2 , if $\rho(q_1, q_2)$ implies for every label $\ell \in \mathcal{L}$:

- If $q_1 \xrightarrow{\ell}_1 q'_1$, then there exists q'_2 such that $q_2 \xrightarrow{\ell}_2 q'_2$ and $\rho(q'_1, q'_2)$.
- If $q_2 \xrightarrow{\ell}_2 q'_2$, then there exists q'_1 such that $q_1 \xrightarrow{\ell}_1 q'_1$ and $\rho(q'_1, q'_2)$.

We say that T_1 and T_2 are *bisimilar*, denoted $T_1 \sim T_2$, if there exists a bisimulation ρ between T_1 and T_2 .

For H_1 and H_2 linear hybrid systems, we write $\langle H_1, q_1 \rangle \sim \langle H_2, q_2 \rangle$ if the transition systems generated respectively from q_1 and q_2 are bisimilar. We write $H_1 \sim H_2$ if there exists a bisimulation ρ between the models of H_1 and H_2 .

We naturally extend the function κ to a relation on states in the following way:

$$\kappa((s, X), (s', X')) \text{ iff } s = s' \text{ and } X' = \kappa_s(X)$$

Proposition 2. For all $\text{CSHS}_{\neq 0}$ H , $H \sim H_K$.

Proof. It is straightforward to prove that the relation κ is a bisimulation between H and H_K . $\square$

Our goal is to prove that H and H_K satisfy the same TCTL formulas modulo the transformation of the predicates over X according to κ . So, for any formula f we define a formula $|f|_K$, such that H satisfies f iff H_K satisfies $|f|_K$. The formula $|f|_K$ is obtained by substitution of any predicate of the form $l \prec x \prec u$ occurring in f by $\bigvee_{s \in S} p_s \wedge l' \prec x' \prec u'$, where p_s is an atomic proposition which holds only at location s and $l' \prec x' \prec u'$ is the condition defined in Sect. 4.1.

Proposition 3. For any H in $\text{CSHS}_{\neq 0}$ and for any TCTL formula f :

$$H \models_M f \text{ iff } H_K \models_M f|_K$$

Proof. By induction on the structure of the formula based on the fact that $H \sim H_K$. $\square$

Example (continued). We consider the CSHS H obtained from the LHS of Fig. 1 by taking the rate of θ equal to 6 at location *Heating*, to -4 at location *Cooling₁* and to -3 at location *Cooling₂*. The evolution laws for θ are different at all locations. However, since all conditions are of the form $\theta = c$ (for a constant c) we can add the assignment $\theta := c$ without changing the semantics of the system. Thus, $H \in \text{CSHS}_{\neq 0}$ and the K -transformation is applicable to obtain the K -TG of Fig. 2. The set of functions κ is given by:

$$\begin{aligned} \theta' &= 2\theta + 180 && \text{at location Heating} \\ \theta' &= -3\theta + 270 && \text{at location Cooling}_1 \\ \theta' &= -4\theta + 360 && \text{at location Cooling}_2 \\ x'_1 &= 12x_1 + 84 && \text{at all locations} \\ x'_2 &= 12x_2 + 60 && \text{at all locations} \end{aligned}$$

Fig. 2. Temperature control system (K -TG)

The transformation of formulas (1), (2) and (3) according to $|\dots|_K$ gives:

$$(Heating \wedge \theta' = 338 \wedge x'_1 = 168 \wedge x'_2 = 120) \Rightarrow \forall \square \neg Shutdown \quad (4)$$

$$(Heating \wedge \theta' = 338 \wedge x'_1 = 84) \Rightarrow \neg \exists \diamond_{<7} Cooling_1 \quad (5)$$

$$(Heating \wedge \theta' = 338 \wedge x'_2 = 60) \Rightarrow \neg \exists \diamond_{<5} Cooling_2 \quad (6)$$

By Proposition 3, H satisfies formulas (1), (2) and (3) iff H_K satisfies formulas (4), (5) and (6), respectively.

4.2 Translating K -TG into TG

The problem of verifying TCTL formulas on K -timed graphs can be reduced to the problem of verifying TCTL formulas on timed graphs. Given a K -timed graph H_K a timed graph H_1 can be obtained such that the satisfaction of a TCTL formula f on H_K is equivalent to the satisfaction of a formula $|f|_{\frac{1}{K}}$ obtained by transforming f .

The $\frac{1}{K}$ -transformation. Given a K -timed graph $H_K = \langle S, X, E, \mathbf{K}, \phi \rangle$, this transformation gives a timed graph $H_1 = \langle S, X, E, \mathbf{1}, \phi \rangle$. That is, an evolution law of the form $x = x_0 + Kt$ in H_K is replaced by the law $x = x_0 + t$ in H_1 .

Property Preservation by $\frac{1}{K}$ -transformation. Given a formula f , we define a formula $|f|_{\frac{1}{K}}$ by replacing the subscripts $\#n$ by $\#Kn$ in the temporal operators.

Proposition 4. For any K -timed graph H_K and any TCTL formula f :

$$H_K \models_M f \text{ iff } H_1 \models_M |f|_{\frac{1}{K}}$$

Example (continued). The $\frac{1}{K}$ -transformation is applicable to the system H_K of Fig. 2 to obtain a TG H_1 . The transformation of formulas (4), (5) and (6) according to $|\dots|_{\frac{1}{K}}$ gives:

$$(Heating \wedge \theta' = 338 \wedge x'_1 = 168 \wedge x'_2 = 120) \Rightarrow \forall \square \neg Shutdown \quad (7)$$

$$(Heating \wedge \theta' = 338 \wedge x'_1 = 84) \Rightarrow \neg \exists \Diamond_{<84} Cooling_1 \quad (8)$$

$$(Heating \wedge \theta' = 338 \wedge x'_2 = 60) \Rightarrow \neg \exists \Diamond_{<60} Cooling_2 \quad (9)$$

By Proposition 4 H_K satisfies formulas (4), (5) and (6) iff H_1 satisfies formulas (7), (8), and (9), respectively.

The following results were obtained using the tool KRONOS [NSY92, OY92] to verify the formulas (7), (8) and (9) on the timed graph H_1 .

formula	time in sec.	iterations	result
(7)	0.050	4	true
(8)	0.066	4	true
(9)	0.083	4	true

Then H_K satisfies formulas (4), (5) and (6) and by Proposition 3 the CSHS $_{\neq 0}$ H satisfies formulas (1), (2) and (3).

Synthesis of the results. By combination of Propositions 3 and 4, we obtain:

Proposition 5. *For any CSHS_{≠0} H and any TCTL formula f :*

$$H \models_M f \text{ iff } H_K \models_M |f|_K \text{ iff } H_1 \models_M ||f|_K|_{\frac{1}{K}}$$

This proposition shows that the model-checking problem for CSHS_{≠0} and TCTL is reduced to the problem of model-checking for TG. Now, the model-checking algorithm in [HNSY92] can be adapted for the timed graphs considered here, that is timed graphs having assignments of intervals instead of resets to zero. It can be shown that in this case the meaning of the formulas can be expressed with the same class of state predicates.

5 From Linear Hybrid Systems to Timed Graphs

In this section we extend the translation method proposed for CSHS to LHS. We first define a subclass of LHS, denoted LHS_{≠0}, such that:

- For all locations $s \in S$ and variables $x \in X$, the end-points of the rate interval $A_s(x)$ and $B_s(x)$ are both positive or both negative, that is $0 \notin [A_s(x), B_s(x)]$.
- The invariant and the guards are boolean combinations of inequalities of the form $l < x < u$ with $< \in \{<, \leq\}$.
- For all edges $e \in E$ and for all variables $x \in X$, if the rates of change of x are different at the source location and at the target location, then x must be in the assignment.
- At any location s any variable with positive (negative) slope has a finite lower (upper) bound in the invariant ϕ_s .

5.1 Translating LHS_{≠0} into K-TG

The following proposition shows that unfortunately the result that for any H in CSHS_{≠0} there exist a bisimilar K -TG cannot be extended to LHS_{≠0}.

Proposition 6. *Given a $H = \langle S, X, E, A, B, \phi \rangle$ in LHS_{≠0} such that for some $s \in S$*

- $A_s < B_s$
- *there exists $e \in E$ such that $e = \langle s, a, [L, U], v, s' \rangle$*

there exists no H_K in K -TG with initial state q_K such that $\langle H, q \rangle \sim \langle H_K, q_K \rangle$ for all the states q from which there exists a trajectory crossing the edge e .

The Generalized K -transformation. Proposition 6 states that it is not possible (in general) to transform a given LHS_{≠0} into a bisimilar K -TG. However, we define a transformation of H into an abstract H' , in the sense that every transition in the model of H corresponds to a transition in the model of H' (the model of H simulates the model of H').

Given $H = \langle S, X, E, A, B, \phi \rangle$ in $\text{LHS}_{\neq 0}$, and a set of variables X' in bijection with X , we define the relation κ as the set $\kappa = \{\kappa_s \mid s \in S\}$ relating X and X' as follows:

$$\kappa_s(X, X') \text{ iff } \frac{K}{B_s}X + C_s \leq X' \leq \frac{K}{A_s}X + C_s,$$

where:

- $K = \text{lcm}\{\text{abs}(A_s(x)), \text{abs}(B_s(x)) \mid x \in X, s \in S\}$, that is K is the least common multiple of the absolute values of all different end-points of the rate intervals.
- C_s is a tuple such that $C_s(x)$ is the maximum of the absolute values of all the constants appearing in assignments to x in edges with target location s or in conditions on x in guards of edges with source location s , multiplied by $\max(\text{abs}(\frac{K}{A_s(x)}), \text{abs}(\frac{K}{B_s(x)}))$.

From H and κ the K -timed graph $H_K = \langle S, X', E', \mathbf{K}, \phi' \rangle$ is defined as follows:

- X' is in bijection with X .
- For each $e = \langle s, a, \psi, v, s' \rangle \in E$ we construct the edge $e' = \langle s, a, \psi', v', s' \rangle \in E'$ where ψ' is obtained replacing each condition of the form $l \prec x \prec u$ appearing in ψ by a condition $l' = \frac{K}{B_s(x)}l + C_s(x) \prec x' \prec \frac{K}{A_s(x)}u + C_s(x) = u'$ when $A_s(x)$ and $B_s(x)$ are positive, or $u' \prec x' \prec l'$ otherwise. The assignment v' is obtained replacing each $x := [\alpha, \beta] \in v$ by $x' := [\frac{K}{B_s(x)}\alpha + C_s(x), \frac{K}{A_s(x)}\beta + C_s(x)] = [\alpha', \beta']$ when $A_s(x)$ and $B_s(x)$ are positive, or $x' := [\beta', \alpha']$ otherwise.
- ϕ'_s is obtained from ϕ_s exactly as ψ' is obtained from ψ .

The remark of Sect. 4.1 is also applicable in this case. Is important to note that when $A = B$, H is a $\text{CSHS}_{\neq 0}$ and the generalized K -transformation coincides with the K -transformation defined in Sect. 4.1.

Simulation. We want to prove that H simulates H_K . We first recall the definition of simulation.

Definition 7. Let $T_1 = \langle Q_1, \rightarrow_1 \rangle$ and $T_2 = \langle Q_2, \rightarrow_2 \rangle$ be two labeled transition systems over a set $\mathcal{L}$ of labels and a relation $\rho \subseteq Q_1 \times Q_2$.

The relation ρ is a *simulation* from T_1 to T_2 , if $\rho(q_1, q_2)$ implies for every label $\ell \in \mathcal{L}$:

- If $q_1 \xrightarrow{\ell} q'_1$, then there exists q'_2 such that $q_2 \xrightarrow{\ell} q'_2$ and $\rho(q'_1, q'_2)$.

We say that T_1 *simulates* T_2 , denoted $T_1 \sqsubseteq T_2$, if there exists a relation ρ such that ρ is a simulation from T_1 to T_2 .

For H_1 and H_2 linear hybrid systems, we write $H_1 \sqsubseteq H_2$ if there exists a simulation ρ from the model of H_1 to the model of H_2 .

We naturally extend the relation κ on states in the following way:

$$\kappa((s, X), (s', X')) \text{ iff } s = s' \text{ and } \kappa_s(X, X')$$

Lemma 8. For all $\text{LHS}_{\neq 0}$ H , $H \sim_{\kappa} H_K$ if only timed transitions are considered.

Lemma 9. For all $LHS_{\neq 0} H$, $H \sqsubseteq H_K$ if only discrete transitions are considered.

Proposition 10. For all $LHS_{\neq 0} H$, $H \sqsubseteq H_K$.

Proof. By combining Lemmas 8 and 9. □

Property Preservation by K -transformation. We show that an interesting subset of TCTL, called $\forall TCTL$, is preserved by the relation K from H_K to H , that is, if $f \in \forall TCTL$ is satisfied by H_K then H satisfies a corresponding formula $|f|_K \in \forall TCTL$. $\forall TCTL$ is the subset of TCTL defined as follows:

$$f ::= p \mid \neg p \mid l \prec x \prec u \mid f_1 \wedge f_2 \mid f_1 \vee f_2 \mid f_1 \forall W_{\#n} f_2 \mid f_1 \forall U_{\prec n} f_2$$

where $f_1 \forall W_{\#n} f_2$ and $p, x, l, u, \prec, n$ and $\#$ are as in definition of Sect. 3.1.

For any formula f in $\forall TCTL$ the formula $|f|_K$ is obtained by replacing any predicate of the form $l \prec x \prec u$ by $\frac{l - C_s(x)}{K} A_s(x) \prec x' \prec \frac{u - C_s(x)}{K} B_s(x)$.

Proposition 11. For any $LHS_{\neq 0} H$ and any $\forall TCTL$ formula f ,

$$\text{if } H_K \models_M f \text{ then } H \models_M |f|_K.$$

Example 3. Consider the example of Sect. 1 modeled by the LHS H of Fig. 1. The relation κ for H is:

$$\begin{aligned} 2\theta + 270 &\leq \theta' \leq 3\theta + 270 && \text{at location Heating} \\ -12\theta + 1080 &\leq \theta' \leq -3\theta + 1080 && \text{at location Cooling}_1 \\ -6\theta + 540 &\leq \theta' \leq -4\theta + 540 && \text{at location Cooling}_2 \\ x'_1 &= 12x_1 + 84 && \text{at all locations} \\ x'_2 &= 12x_2 + 60 && \text{at all locations} \end{aligned}$$

Figure 3 shows the K -TG H_K obtained. The $\frac{1}{K}$ -transformation is applicable to the system H_K to obtain a TG H_1 . Consider the following formulas:

$$(Heating \wedge 428 \leq \theta' \leq 507 \wedge x'_1 = 168 \wedge x'_2 = 120) \Rightarrow \forall \square \neg Shutdown \quad (10)$$

$$(Heating \wedge 428 \leq \theta' \leq 507 \wedge x'_1 = 84) \Rightarrow \neg \exists \Diamond_{<84} Cooling_1 \quad (11)$$

$$(Heating \wedge 428 \leq \theta' \leq 507 \wedge x'_2 = 60) \Rightarrow \neg \exists \Diamond_{<60} Cooling_2 \quad (12)$$

It can be shown that if H_1 satisfies formulas (10), (11) and (12) then the LHS $\neq 0$ H of Fig. 1 satisfies formulas (1), (2) and (3).

The following results were obtained using the tool KRONOS to verify the formulas (10), (11) and (12) on the TG H_1 .

formula	time in sec.	iterations	result
(10)	0.084	7	false
(11)	0.117	4	true
(12)	0.084	4	true

Notice that formulas (11) and (12) are verified which implies that formulas (2) and (3) are satisfied by the system of Fig. 1. On the contrary, formula (10) is not verified and nothing can be said about the validity of formula (1).

Fig. 3. Temperature control system (H_K)

6 Conclusion

As mentioned in the introduction, the paper contributes in two different directions to the development of a verification methodology for LHS:

- it shows that the model checking method for TCTL on timed graphs can be extended to a non-trivial class of CSHS, the so called Non-Zero Constant Slope Hybrid Systems $CSHS_{\neq 0}$. Even though the latter class is semantically equivalent to timed graphs, we noticed that several examples of LHS in the literature happen to belong to $CSHS_{\neq 0}$.
- it shows that well-known results of property preservation by abstractions can be adapted to the case of $LHS_{\neq 0}$. The generalized K -transformation presented in Sect. 5 maps this class to timed graphs and, consequently, the verification of a large class of properties such as invariance and bounded inevitability, on some element of $LHS_{\neq 0}$, can be reduced to the verification of the same property on the corresponding timed graph. It remains to evaluate through examples the usefulness of this approach, that is how much faithful are the abstractions defined by K .

These results should be related to the results of [PV94] which consider almost the same class of LHS by using two different approaches. The restriction to have assignments for variables whose evolution law changes is essential for both. The differences concern our requirement that at some position the sign of the rate of the variables cannot change while we accept also open intervals for guards and assignments. The restriction to closed intervals is essential for the decidability result presented in [PV94]. This result is certainly an important one and stronger than our results if they are restricted to decidability of the reachability problem. However, a general comparison is difficult for at least two reasons. The result

in [PV94] is obtained by digitization which means that it is hardly exploitable in practice and it concerns only reachability properties; it is not obvious that it can be extended to other properties, for instance inevitability properties.

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