Generating BDDs for Symbolic Model Checking in CCS

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Abstract Finite transition systems can easily be represented by binary decision diagrams (BDDs) through the characteristic function of the transition relation. Burch et al. have shown how model checking of a powerful version of the μ -calculus can be performed on such BDDs.

In this paper we show how a BDD can be generated from elementary finite transition systems given as BDDs by applying the CCS operations of parallel composition, restriction, and relabelling. The resulting BDDs only grow linearly in the number of parallel components. This way bisimilarity checking can be performed for processes out of the reach of conventional process algebra tools.

1 Introduction

A binary decision diagram as described by Bryant [1] is a normal form representation of boolean functions $f : \mathcal{B}^n \rightarrow \mathcal{B}$, where $\mathcal{B} = \{0, 1\}$. It is often much smaller than other normal form representations. Moreover boolean operations can be applied efficiently.

Transition systems have states and transitions leading from one state via some action to another state. They underly many semantics for concurrent systems, e.g. bisimulation equivalence for CCS [10]. They also underly semantics of modal logics, which allow to reason about concurrent systems.

Burch, Clarke, McMillan, Dill, and Hwang [2] have shown how model checking of a powerful version of the μ -calculus including existential quantification and λ -abstraction can be performed for a boolean domain using BDDs for internal representation. They call the approach *symbolic model checking*. They also indicate how other finite domains can be treated through binary encoding.

In particular for a finite transition system the set of states S and the set Act of actions can be encoded as subsets of $\mathcal{B}^{\lceil \log_2 |S| \rceil}$ and $\mathcal{B}^{\lceil \log_2 |Act| \rceil}$ respectively. The transition relation $D \subseteq S \times Act \times S$ can be encoded as a function $\chi : \mathcal{B}^{2 \cdot \lceil \log_2 |S| \rceil + \lceil \log_2 |Act| \rceil} \rightarrow \mathcal{B}$ which in turn can be represented as a BDD. Burch et al. show how weak and strong bisimilarity can be expressed as formulas in their μ -calculus and hence can be checked for transition systems given as BDDs.

However they do not indicate how the BDDs are generated. If they were generated from the list of state-action-state triples of the transition relation (which is straightforward) the approach would immediately suffer the well-known explosion problem, namely that a system of N parallel components, with n states each, may have n^N transitions.

The question arises whether the BDD which represents the transition system of the compound system could be generated more efficiently directly from the N components given as BDDs without enumerating all resulting transitions. This question is the topic of this paper. It is answered in the following way: We give a certain encoding of transition systems as BDDs. We show how the CCS [10] operators of parallel composition, restriction, and relabelling can be applied to BDDs. We prove that for a fixed set of actions the BDD for representing N parallel communicating processes grows only linearly in N . Worst case boundaries for the size of BDDs are rare in literature, hence this bound is interesting. However it has to be pointed out that it concerns only the resulting BDD which represents the system of N parallel component processes. It does not concern intermediate BDDs which are used during the generation or during the model checking. Nevertheless a benchmark example shows that bisimilarity checking can be performed for systems out of the reach of conventional tools such as [3, 4, 7, 9, 6].

Note that the μ -calculus of [2] is more powerful than that of e.g. [5, 11]. Hence all formulas checked with the latter approaches can be checked in the framework of this paper. We expect to be able to handle larger transition systems.

On the other hand it should be noted that BDDs are efficient only in a heuristic sense, the worst case may still be catastrophic. For circuit verification experience has shown that indeed BDDs can serve as an efficient representation in many practical cases. This paper gives evidence that this is also the case for verification of parallel processes. Possible applications are communication protocols, operating system tasks, and distributed control systems.

Before we start with the technical part of this paper let us explain with an example the basic idea, i.e. how to represent relations by BDDs and why this is promising for the parallel composition of transition systems. Consider the relation D of Fig. 1 where $D \subseteq S \times S$ for $S = \{0, 1, 2, 3\}$. We omit the actions for the moment. An obvious boolean encoding of S is $0 \mapsto 00, 1 \mapsto 01, 2 \mapsto 10, 3 \mapsto 11$. The relation D may then be represented by a function $\chi : B^4 \rightarrow B$, such that $\chi(r_1, r_2, s_1, s_2) = 1$ if and only if the state encoded by $r_1 r_2$ has an edge to the state encoded by $s_1 s_2$. This boolean function χ can in turn be represented as a BDD.

Figure 1: A relation (left) and its BDD representation (right)

For a given set of boolean variables which are totally ordered by $<$ a BDD is a rooted directed acyclic graph, each node is either terminal, then it is labelled by a truthvalue and has no successor, or it is nonterminal, then it is labelled by some variable and has two successors which are terminal or labelled by a larger ($<$) boolean variable. One successor corresponds to 0, the other to 1.

Taking the ordering $r_1 < r_2 < s_1 < s_2$ the function χ is represented by the BDD B given in Fig. 1.

Now assume we want to represent the relation¹

$$\{ \langle \langle r, r' \rangle, \langle s, s' \rangle \rangle \in S \times S \mid \langle r, s \rangle \in D \wedge \langle r', s' \rangle \in D \}$$

as a BDD C . We can take a copy B' of B with fresh variables r'_1, r'_2, s'_1, s'_2 and calculate C as $B \wedge B'$ where the conjunction of BDDs is as described in [1].

However the size of C depends on the chosen variable ordering. If $r_1 < r_2 < s_1 < s_2 < r'_1 < r'_2 < s'_1 < s'_2$ is chosen it has 20 nodes, approximately $2 \cdot |B|$, where $|B|$ denotes the number of nodes in B . It is formed simply by attaching B' to the 1-exit of B . If, on the other hand, $r_1 < r_2 < r'_1 < r'_2 < s_1 < s_2 < s'_1 < s'_2$ is chosen it has 47 nodes (approximately $|B|^2/2$).

This ends our introductory example. We wanted to indicate that if the ordering of variables is chosen carefully the BDD only grows additively while the number of elements in the represented relation grows multiplicatively. However be aware that the encoding as presented in Section 4 uses a different ordering in order to also serve the asynchronous case well.

¹This relation resembles the synchronization part of parallel composition, cf. the next section. In general, of course, one wants to combine two relations and not twice the same.

2 Operators on transition systems

Our language for composing parallel processes is taken from the process algebra CCS [10]. Let Act be the finite set of actions which contains the invisible action τ and all other actions in two copies, a and $\bar{a}$, which are complementary, i.e. $\bar{\bar{a}} = a$. A transition system $T = \langle S, D, z \rangle$ has states S , initial state $z \in S$, and transitions $D \subseteq S \times Act \times S$ leading from one state with some action to another state. Throughout we assume S to be finite.

Next we give operators on transition systems which correspond to CCS parallel composition, restriction, and relabelling. In [12] it is shown that they are correct (consistent) in the following sense. Given two closed CCS terms P_1 and P_2 and let T_1, T_2 be their respective transition systems according to the transitional semantics [10], then $T_1 \mid T_2$ as defined below is strongly bisimilar to the transition system of $P_1 \mid P_2$ according to the transitional semantics. Similar results hold for restriction and relabelling.

The *CCS parallel composition* for given $T_1 = \langle S_1, D_1, z_1 \rangle$ and $T_2 = \langle S_2, D_2, z_2 \rangle$ is defined as $T_1 \mid T_2 = \langle S_1 \times S_2, D, \langle z_1, z_2 \rangle \rangle$ where

$$D = \left\{ \langle \langle r_1, r_2 \rangle, \alpha, \langle s_1, s_2 \rangle \rangle \mid \begin{array}{l} \langle r_1, \alpha, s_1 \rangle \in D_1 \wedge r_2 = s_2 \vee \langle r_2, \alpha, s_2 \rangle \in D_2 \wedge r_1 = s_1 \\ \vee \alpha = \tau \wedge \exists a \neq \tau : \langle r_1, a, s_1 \rangle \in D_1 \wedge \langle r_2, \bar{a}, s_2 \rangle \in D_2 \end{array} \right\}.$$

The condition $\langle r_1, \alpha, s_1 \rangle \in D_1 \wedge r_2 = s_2$ and its symmetric version represent an *asynchronous* move, only one component proceeds. On the other hand the condition $\langle r_1, a, s_1 \rangle \in D_1 \wedge \langle r_2, \bar{a}, s_2 \rangle \in D_2$ represents a *synchronous* move. The latter in a simplified version has already been discussed in the introduction.

Let $T = \langle S, D, z \rangle$ be given. *Restriction* of a subset $A \subseteq Act - \{\tau\}$ is defined as

$$T \backslash A = \langle S, \{ \langle r, \alpha, s \rangle \in D \mid \alpha \notin A \wedge \bar{\alpha} \notin A \}, z \rangle.$$

Relabelling of a visible action a ($a \neq \tau$) into the action b is defined as $T[b/a] = \langle S, D', z \rangle$ where

$$D' = \{ \langle r, \beta, s \rangle \mid \exists \alpha : \langle r, \alpha, s \rangle \in D \wedge (\beta = \alpha \notin \{a, \bar{a}\} \vee \beta = b \wedge \alpha = a \vee \beta = \bar{b} \wedge \alpha = \bar{a}) \}.$$

3 Symbolic model checking using BDDs

Burch, Clarke, McMillan, Dill, and Hwang [2] show how model checking can be performed using BDDs. They call their approach 'symbolic' as it proceeds without supplying an interpretation for the individual variables. The result of model checking a formula is a BDD which represents the result of the formula for all interpretations. Given a particular interpretation taking choices according to the values of the variables under this interpretation leads from the root of the BDD to a truthvalue. This is the truthvalue of the formula under the particular individual variable interpretation. Additionally in [2] BDDs are used to represent relations internally with the intention that this is more efficient than the corresponding lists of tuples.

The version of the μ -calculus used in [2] includes individual variables, n -ary relational variables ($n > 0$), existential quantification, abstraction (λ -binding) of individual variables, and fixpoint binding of relational variables. The reader is referred to [2] for the details of syntax and semantics.

In general, given a structure of a domain and interpretations I_P and I_D of relational and individual variables, model checking performs the task of checking, whether a formula F is true in this structure (i.e. $\mathcal{D}(F)(I_P)(I_D) = 1$), in other words, whether the structure is a model for F .

From a different point of view the model checker of Burch et al. may be seen as a certain approach to work with relations which are represented as BDDs. The actual model checking of a formula F proceeds by first calculating as a BDD the relation which contains a tuple $\langle d_1, \dots, d_n \rangle$ iff F is true for the individual variable interpretation which maps the first variable to $d_1 \dots$ and the last variable to d_n . The model checking is finished (and yields a truthvalue) by instantiating this BDD to a particular individual variable interpretation.

However the interpretation of the relational variables have to be supplied before the symbolic model checker can start. They have to be supplied as BDDs. In particular for model checking CCS-terms

[2, Sect. 8] the transition relations of the transition systems to be checked have to be supplied by the relational variable interpretation, i.e. as BDDs. Burch et al. leave open where the BDDs for the transition relations come from. The following two sections propose an approach for generating these BDDs.

4 Encoding of transition systems as BDDs

In principle given a certain enumeration $e_0, e_1, \dots, e_{|S|-1}$ of a finite set S the boolean encoding is obvious, one needs $\#S := \lceil \log_2 |S| \rceil$ boolean variables and lets $s_1, \dots, s_{\#S}$ denote e_i where i is the number one gets when interpreting $s_1 \dots s_{\#S}$ as a binary digit.

We will use this encoding for the state sets of elementary² transition systems and for the set of actions. For the latter we additionally assume that Act is enumerated $\tau, a, b, \dots, \bar{a}, \bar{b}, \dots$, i.e. that τ is encoded as all zeros and any visible action a as $0x_2 \dots x_{\#A}$ and its complement $\bar{a}$ as $1y_2 \dots y_{\#A}$ such that for all $i \in \{2, \dots, \#A\}$ we have $x_i = y_i$. Here $\#A := \lceil \log_2 |Act| \rceil$.

In order to work with BDDs one has to fix a global ordering on the boolean variables used to encode the information. This ordering has great influence on the size of the BDDs. If nothing is known on the structure of a relation to be represented as a BDD not much can be done. However in our case we can use knowledge about the transition relation and in particular about the operations performed on them, most importantly the parallel composition.

For explanation let us consider a binary relation $D \subseteq S \times S$ as in the introduction. If we need $\#S$ bits to encode S we need $2 \cdot \#S$ bits to encode D , say $r_1, \dots, r_{\#S}$ to encode the first element of a pair in D and $s_1, \dots, s_{\#S}$ for the second. In the introduction we demonstrated for the synchronization case of parallel composition of two such relations that we get small BDDs if all variables of one relation are ordered before those of the other.

However in the asynchronous case where only one component proceeds while the other (say the first) stays in its state we have to check for $\langle (r, r'), (s, s') \rangle$ whether $r = s \wedge \langle r', s' \rangle \in D$, i.e. we have to check whether $r_1 = s_1 \wedge r_2 = s_2 \wedge \dots \wedge r_{\#S} = s_{\#S}$. But with the variable ordering $r_1 \prec r_2 \prec \dots \prec r_{\#S} \prec s_1 \prec \dots \prec s_{\#S}$ for the bits of a relation the BDD for this check explodes.

Figure 2: BDD for $(r_1 = s_1) \wedge (r_2 = s_2) \wedge (r_3 = s_3)$ with poor (left) and good (right) variable ordering

Figure 2 gives an example for $\#S = 3$. In general it has $\Omega(2^{\#S})$ nodes. Therefore we choose a different ordering, namely

$$r_1 \prec s_1 \prec r_2 \prec s_2 \prec \dots \prec r_{\#S} \prec s_{\#S}.$$

²We call a transition system elementary if it is not formed by parallel composition, restriction, or relabelling but is given as a list of transitions.

With this improved ordering the BDD for the above example is as given on the right of Figure 2, i.e. it grows only linearly.

We are now ready to present our encoding for a transition relation $D \subseteq S \times Act \times S$. We choose the variable ordering

$$a_1 \prec a_2 \prec \dots \prec a_{\#A} \prec r_1 \prec s_1 \prec r_2 \prec s_2 \prec \dots \prec r_{\#S} \prec s_{\#S}$$

where $a_1 \dots a_{\#A}$ encodes the action of the transition, $r_1 \dots r_{\#S}$ encodes the source state of the transition, and $s_1 \dots s_{\#S}$ encodes the target state. We always understand that $00 \dots 0$ is the initial state. Fig. 3 gives an example transition system and Fig. 4 shows the representation of its transition relation as a BDD.

Figure 3: A transition system and the encoding of its states and actions

Figure 4: BDD encoding of the transition relation of the system of Fig. 3

The reason to put the bits for the actions above those for the states is that in the worst case this leads to only $|Act|$ r_1 -labelled nodes in the BDD, each corresponding to one action. If the bits were below those of the states the worst case of a_1 -labelled nodes would be $2^{|Act|}$, each corresponding to a subset of Act . However this is an intuitive reason only. As it stands it does not imply that the resulting BDD is smaller.

5 Operators on BDDs

In this section we describe the operators of CCS parallel composition, restriction, and relabelling on BDDs. Note that this comprises the practically most important case of a parallel composition of N processes

$$(P_1 \mid P_2 \mid \dots \mid P_N) \setminus A$$

where the restriction imposes the wanted synchronization.

We assume that the elementary processes (transition systems) are given as BDDs. The BDD for an elementary transition system may easily be formed from the list of triples in the transition relation as follows. Each transition $\langle r, a, s \rangle$ which has encodings $br_1, \dots, br_{\#S}$, $ba_1, \dots, ba_{\#A}$, and $bs_1, \dots, bs_{\#S}$, is considered as one BDD with only the path $a_1 \xrightarrow{ba_1} a_2 \xrightarrow{ba_2} \dots a_{\#A} \xrightarrow{ba_{\#A}} r_1 \xrightarrow{br_1} s_1 \xrightarrow{bs_1} \dots r_{\#S} \xrightarrow{br_{\#S}} s_{\#S} \xrightarrow{bs_{\#S}} 1$ leading to 1. All other branches directly lead to 0. The BDD for the transition relation is calculated by performing the disjunction (cf. [1]) of all these BDDs. In the worst case this yields a BDD with approximately $2^{2 \cdot \#S + \#A}$ nodes.

CCS parallel composition

Let B_1 and B_2 be two BDDs representing transition relations D_1 and D_2 over the same set of actions but over disjoint sets of states encoded as described in the previous section. The BDD $B_1 \mid B_2$ representing the transition relation of the parallel composition is calculated as follows.

Let $a_1, \dots, a_{\#A}$ be the boolean variables for the actions in both, B_1 and B_2 . Let r_i, s_i ($i \in \{1, \dots, \#S_1\}$) and r'_j, s'_j ($j \in \{1, \dots, \#S_2\}$) be the boolean variables for the source and target states of D_1 and D_2 respectively. According to the previous section we have the following ordering dependencies

$$a_1 \prec \dots \prec a_{\#A} \prec r_1 \prec s_1 \prec \dots \prec r_{\#S_1} \prec s_{\#S_1}$$

and

$$a_1 \prec \dots \prec a_{\#A} \prec r'_1 \prec s'_1 \prec \dots \prec r'_{\#S_2} \prec s'_{\#S_2}.$$

We additionally impose that

$$s_{\#S_1} \prec r'_1$$

this leaves the previous orderings and hence B_1, B_2 unchanged.

Note that this choice of combined ordering ensures that the BDD for $B_1 \mid B_2$ again fulfills our encoding convention for transition systems, i.e. we use the first $\#A$ variables for the actions and then alternatingly one bit for the source and one for the target state of the transition.

We now may calculate $B_1 \mid B_2$ as

$$B_1 \wedge Stab_2 \vee B_2 \wedge Stab_1 \vee C$$

where the missing components are explained below. The $\wedge$ and $\vee$ are operations on BDDs as described in [1]. $Stab_1$ is the BDD for $r_1 = s_1 \wedge \dots \wedge r_{\#S_1} = s_{\#S_1}$ (cf. Fig. 2). It corresponds to the condition that the first component stays in its state ($Stab_2$ is analogous). Let E be the BDD calculated as

$$B_1|_{a_1=0} \wedge B_2|_{a_1=1} \vee B_1|_{a_1=1} \wedge B_2|_{a_1=0}.$$

$B_1|_{a_1=0}$ denotes the subgraph of B_1 one gets when restricting the variable a_1 to 0 (see [1] for details) again $\wedge$ and $\vee$ are operations on BDDs. The BDD E expresses the condition that complementary actions match. The last component C of the parallel composition above is calculated as

$$(a_1 = 0) \wedge \dots \wedge (a_{\#A} = 0) \wedge (\exists a_2 \exists a_3 \dots \exists a_{\#A} E).$$

Here $\exists a_i G$ for some BDD G is short for $G|_{a_i=0} \vee G|_{a_i=1}$, i.e. it represents an existential quantification of the boolean variable a_i . The existential quantification of $a_2, \dots, a_{\#A}$ can be implemented directly as a BDD operator. Applied to E it yields a BDD without nodes labelled by variables $a_1, \dots, a_{\#A}$. The first part of C puts this BDD below the encoding of τ .

Restriction

Given a BDD B representing the transition relation of T and a BDD C for the set of actions A (it may be generated straightforward from the list of actions in A) the BDD for the transition relation of $T \setminus A$ is simply $B \wedge \neg(C|_{a_1=0} \vee C|_{a_1=1})$, where $\wedge, \vee, \neg$, and $|_{a_i=b}$ on BDDs are as described in [1].

Relabelling

For B as above let the binary encoding of actions a and b be $ba_1, \dots, ba_{\#A}$ and $bb_1, \dots, bb_{\#A}$ respectively. The BDD for the transition relation of $T[b/a]$ is

$$B \wedge \neg(a_2 = ba_2 \wedge \dots \wedge a_{\#A} = ba_{\#A}) \\ \vee (a_2 = bb_2 \wedge \dots \wedge a_{\#A} = bb_{\#A}) \wedge B|_{a_2=ba_2|a_3=ba_3|\dots|a_{\#A}=ba_{\#A}}.$$

6 Complexity of resulting BDDs

We have the following results on the size of the generated BDDs.

Theorem 6.1 Let $T_i = \langle S_i, D_i, z_i \rangle$ for $i \in \{1, \dots, N\}$ be transition systems where the transition relation is represented as BDD B_i according to Section 4. The number of nodes of the BDD $B := B_1 | B_2 | \dots | B_N$ is

$$O(2^{|Act|} \cdot \sum_{i=1}^N |S_i|^2).$$

Proof We will use $\vec{a}, \vec{r}_i, \vec{s}_i$ as abbreviations such that $\vec{a} = \langle a_1, \dots, a_{\#A} \rangle$ is the variable vector encoding the actions of Act and such that $\vec{r}_i = \langle r_{i,1}, \dots, r_{i,\#S_i} \rangle$ and $\vec{s}_i = \langle s_{i,1}, \dots, s_{i,\#S_i} \rangle$ are the variable vectors encoding source and target state of the transition relation D_i . The BDD B_i ranges over the variables $\vec{a}, \vec{r}_i, \vec{s}_i$ and the BDD B over the variables $\vec{a}, \vec{r}_1, \vec{s}_1, \dots, \vec{r}_N, \vec{s}_N$. The variable ordering is $\vec{a} \prec \vec{r}_1, \vec{s}_1 \prec \dots \prec \vec{r}_N, \vec{s}_N$ where the ordering within $\vec{r}_i, \vec{s}_i$ is as described in Section 4.

For the proof we have a layered view of B . Let us say that all variables for states of transition system T_l belong to level l .

For instances $\vec{ba} \in B^{\#A}$ and $\vec{br}_i, \vec{bs}_i \in B^{\#S_i}$ let σ_l denote the instantiation $\langle \vec{ba}, \vec{br}_1, \vec{bs}_1, \dots, \vec{br}_{l-1}, \vec{bs}_{l-1} \rangle$. Let $\sigma_l(B)$ denote the BDD one gets by instantiating B to σ_l , i.e.

$$\sigma_l(B) := B|_{\vec{a}=\vec{ba}, \vec{r}_1=\vec{br}_1, \vec{s}_1=\vec{bs}_1, \dots, \vec{r}_{l-1}=\vec{br}_{l-1}, \vec{s}_{l-1}=\vec{bs}_{l-1}}.$$

This BDD $\sigma_l(B)$ denotes a boolean function of arity $2 \cdot \sum_{i=l}^N \#S_i$.

To count the nodes of B on level l we determine the number of different such functions one gets by varying σ_l over all instances. This number is called $width(l)$. Since BDDs are a normal form for boolean functions and they are reduced (i.e. have no two isomorphic subgraphs) [1] this number immediately gives an upper bound for the number of nodes in B which are labeled with variables of level l . For $j \in \{1, \dots, \#S_l\}$ this bound is $width(l) \cdot 2^{2 \cdot (j-1)}$ for the label $r_{l,j}$ and $width(l) \cdot 2^{2 \cdot (j-1)+1}$ for the label $s_{l,j}$. Note that we do not make any assumptions about the part of B within level l hence we have to allow the exponential growth in this part. Knowing an upper bound W for $\max_l width(l)$ we get an upper bound for $|B|$, the number of nodes in B .

$$\begin{aligned} |B| &\leq 2^{\#A} + \sum_{l=1}^N \sum_{j=1}^{\#S_l} width(l) \cdot (2^{2 \cdot (j-1)} + 2^{2 \cdot (j-1)+1}) \\ &\leq 2^{\#A} + \sum_{l=1}^N width(l) \cdot 2^{2 \cdot \#S_l} \\ &\leq 2^{\#A} + W \cdot \sum_{l=1}^N 2^{2 \cdot \#S_l} \\ &\leq 2 \cdot |Act| + W \cdot \sum_{l=1}^N 4 \cdot |S_l|^2 \\ &= O(|Act| + W \cdot \sum_{l=1}^N |S_l|^2) \end{aligned}$$

The rest of the proof is given in the appendix. It calculates W to be $4 \cdot 2^{|Act|}$ which in turn proves the stated bound. $\square$

For a fixed set of actions and $n := \max_i |S_i|$ this bound is simply $O(N \cdot n^2)$ which compares favourably to the straightforward worst case bound $O(n^{N+2} \cdot N^2)$ for the number of transitions in $T_1 | \dots | T_N$.

If we know more on the structure of the transition systems we are able to give the following tighter bound. See the appendix for details.

Corollary 6.2 For T_i and B as above let c be the number of sets of visible actions which occur in transitions between any two states of any component, i.e.

$$c := \left| \left\{ A \subseteq \text{Act} \mid \exists j \in \{1, \dots, N\} : \exists r, s \in S_j : A = \{\alpha \mid \alpha \neq \tau \wedge \langle r, \alpha, s \rangle \in D_j\} \right\} \right|.$$

If no component contains a visible self-loop, i.e. if for all $j \in \{1, \dots, N\}$ there exists no $s \in S_j$, $\alpha \in \text{Act} - \{\tau\}$ such that $\langle s, \alpha, s \rangle \in D_j$, then

$$|B| = O\left((c + |\text{Act}|) \cdot \sum_{i=1}^N |S_i|^2\right).$$

□

We expect c to be close to $|\text{Act}|$ in typical cases.

Let us remark that the same bounds are true for any relabelling of the B_i 's and for arbitrary restrictions (as for example in $((B_1 \mid B_2) \setminus A_1 \mid (B_3 \mid B_4) \setminus A_2) \setminus A_3$). In particular this comprises the most important practical case given at the beginning of the previous section.

7 Implementation

We have implemented the described generation of BDDs for elementary transition systems and the operators of CCS parallel composition, restriction, and relabelling. Furthermore we implemented the symbolic model checker of Burch et al. [2]. The implementations are based on a Prolog system extended by unification in finite algebras [8].

Our implementation captures the powerful version of the μ -calculus as used in [2]. In particular the μ -calculus formulas for strong and weak bisimilarity can be checked. Note that special care is needed for the ordering of the boolean variables encoding the variables in these μ -calculus formulas in order to avoid exponential explosion of the involved BDDs.

8 Example

Milner's example of a simple distributed scheduler has become a benchmark for process algebra tools [7, 9]. The scheduler consists of one starter process and N processes which are scheduled. The communication is organized in a ring. Expressed in CCS [10] the processes are as follows.

$$\begin{aligned} \text{Starter} &\stackrel{\text{def}}{=} \overline{c_0}.0 \\ C_0 &\stackrel{\text{def}}{=} c_0.a_0.(\tau.\overline{c_1}.C_0 + \overline{c_1}.\tau.C_0) \\ C_1 &\stackrel{\text{def}}{=} c_1.a_1.(\tau.\overline{c_2}.C_1 + \overline{c_2}.\tau.C_1) \\ &\vdots \\ C_{N-1} &\stackrel{\text{def}}{=} c_{N-1}.a_{N-1}.(\tau.\overline{c_0}.C_{N-1} + \overline{c_0}.\tau.C_{N-1}) \end{aligned}$$

Each cyler process C_i awaits the permit c_i to start, performs action a_i , and passes the permit to the next cyler either before or after some internal computation. The transition system for C_0 is given in Fig. 3. The compound process for N cyclers is

$$\text{SCHED}_N \stackrel{\text{def}}{=} (\text{Starter} \mid C_0 \mid C_1 \mid \dots \mid C_{N-1}) \setminus \{c_0, \dots, c_{N-1}\}$$

it is weakly bisimilar to $\text{SPEC}_N \stackrel{\text{def}}{=} a_0.a_1 \dots a_{N-1}.\text{SPEC}_N$.

Table 1 shows the size of the BDD for SCHED_N and the times needed to check the weak bisimilarity between SCHED_N and SPEC_N compared to conventional tools. The number of states and transitions for $N > 12$ as well as the times for the conventional tools for $N = 20$ are extrapolated estimates. The times for AUTO and BB are taken from [9], those for Aldébaran from [7]. Times for our system were obtained on an Apollo with 16MBs memory. It is approximately as fast as a SUN 3/60.

To be better comparable one would have to add the times needed for computing the transition relation of SCHED_N to the columns for AUTO, Aldébaran, and BB. Additionally the time needed to compute the transitive closure with respect to τ -transitions has to be added for AUTO and Aldébaran, it is not needed for branching bisimulation. Only the column for this paper includes all these times.

Applying Corollary 6.2 to SCHED_N yields $c = O(|\text{Act}|)$ and hence the size of the BDD representing the transition system of SCHED_N is $O(N \cdot \sum_{i=0}^{N-1} |S_i|^2) = O(N^2)$.

N	states	transitions	nodes in BDD	AUTO [4]	Aldébaran [7]	BB [9]	BDD
6	577	2017	424	3.3s	1.9s	0.2s	177s
8	3073	13825	654	57s	24s	1.2s	345s
10	15361	84481	908	-	-	7.4s	665s
12	73729	479233	1213	-	-	53s	1147s
14	300000	2000000	1549	-	-	-	1928s
16	1200000	8000000	1960	-	-	-	2972s
18	4800000	32000000	2337	-	-	-	3259s<1h
20	18000000	128000000	2810	(10 ⁹ s)	(7 · 10 ⁷ s)	(50000s)	4927s<1.5h

Table 1: Benchmarks for the scheduler

9 Conclusion

We have presented the missing link, namely the generation of transition relations as BDDs, for exploiting symbolic model checking as a tool for process algebras. The example shows that processes out of the reach of conventional tools can be checked for bisimilarity this way.

We have shown that the BDD for N parallel transition systems grows only linearly in N . Such a worst case bound for BDDs is rare in the literature. On the other hand it is not clear how the BDDs grow during the symbolic model checking.

Another open problem is the question whether other process algebra operators, in particular recursion can be performed on BDDs similarly efficiently.

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Appendix

This appendix shows how to calculate the bound W needed for the proof of Theorem 6.1. For the BDD B recall that $\sigma_l(B) = B|_{\vec{a}=\vec{b}\vec{a}, \vec{r}_1=\vec{b}\vec{r}_1, \vec{s}_1=\vec{b}\vec{s}_1, \dots, \vec{r}_{l-1}=\vec{b}\vec{r}_{l-1}, \vec{s}_{l-1}=\vec{b}\vec{s}_{l-1}}$. For arbitrary l we want W to be an upper bound for the number of different relations (i.e. characteristic functions) $\sigma_l(B)$ for any instantiation $\sigma_l = \langle \vec{b}\vec{a}, \vec{b}\vec{r}_1, \vec{b}\vec{s}_1, \dots, \vec{b}\vec{r}_{l-1}, \vec{b}\vec{s}_{l-1} \rangle$.

In the following the relations $\sigma_l(B)$ will be stated in terms of the relations $stab_l$, $move_l(A)$ and $SyncAsync_l$. The relation $stab_l$ expresses that all components T_i on level l or lower are stable, i.e. perform no change of state.

$$stab_l \stackrel{\text{def}}{=} \{ \langle \vec{b}\vec{r}_l, \vec{b}\vec{s}_l, \dots, \vec{b}\vec{r}_N, \vec{b}\vec{s}_N \rangle \mid \forall i, l \leq i \leq N : \vec{b}\vec{r}_i = \vec{b}\vec{s}_i \}$$

For a set A of actions $move_l(A)$ expresses that one component T_j on level l or lower performs a transition with an action of A while all other processes are stable.

$$move_l(A) \stackrel{\text{def}}{=} \{ \langle \vec{b}\vec{r}_l, \vec{b}\vec{s}_l, \dots, \vec{b}\vec{r}_N, \vec{b}\vec{s}_N \rangle \mid \exists \alpha \in A, j, l \leq j \leq N : \langle \vec{b}\vec{r}_j, \alpha, \vec{b}\vec{s}_j \rangle \in D_j \\ \wedge \forall i, i \neq j, l \leq i \leq N : \vec{b}\vec{r}_i = \vec{b}\vec{s}_i \}$$

The relation $SyncAsync_l$ states that some process T_j on level l or lower performs an asynchronous τ -transition or that two processes T_j, T_k perform synchronizing transitions. In both cases all other processes are stable.

$$SyncAsync_l \stackrel{\text{def}}{=} move_l(\{\tau\}) \cup \{ \langle \vec{b}\vec{r}_l, \vec{b}\vec{s}_l, \dots, \vec{b}\vec{r}_N, \vec{b}\vec{s}_N \rangle \mid \exists \alpha \in Act, j, k, j \neq k, l \leq j, k \leq N : \\ \langle \vec{b}\vec{r}_j, \alpha, \vec{b}\vec{s}_j \rangle \in D_j \wedge \langle \vec{b}\vec{r}_k, \bar{\alpha}, \vec{b}\vec{s}_k \rangle \in D_k \wedge \\ \forall i, k \neq i \neq j, l \leq i \leq N : \vec{b}\vec{r}_i = \vec{b}\vec{s}_i \}$$

Instantiations σ_l can be divided into non- τ - and τ -transitions (cases a and b below). The former are always asynchronous. Depending on the number of unstable components in σ_l (a component k of σ_l is unstable if $\bar{b}r_k \neq \bar{b}s_k$) we distinguish the cases a0 (no unstable components), a1 (one unstable component), and a2 (two or more unstable components).

Case a $\bar{b}a \neq \tau$

Case a0 $\bar{b}r_i = \bar{b}s_i$ for all $i \in \{1, \dots, l-1\}$

$$\sigma_l(B) = \begin{cases} \text{move}_l(\{\bar{b}a\}) \cup \text{stab}_l & \text{if } \exists j, 1 \leq j < l : \langle \bar{b}r_j, \bar{b}a, \bar{b}r_j \rangle \in D_j \\ \text{move}_l(\{\bar{b}a\}) & \text{otherwise} \end{cases}$$

Case a1 $\exists j : \bar{b}r_j \neq \bar{b}s_j$ and $\forall i, i \neq j : \bar{b}r_i = \bar{b}s_i$ where $1 \leq i, j < l$

$$\sigma_l(B) = \begin{cases} \text{stab}_l & \text{if } \langle \bar{b}r_j, \bar{b}a, \bar{b}s_j \rangle \in D_j \\ \emptyset & \text{otherwise} \end{cases}$$

Case a2 $\exists j, k, j \neq k : \bar{b}r_j \neq \bar{b}s_j \wedge \bar{b}r_k \neq \bar{b}s_k$ where $1 \leq j, k < l$

$$\sigma_l(B) = \emptyset$$

The τ -transitions are more complicated because we have to consider transitions resulting from two synchronizing processes as well as asynchronous τ -transitions of a single process. Depending on the number of unstable components in σ_l we distinguish cases b0, b1, b2 and b3. The case b3 covers those σ_l which contain three or more unstable components.

The cases b0, b1 and b2 have to check whether for σ_l a process can proceed asynchronously and whether two processes can synchronize. For this purpose we define the predicates Async- τ and Sync- τ .

$$\begin{aligned} \text{Async-}\tau(j) &\stackrel{\text{def}}{=} \langle \bar{b}r_j, \tau, \bar{b}s_j \rangle \in D_j \\ \text{Sync-}\tau(j, k) &\stackrel{\text{def}}{=} \exists \alpha \in \text{Act} : \langle \bar{b}r_j, \alpha, \bar{b}s_j \rangle \in D_j \wedge \langle \bar{b}r_k, \alpha, \bar{b}s_k \rangle \in D_k \end{aligned}$$

Case b $\bar{b}a = \tau$

Case b3 $\exists M \subseteq \{1, \dots, l-1\}, |M| > 2 : \forall i \in M : \bar{b}r_i \neq \bar{b}s_i$

This means that three or more processes perform a transition in B , which is impossible from the definition of the parallel composition operator.

$$\sigma_l(B) = \emptyset$$

Case b2 $\exists j, k, j \neq k : \bar{b}r_j \neq \bar{b}s_j, \bar{b}r_k \neq \bar{b}s_k, \forall i, k \neq i \neq j : \bar{b}r_i = \bar{b}s_i$ where $1 \leq i, j, k < l$

If two processes are making a transition they are synchronizing, i.e. the remaining components must be stable.

$$\sigma_l(B) = \begin{cases} \text{stab}_l & \text{if Sync-}\tau(j, k) \\ \emptyset & \text{otherwise} \end{cases}$$

Case b1 $\exists j : \bar{b}r_j \neq \bar{b}s_j \forall i, i \neq j : \bar{b}r_i = \bar{b}s_i$ where $1 \leq i, j < l$

If process T_j can perform an asynchronous τ -transition or synchronize with a process $T_i, i < l$ the relation stab_l is contained in $\sigma_l(B)$. The other part of $\sigma_l(B)$ stems from synchronisations of T_j with a process $T_i, i \geq l$.

For the latter case we define the set $A_1 := \{\bar{\alpha} \mid \langle \bar{b}r_j, \alpha, \bar{b}s_j \rangle \in D_j \wedge \alpha \neq \tau\}$.

$$\sigma_l(B) = \begin{cases} \text{move}_l(A_1) \cup \text{stab}_l & \text{if Async-}\tau(j) \text{ or } \exists k, j \neq k < l : \text{Sync-}\tau(j, k) \\ \text{move}_l(A_1) & \text{otherwise} \end{cases}$$

Case b0 $\bar{b}r_i = \bar{b}s_i$ for all $i \in \{1, \dots, l-1\}$

Again we have the same possibilities as in case b1, but the value of j is not fixed, it may be in the range of 1 to $l-1$. As components 1 to $l-1$ are stable we also have the possibility that processes $T_1, \dots, T_N$ perform an asynchronous transition or a pair of synchronizing transitions.

Let $A_0 := \{\bar{\alpha} \mid \exists j, 1 \leq j < l : \langle \bar{b}r_j, \alpha, \bar{b}r_j \rangle \in D_j \wedge \alpha \neq \tau\}$.

$$\sigma_l(B) = \begin{cases} \text{move}_l(A_0) \cup \text{SyncAsync}_l & \text{if } \exists j, 1 \leq j < l : \text{Async-}\tau(j) \text{ or} \\ \cup \text{stab}_l & \exists j, k, j \neq k, 1 \leq j, k < l : \text{Sync-}\tau(j, k) \\ \text{move}_l(A_0) \cup \text{SyncAsync}_l & \text{otherwise} \end{cases}$$

By counting all possibilities for $\sigma_l(B)$ in cases **a0** to **a2** and **b0** to **b3** and allowing any subset of Act for A_0 and A_1 we get the upper bound $W = 4 \cdot 2^{|Act|}$.

Critical cases for a better approximation are **b0** and **b1** from which we can get a smaller upper bound W . If no process contains a visible self-loop the set A_0 in case **b0** is empty. We then get $W = 2 \cdot |Act| + 4 + 2 \cdot c$ with c as stated in Corollary 6.2.

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