Domains in a Realizability Framework¹

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Abstract

We aim to relate and develop various approaches to a theory of domains in a realizability framework.

In the first part we develop the theory in an arbitrary *partial cartesian closed category* and, as a particular instance, in the category of *partial equivalence relations* (per) over an arbitrary *partial combinatory algebra* (pca). In this way one obtains the category of *separated* pers that may be regarded as a universe of partially ordered sets and monotone maps with strong closure properties.

Pursuing the analogy with domain theory one tries to build a category of directed complete partial orders and continuous maps. We consider two main approaches: one based on *Kleene's pca* of numbers and partial recursive functions and the other based on *Scott's D_∞ λ -model*.

Contents: 0. Introduction, 1. Dominance and Intrinsic Preorder, 2. Separated Partial Equivalence Relations (Σ per), 3. N-completeness, 4. Σ -connection and Extensionality, 5. D-completeness and Uniformity, 6. Conclusion.

0. Introduction

The general and vague problem is to develop a theory in which data types can be regarded as sets. A number of examples show that classical set theory is not well-suited to this purpose. However it has become more and more clear that this program can be achieved if one takes a constructive attitude. In particular we will be interested in set-theoretic structures based on realizability.

A model for a programming language should include some notion of effectivity and/or computability. In the kind of models we are going to present the underlying realizability structure plays the role of an abstract machine. A data is represented as a collection of extensionally equivalent implementations. A map from data A to data B is a function that can be actually implemented by a realizer and that transforms equivalent implementations of a data in A into equivalent implementations of a data in B. This aspect of the model is relevant for the interpretation of subtyping.

The fact that all operations have to be realized leads to very interesting properties of the model reflecting its computational nature. A typical example is the validity of a "Uniformity Principle" which plays an important role in the interpretation of second order quantification as intersection.

There have been several attempts in this direction. It seems fair to say that this program was first pushed by Dana Scott and his students: McCarty [84], and Rosolini [86], whose work relates in particular to the *effective topos* (Hyland [82], but see also Mulry [81] for another approach). Recently there has been another wave of results in Amadio [89], Abadi & Plotkin [90], Freyd & al. [90], Phoa [90]. We wish to analyse them from two points of view:

(1) *Success* in combining interesting properties of domain-theoretic and realizability models. For example, for the former: solution of recursive domain equations (Scott [72], Wand [80], Smyth & Plotkin [82]), existence of a universal (homogeneous) domain (Scott [76], Gunter & Jung [90], Droste & Göbel [90]), abstract account of effective computations (Smyth [77]); and for

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the latter: simple interpretations of polymorphism (Hyland[89], Carboni&al.[88], Longo&Moggi[88]) and subtyping (Bruce & Longo[90]).

(2) *Conceptual economy* of the solution, that is trying to have a theory that is at once general in that it allows to explain and relate different approaches and simple in that it makes a careful analysis of the axioms introduced.

0.1 External vs. Internal Logic

We have to point-out that the results of such analysis may depend on the tools we use. We will try to work as much as we can with a generic partial combinatory algebra D (henceforth pca , definition in 2.1.1) and in full (reflective) subcategories of the category of partial equivalence relations over D (henceforth per_D).

We will not be able to carry on all the program at this level of generality, however, in view of new developments, we think that it is important to stress which results are general and which results depend on the specific $pers$ and $pcas$ we are dealing with.

A subtle point is the choice between an external classical logic (say a system like ZF) and the internal logic of a realizability universe. Living in a set-theoretic realizability framework it is clear after a few examples that usual definitions (e.g. that of N -completeness, see 3.1) do not lead to the expected theorems unless they are given a constructive reading.

In this sense one receives benefits by thinking at the constructive interpretation of logical operators. This approach can be made rigorous by using the internal logic of the effective topos, basically a higher order intuitionistic type theory that includes principles like the countable axiom of choice (AC_ω), the computability of all the maps on natural numbers (Church Thesis), the Uniformity Principle and Markov Principle (see Hyland[82]). Reasoning in this non-standard logic is prone to errors induced by our habit of working in classical logic. Moreover there is an obvious problem of communication with the overwhelming majority of the mathematical community.

The moral that we draw from this situation is that when developing the theory one should make a sensible use of the internal logic. In particular one should try to understand what is the effect of a constructive interpretation of a definition on the structure of the mathematical object we are dealing with. This may involve some tiresome computation on realizers, but we will be rewarded by a stronger intuition of "what is going on" that may lead to new insights. Every logical statement that we formulate in the following part of the paper should be read classically as a sentence of, say, ZF set theory.

0.2 Overview

The paper is supposed to be a gentle and systematic introduction, requiring a minimum of logical and categorical apparatus, to a series of concepts that were mostly introduced or suggested by other people. However, to the best of our knowledge, this is the first place where a systematic analysis of the various approaches mentioned above and an investigation of their relationships is started. In a sense we feel that this is our main contribution or, at least, the one that took the greatest effort.³

In §1 we develop a notion of *intrinsic pre-order* working in an arbitrary *partial cartesian closed category* (pccc). Every pccc has an object Σ that classifies the *admissible monos* and that can be regarded as an abstraction of Sierpinski space. The following isomorphism holds: $a \rightarrow \Sigma \cong a \rightarrow 1^4$; hence given $f: a \rightarrow \Sigma$ and $x: 1 \rightarrow a$ we have an induced notion of *convergence* for fx . Given two points of an object a , say $x, y: 1 \rightarrow a$, x is *intrinsically less than* y if any map $f: a \rightarrow \Sigma$ that converges on x also converges on y . Maps

³ In this process of revision and generalization a number of conceptual simplifications have been made.

⁴ $a \rightarrow 1$ are the "partial" maps from a into the terminal object.

automatically preserve the intrinsic pre-order. A separated object is an object in which the intrinsic pre-order is a partial order (i.e. anti-symmetry holds).

In §2 we restrict our attention to a pccc of *pers over an arbitrary pca*. The convergence predicate of the pca naturally induces on any given per a family of *semi-computable* predicates, that, on the other hand, gives rise to a family of *admissible monos* generating the pccc. *Separated pers* (Σper) are those pers whose points are separated by the topology induced by the semi-computable subsets. They are shown to be a full reflective sub-category of per and therefore they enjoy all the completeness properties of the larger category. The category of separated pers can be seen as a universe of partially ordered sets and monotone maps.

In §3 the notion of *N-completeness* is introduced. Σper contains a natural number object (N). Given a Σper A we will say that A is N-complete if every chain (w.r.t. the intrinsic order) described by a map from N to A has a supremum in A. When working on Kleene's pca ($\omega, \cdot$) it will turn out that all realized maps between N-complete Σper ($c_N\Sigma\text{per}$) are continuous in the sense of preserving suprema of such chains.

In §4 the notion of Σ -connected object (or equivalently Σ -linked object) is presented. Its motivation arises from the need of giving a sufficient condition for the coincidence of intrinsic and pointwise orders on products. On the other hand the problem of having strong completeness properties for $c_N\Sigma\text{per}_\omega$ leads to the introduction of *extensional pers* (exper). The notion of extensionality implies those of separation and Σ -connection. By definition every extensional per A can be regarded as a sub-poset of a "functional" per $B \rightarrow \Sigma$. This assumption has two important consequences: (a) it is possible to exhibit an operator that, uniformly, given any (N-)ascending chain computes its lub; (b) since $B \rightarrow \Sigma$ is N-complete it is possible to extend every extensional per to an N-complete extensional per hence showing the reflection of $c_N\text{exper}_\omega$ into exper_ω .

In §5 we take the point of view that the notion of computation should be modelled at a suitable level of *abstraction*, say in terms of ordered sets and continuous transformations, rather than at the level of indices of Turing machines. Working over a pca D_{dcpo} with the structure of a directed complete partial order we discuss the notion of D-complete Σper ($c_D\Sigma\text{per}$). A per is *D-complete* if, when regarded as a relation, it is closed under directed sets w.r.t. the D-order. The category of $c_D(\Sigma)\text{per}_{D_{\text{dcpo}}}$ is *reflective* in $(\Sigma)\text{per}_{D_{\text{dcpo}}}$ and its *pointed* objects interpret a least fixed point combinator. Next we consider a pca D_∞ built via the inverse limit technique and we impose a condition of *uniformity* on the $c_D\Sigma\text{pers}$ ($c_{\text{Du}}\Sigma\text{per}$). Under this condition $c_{\text{Du}}\Sigma\text{per}_{D_\infty}$ turns out to be a *complete ultra-metric space*. Thanks to the contractiveness of the first-order type constructors it is possible to solve domain equations up-to-equality. Moreover if we impose that D_∞ is in sfp^5 , D_{sfp} , we can look at the full subcategory $c_{\text{Du}}\Sigma\text{per}_{D_{\text{sfp}}}$ as at *sfp* domains and continuous maps (w.r.t. the intrinsic order). The crucial theorem shows that the quotient space is a cpo. Its proof relies on techniques apparently unrelated to those employed in §3.

The following diagram summarizes the relationships among the main categories discussed above:

$$\begin{array}{c}
 c_N\text{exper}_\omega \supset \text{exper}_D \\
 \downarrow \cap \\
 c_N\Sigma\text{per}_\omega \subset \Sigma\text{per}_D \supset \text{per}_D \\
 \uparrow \cup \\
 c_{\text{Du}}\Sigma\text{per}_{D_{\text{sfp}}} \subset c_D\Sigma\text{per}_{D_{\text{dcpo}}}
 \end{array}$$

where:⁶ $\subset \equiv$ full subcategory $\supset \equiv$ full reflective subcategory.

⁵ *sfp* stands for Sequences of Finite inductive Partial orders. The category of *sfp* domains and continuous maps was introduced by G. Plotkin. *sfp* domains are also referred to with the name of *bifinite* domains.

⁶ For the reader's convenience we recall: $D \equiv \text{pca}$. $\omega \equiv$ Kleene's pca. $D_{\text{dcpo}} \equiv$ a specific pca in *dcpo*. $D_{\text{sfp}} \equiv$ a specific pca in *sfp*. $\text{per} \equiv \text{pers}$. $\Sigma \equiv$ separated. $\text{ex} \equiv$ extensional. $c_N \equiv$ N-completeness. $c_D \equiv$ D-completeness.

1. Dominance and Intrinsic Preorder

In this section we introduce some basic notions about *partial cartesian closed categories* (pccc) and their properties.

Every pccc has an object Σ , called *dominance*, that classifies the *admissible subobjects* in the same sense as the object of truth-values Ω classifies arbitrary subobjects in a topos.

The object Σ induces a *preorder* $\leq_a$ on the points of every object a (i.e. maps from the terminal object into a). Following Rosolini[86] we focus on the full subcategory of *separated objects*, that is composed of those objects for which $\leq_a$ is antisymmetric.

We collect some basic properties of this subcategory that will be extensively used in the following sections.

1.1 Partial Cartesian Closed Categories (pccc)⁷

It is standard to consider an equivalence class of monos on an object as a generalized notion of subset. A partial map from a to b can then be represented as a total map from a subset of a to b . In many interesting examples the domain of convergence of a partial map is not arbitrary. E.g. it is open, recursively enumerable, etcetera. It is then reasonable to look for a corresponding categorical notion of admissible mono as specified by the following definition.

1.1.1 Definition (Admissible family of monos)⁸

An admissible family of monos M for a category C is a collection $\{M(a) / a \in \text{Ob } C\}$ s.t.:

- (1) If $m \in M(a)$ then m is a *mono* $m: d \rightarrow a$.
- (2) The *identity* on a is in $M(a)$: $\text{id}_a \in M(a)$.
- (3) M is closed under *composition* i.e.
if $m_1: a \rightarrow b \in M(b)$ and $m_2: b \rightarrow c \in M(c)$ then $m_2 m_1: a \rightarrow c \in M(c)$.
- (4) M is closed under *pullbacks* i.e.
if $m: d \rightarrow b \in M(b)$ and $f: a \rightarrow b$ then $f^1(m) \in M(a)$.

1.1.2 Fact (the related category of partial maps)

An admissible family of monos M on C enjoys properties sufficient for the construction of a related category of partial maps pC . A representative for a partial map from a to b is a pair of maps in C , (m, f) , such that $m: d \rightarrow a \in M(a)$ determines the domain and $f: d \rightarrow b$ the functional behavior. The category pC has the same objects as C and as morphisms equivalence classes of representatives of partial maps, namely:

$$pC[a, b] \triangleq \{[m, f] / m: d \rightarrow a \in M(a), f: d \rightarrow b\}$$

where (m, f) is equivalent to (m', f') iff they have the same domain and codomain, and there is a map that makes m isomorphic to m' in the slice category C/a , and f isomorphic to f' in the slice category C/b . To specify domain and codomain of a (representative) for a partial map we write $[m, f]: a \rightarrow b$ ($(m, f): a \rightarrow b$).

1.1.3 Definition (Pccc)

Let M be an admissible collection of monos on the category C . (C, M) is a *pccc* (partial cartesian closed category) iff C is *cartesian* and for any b there is a functor *partial exponent* on b $\text{pexp}_b: pC \rightarrow C$ that is right adjoint to the product functor $_ \times b: C \rightarrow pC$ that is: $pC[_ \times b, _] \cong C[_, \text{pexp}_b]$.

$u \equiv$ uniformity. This is just a mnemonic schema. Note that we are dealing with different pcas. In general the *pca* of the contained category is an instance of the *pca* of the containing category. Things make sense only after instantiation of the latter (containing) with the former (contained).

⁷Refer to Curien&Obtulowicz[88], Moggi[88] and Robinson&Rosolini[88] for extended informations on the origins and the development of the theory. Longo&Moggi[84] is the first place where we found definition 1.1.3.

⁸We often abbreviate functional composition $f \circ g$ with fg .

By instantiating the natural isomorphism one obtains the following version of “currying”: $a \times b \rightarrow c \cong a \rightarrow (b \rightarrow c)$, where we write more suggestively $b \rightarrow c$ for $\text{pexp}_b(c)$. Indeed by virtue of the previous isomorphism we can safely confuse $b \rightarrow c$ with $\text{pC}[b, c]$. Finally the standard notion of *lifting* can be defined as $(a)_\perp \triangleq 1 \rightarrow a$.

1.1.4 Fact (Dominance)

In every pccc there is an object $\Sigma \triangleq (1)_\perp \triangleq 1 \rightarrow 1$, called dominance, that classifies the monos in $\mathbf{M}$: $\mathbf{M}(_) \equiv \mathbf{C}[_, \Sigma] : \mathbf{C}^0\mathbf{P} \rightarrow \mathbf{Set}$. The following isomorphism holds: $a \rightarrow \Sigma \cong a \rightarrow 1$.

1.2 Intrinsic Pre-order

In a pccc the maps from an object a to the dominance Σ play the role of convergence tests. Such tests induce a pre-order on the points of an object. The idea of ordering points by tests bears a striking analogy with the one encountered in, so-called, operational semantics of ordering terms by observations. However a crucial difference is that in our case the pre-order does not depend on the language at hand but only on a global category on which one can interpret a variety of constructs.

1.2.1 Convention (Points and convergence predicate)

We write $x: a$ to indicate that x is a *point* of a , that is a morphism $x: 1 \rightarrow a$. Since we will be dealing with (p)ccc we confuse points in the objects $a \rightarrow b$ and $a \rightarrow b$ with morphisms, respectively, in $\mathbf{C}[a, b]$ and $\text{pC}[a, b]$. E.g. $f: a \rightarrow b$ can be seen both as a morphism from a to b and as a point in $a \rightarrow b$.

We introduce a *convergence predicate*, $\downarrow$, as follows: if $x: a$, $p \equiv [m, f]: a \rightarrow b$, with $m: d \rightarrow a$, $f: d \rightarrow b$ then we write: $px \downarrow \Leftrightarrow \exists h: 1 \rightarrow d. (mh = x)$. By virtue of the isomorphisms in 1.1.4 we write both $p: a \rightarrow b$ and $p: a \rightarrow (b)_\perp$.

1.2.2 Definition (Intrinsic preorder)⁹

Let $(\mathbf{C}, \mathbf{M})$ be a pccc and a an object. Define a preorder $\leq_a$ on the points of a as:

if $x, y: a$ then $x \leq_a y$ iff $\forall p: a \rightarrow \Sigma. (px \downarrow \supset py \downarrow)$.

The intuition is that x is less than y in a if every convergence test $p: a \rightarrow \Sigma$ that succeeds on x also succeeds on y . We also write $px \preceq py$ for $px \downarrow \supset py \downarrow$, and $px \equiv py$ for $px \downarrow \Leftrightarrow py \downarrow$.

1.2.3 Definition (category of Σ -objects)

Given a pccc $(\mathbf{C}, \mathbf{M})$ with dominance Σ we denote with $\Sigma\mathbf{C}$ the full subcategory of $\mathbf{C}$ whose objects enjoy the property that the intrinsic preorder is anti-symmetric. An object a such that $\leq_a$ is a partial order is called Σ -object or, equivalently, *separated object*.

1.2.4 Proposition (Basic properties)

Let $(\mathbf{C}, \mathbf{M})$ be a pccc with dominance Σ . Then:

- (1) Maps preserve the intrinsic preorder.
- (2) Σ -objects are closed under subobjects.
- (3) Moreover if $(\mathbf{C}, \mathbf{M})$ has enough points and a is an object then Σ^a is a Σ -object.

Proof

(1) Let $f: a \rightarrow b$ and $x, y: a$. Suppose $x \leq_a y$, then given any $p: b \rightarrow \Sigma$ we have by hypothesis $\text{pfx} \preceq \text{pfy}$, since $\text{pf}: a \rightarrow \Sigma$. Hence $\text{fx} \leq_b \text{fy} : b$.

(2) Let $m: a \rightarrow b$ be a mono and b be a Σ -object. If x and y are two distinct points in a then mx and my are two distinct points in b . Hence, since b is a Σ -object, they are separable by a

⁹ Every partial category pC has a partial ordering on morphisms given by the containment of the convergence spaces. In such partial ordering the total morphisms represent the maximal elements. Therefore the points of an object a , i.e. the total maps $x: 1 \rightarrow a$, turn out to be all incomparable. On the other hand the intrinsic preorder may give non-trivial partial orders on the points.

map $p: b \rightarrow \Sigma$. Then the map pm separates the points x and y .

(3) If $f, g: \Sigma^a$ and $f \neq g$ then, by the enough point assumption, there is a $x: a$ s.t. $\sim fx \equiv gx$. Take $\lambda h: \Sigma^a. hx: \Sigma^a \rightarrow \Sigma$ as separator for f and g . $\square$

1.2.5 Note (on category theoretic notions)

We recall the notion of *full reflective subcategory* that will play an important role in the following sections. Suppose that C is a full subcategory of D . We say that C is *reflective* in D if the inclusion functor $\text{Incl}: C \rightarrow D$ has a *left adjoint* $L: D \rightarrow C$ (called *reflector*), hence satisfying: $C[L_ , _] \cong D[_, \text{Incl}_]$. If C is a full reflective subcategory of D then every functor $S: J \rightarrow C$ having limit in D has (an isomorphic) limit in C (see MacLane[72] for a sketch of the proof). Hence C inherits all the limits and colimits (L has a right adjoint) of D .

2. Separated Partial Equivalence Relations (Σper)

One can build over every partial combinatory algebra (pca) D the category of partial equivalence relations, per_D , and a family of admissible monos, M_D . (per_D, M_D) turns out to be a *pccc with enough points*. Besides, Σper_D , the category of separated pers over D , is *reflective* in per_D .

2.1 Partial Combinatory Algebras and PER

The category of pers built over a pca has very relevant completeness properties (see Hyland[88], Carboni&al.[87], Longo&Moggi[88]). In the following we will take the notion of morphism as fundamental and we will restrict our attention to full, possibly reflective, subcategories.

2.1.1 Definition (partial combinatory algebra)

A partial combinatory algebra (pca) is a partial algebraic structure $(D, k, s, \cdot)$ where:

$$k, s \in D, \cdot : D \times D \rightarrow D, \text{ and } kxy = x, sxy \downarrow, sxyz \equiv xz(yz).$$

As usual $\rightarrow$ denotes the partial functional space. $t \downarrow$ denotes the fact that the expression t is defined. $\equiv$ is Kleene's equality: $t \equiv s$ iff $(t \downarrow \Leftrightarrow s \downarrow) \wedge (t \downarrow \Rightarrow t = s)$. $t = s \Rightarrow t \downarrow \wedge s \downarrow$, $ts \downarrow \Rightarrow t \downarrow \wedge s \downarrow$.

Proviso

For the sake of readability we will write programs using λ -notation. It is intended that terms have to be compiled in the language of combinators according to some translation that simulates conveniently λ -abstraction.

2.1.2 Definition (partial equivalence relations)

Given a pca D the category of pers over D is defined as follows:

$\text{Obper}_D \triangleq \{A \mid A \subseteq D \times D \text{ and } A \text{ is symmetric and transitive}\}$

$\text{per}_D[A, B] \triangleq \{f: D/A \rightarrow D/B \mid \exists \phi \in D. \forall d \in D. dAe \Rightarrow \phi d \in f([d]_A)\}$

where conventionally: $dAe \equiv_{\text{abr.}} (d, e) \in A$, $[d]_A \triangleq \{e \in D \mid dAe\}$, $D/A \triangleq \{[d]_A \mid dAe\}$,

$|A| \triangleq \{d \mid d \in D, dAe\}$, if ϕ is a realizer for the map $f: A \rightarrow B$ we may denote f with $[\phi]_{A \rightarrow B}$.

2.1.3 Semi-computable predicates on D

Partiality is explicitly given in D and by generalizing basic facts of recursion theory (i.e. sets are exactly the domain of computable functions) it also provides a notion of *semi-computable predicate* on D .

(1) $\Sigma(D) \triangleq \{W \mid W \subseteq D, \exists d \in D. W = \text{Dom}(d)\}$, where $\text{Dom}(d) \triangleq \{e \mid e \in D, de \downarrow\}$.

(2) The collection of predicates $\Sigma(D)$ induces a *refinement pre-order* on D defined as:
 $d \leq_D e$ iff $\forall W \in \Sigma(D). (d \in W \Rightarrow e \in W)$

(3) Observe that the operation of application preserves this pre-order:

$$\forall e \in D. (d \leq_D d' \Rightarrow ed \leq_D ed').$$

(4) If the pca is not total then $\Sigma(D)$ can be seen as a *basis for a topology* as:

(a) $\emptyset, D \in \Sigma(D)$. (b) If $W, W' \in \Sigma(D)$ then $W \cap W' \in \Sigma(D)$

Hints: (a) Take respectively the always divergent and always convergent map. (b) If $W = \text{Dom}(e)$ and $W' = \text{Dom}(e')$ then

$$W \cap W' = \text{Dom}(\lambda d. \text{if } (ed \downarrow \wedge e'd \downarrow) \text{ then } \downarrow) = \text{Dom}(\lambda d. (\lambda x. \lambda y. c)(ed)(e'd)).$$

2.1.4 Examples (pcas)

(1) An important example of pca is *Kleene's* $(\omega, \cdot)$ where ω are the natural numbers and $n \cdot m$ is the n -th Turing machine applied to the input m , (given some standard enumeration). In this case: $\Sigma(\omega, \cdot) = \text{RE}$, the recursively enumerable sets.

(2) Another canonical example of pca is that of a non-trivial domain D that is a retract¹⁰ of the partial functional space, written $D \rightarrow D \triangleleft D$, in the category of directed complete partial orders and partial continuous maps (see Plotkin[85]). Then $\Sigma(D)$ is the collection of open sets in Scott's topology.

(3) Examples of pca's in which the collection of points is not separable can be built out of term models. For example if one takes a $\lambda\beta$ term model where the notion of convergence corresponds to reduction to a λ -abstraction one finds that all terms that are strongly unsolvable are undistinguishable.

2.2 PER as a Pccc

We show that given any per, A , $\Sigma(D)$ induces a collection, $\Sigma(A)$, of semi-computable predicates on A . From this structure it is easy to obtain a family M_D of admissible monos on per_D that turns the category into a pccc.

2.2.1 Definition (Semi-computable subsets of a per)

Let $A \in \text{per}_D$, define $\Sigma(A) \triangleq \{B \in \text{per}_D \mid D/B \subseteq D/A, \exists W \in \Sigma(D). |A| \cap W = |B|\}$.

In other terms B belongs to $\Sigma(A)$ if the equivalence classes in B form a subset of those in A and there is a set $W \in \Sigma(D)$ that separates B from the other equivalence classes in D/A . This definition generalizes the notion of semi-computable set in the category of enumerated sets (or equivalently total equivalence relations). In this case an object is a pair $(X, e_X : \omega \rightarrow X)$ where e_X is an onto enumeration of X and a subset Y of X is semi-computable iff $e_X^{-1}(Y)$ is recursively enumerable. Such sets form the basis for what is sometime called Mal'cev topology.

2.2.2 Closure properties of $\Sigma(A)$

Assume D non-total pca. Then $\Sigma(A)$ enjoys closure properties analogous to $\Sigma(D)$ (see 2.1.3.(4)): (a) $\emptyset, A \in \Sigma(A)$. (b) If $B, B' \in \Sigma(A)$ then $B'' \in \Sigma(A)$, where B'' is the per corresponding to the partial partition $D/B \cap D/B'$.

2.2.3 Definition (Admissible family of monos)

Define M_D as the following family of monos:

$m: A' \rightarrow A \in M_D(A)$ iff $A' \in \Sigma(A)$ and m is the inclusion morphism.

Note that the morphism m is realized by the identity (hence m is a coercion in the sense of Bruce&Longo[88]). It is easy to check that this collection of monos is indeed admissible. The conditions for identity and composition are clear. Let us consider the case for the pullbacks:

¹⁰ We write $D \triangleleft E$ if there is a pair of morphisms $i: D \rightarrow E, j: E \rightarrow D$ such that $ji = \text{id}_D$.

assume $f:A \rightarrow B$ and $m:C \rightarrow B$ with ϕ realizer of f and $|B| \cap \text{Dom}(\psi) = |C|$. To construct the pullback consider $W' = \text{Dom}(\lambda d. \psi(\phi d))$ and the related C' admissible subobject of A .

2.2.4 (per_D, M_D) is a pccc

The category (per_D, M_D) of pers and partial maps (see 1.1.2) is equivalent to the category pper_D defined as follows:

$$\text{Ob}_{\text{pper}_D} \triangleq \text{Ob}_{\text{per}_D}$$

$$\text{pper}_D[A, B] \triangleq$$

$$\triangleq \{f: D/A \rightarrow D/B \mid \exists \phi \in D. \forall d. dAd \Rightarrow ((\phi d \downarrow \Leftrightarrow f([d]_A) \downarrow) \wedge (\phi d \downarrow \Rightarrow \phi d \in f([d]_A)))\}$$

The category (per_D, M_D) is a pccc. In particular the *partial exponent* is defined as:

$$f(A \rightarrow B)g \Leftrightarrow \forall d. e. (dAe \Rightarrow fd \approx_B ge).$$

where $t \approx_B s \Leftrightarrow (t \downarrow \Leftrightarrow s \downarrow)$ and $(t \downarrow \Rightarrow tBs)$ i.e. Kleene equality relativized to B . Notice that the category has *enough points*. The *terminal object* is any per with one equivalence class, say $1 \triangleq D \times D$. In this case the *dominance* is $\Sigma \triangleq 1 \rightarrow 1 = \{\perp, \top\}$, where: $\perp \triangleq \{d \in D \mid \forall e. de \uparrow\}$, $\top \triangleq \{d \in D \mid \forall e. de \downarrow\}$.

2.2.5 D-order implies intrinsic order on pers

Applying the constructions in the previous section we can now *define*:

Σper_D as the full subcategory of per_D whose objects are Σ -objects.

What is the relationship between the D-order (2.1.3.(2)) and the intrinsic pre-order (1.2.2) ? If $d \leq_D d'$, $A \in \text{per}_D$, and $d, d' \in |A|$ then a fortiori $[d]_A \leq_A [d']_A$. Suppose $B \in \Sigma(A)$ and $[d]_A \in D/B$ then there is a $W \in \Sigma(D)$ such that $|A| \cap W = |B|$. But by hypothesis $d' \in W$ and therefore $[d']_A \in D/B$.

The sense of this remark is that if two elements cannot be separated in the type free universe of the realizability structure D then a fortiori they cannot be separated in the "typed" structure of pers. An interesting approach can arise by trying to reverse this implication i.e. consider a canonical representative for each equivalence class and find conditions under which the intrinsic order on pers forces the underlying order.

2.3 Reflection of Σper

We give an elementary and direct proof that Σper_D is reflective in per_D . This fact can also be derived following more abstract considerations, see for example proof 4.2.6.

2.3.1 Theorem ($\Sigma \text{per} \subset_{>\text{per}}$)

Σper is a full reflective subcategory of per .

Proof

The very simple idea for obtaining a Σper $L_\Sigma(A)$ from the per A is to collapse equivalence classes that cannot be separated by $\Sigma(A)$. Given a per A and the intrinsic preorder $\leq_A$ we can define, as usual, an equivalence relation, $\approx_A$, on D/A as:

$$[d]_A \approx_A [e]_A \Leftrightarrow d, e \in |A| \wedge [d]_A \leq_A [e]_A \wedge [e]_A \leq_A [d]_A.$$

Now define the reflector $L_\Sigma: \text{per} \rightarrow \Sigma \text{per}$ as follows:

if $A \in \text{per}$ then $L_\Sigma(A)$ is such that: $dL_\Sigma(A)e \Leftrightarrow [d]_A \approx_A [e]_A$

if $f: A \rightarrow B$ then $L_\Sigma(f)([d]_{L_\Sigma(A)}) = f([d]_A)$

One can easily verify that: (1) $L_\Sigma(A)$ is a Σper . Actually it is the least Σper containing A (as a relation). (2) $dL_\Sigma(A)e \Rightarrow f([d]_A) = f([e]_A)$, as B is separated. (3) Every map from a per A to a Σper B can be uniquely extended to a map from $L_\Sigma(A)$ to B . From these facts it is easy to exhibit the natural isomorphism of the adjunction. $\square$

2.3.2 Corollary (Closure properties of Σper)

Σper is a ccc and it has all limits and colimits of per.

Proof

The existence of limits and colimits is guaranteed by 2.3.1 and note 1.2.5. Try, as an exercise, to show that Σper is closed under exponentiation. $\square$

3. N-completeness

One is interested in finding an analogous of the notion of ω -completeness in a realizability framework. In the first place we need an object N that can play the role of the *natural numbers*.¹¹ Next we will concentrate on the *N-complete Σpers* that is those Σpers such that any ascending sequence on them, that is definable as a map in the category, has a lub w.r.t. the intrinsic order.

When restricting the attention to *N-complete Σper over Kleene's pca* (2.1.4.(1)) it is possible to prove a remarkable generalization¹² of a theorem in Myhill&Shepherdson[55] asserting that all the maps preserve lubs of chains definable in the category. This will arise as a corollary of the fact that for any per A $\Sigma(A)$ is a collection of (N-)Scott's opens. The line of the proof is that of Rosolini[86] with the warning that there one works in the internal logic of the effective topos.

The remarkable application of this fact is that the full subcategory of N-complete, separated pers can be seen as a sort of *pre-O-category* (Wand[79], Smyth&Plotkin[82]) in that the morphisms are partially ordered, there are lubs' of (N-)chains, and the operation of composition preserves this structure.

3.1 N-complete Σper ($c_N\Sigma\text{per}$)

In this section we work over an arbitrary pca D , however for getting the main results (3.2) additional assumptions on D seem necessary. When stating the condition of completeness for a Σper A we will only be interested in the existence of the lowest upper bounds (lubs) of the chains, $\chi: N \rightarrow A$, that are definable as maps from the nno N to A , briefly we will write $\chi: AS(A)$ (AS for ascending sequence). Let us observe at once the *existential content* of this condition. Under a constructive reading we are not only requiring the existence of the lub of every ascending sequence but also a method to find such lub given a realizer for the sequence. Indeed as soon as we consider the problem of the closure of the collection of N-complete objects w.r.t. the functional space constructor we realize that it is important to have a realizer that uniformly, for every ascending sequence of a given type, computes the lub. This leads to the following definition.

3.1.1 Definition (N-completeness)

We write $\chi: AS(A)$ iff $\chi: N \rightarrow A \wedge \forall n: N. (\chi n \leq_A \chi n+1)$.

$A \Sigma\text{per}$ A is N-complete iff $\forall \chi: AS(A). \exists \text{lub}_A \chi$,

where the existence of lub has to be interpreted constructively, that is:

(a) $\forall \chi: AS(A). \exists x: A. (\forall n: N. (\chi n \leq_A x) \wedge \forall y: A. (\forall n: N. (\chi n \leq_A y) \Rightarrow x \leq_A y))$ and

(b) $\exists \sigma_A: AS(A) \rightarrow A. \forall \chi: AS(A). \sigma_A(\chi) = \text{lub}_A \chi$.

Since the map σ_A , if it exists, is uniquely determined we will simply indicate with A rather than with (A, σ_A) a N-complete Σper (henceforth $c_N\Sigma\text{per}$).

Notes

(1) Whenever we select a subset of the equivalence classes of a $(\Sigma)\text{per}$ we can naturally

¹¹ More precisely a natural number object (nno) is a diagram $1 \rightarrow^0 N \rightarrow^s N$ that is initial among all diagrams of the shape: $1 \rightarrow^x A \rightarrow^f A$.

¹² See note 3.2.6.(1) for a rephrasing of the original statement of the theorem.

consider it as a $(\Sigma)\text{per}$. E.g. $AS(A)$ is a subset of $D/(N \rightarrow A)$ and $U \in \Sigma(A)$ is a subset of D/A .

(2) Given N it is fairly easy to show that $c_N\Sigma\text{per}$ is a non-trivial category. For example every separated object A in which all elements are incomparable is going to be in $c_N\Sigma\text{per}$ as one can define $\sigma_A \triangleq \lambda\chi:AS(A). \chi(0)$ where 0 is the zero of the nno (every ascending sequence on a flat object is constant).

3.1.2 Definition (Scott open)

Let A be a per. A subset U of D/A is a Scott open and we write $U \in \tau(A)$ iff

(1) $\forall x, y: A. (x \in U \wedge x \leq_A y \Rightarrow y \in U)$. (2) $\forall \chi: AS(A). (\exists \text{lub}_A \chi: U \Rightarrow \exists n: N. (\chi n: U))$.

Note that this definition makes sense in any preorder.

3.2 Realized maps are N-continuous

It is immediate to check that $\tau(A)$ defines a topology over A . Before stating the main result we need to look closer at the nno.

3.2.1 Natural number object in a generic pca.

Denote with $\underline{n}$ the term corresponding to the numeral n in a pca, say the compilation in a pca of Church numerals. Let n^D be the interpretation of the numeral in the pca D . We set:

$$N_{CN} \triangleq \{ \{ n^D \} \mid n \in \omega \}$$

In a pca all the basic arithmetic operations are definable on the numerals. As a consequence of this fact if D is non-trivial and $n \neq m$ then $n^D \neq m^D$. Moreover $N_{CN} \in \Sigma\text{per}$. Finally one can show that N_{CN} is a nno in the category.

3.2.2 Natural number object in $(\omega, \cdot)$.

When working in $(\omega, \cdot)$ we can define as nno:

$$N_\omega \triangleq \{ \{ n \} \mid n \in \omega \}$$

In the following we will use the fact that $\{\{n\} \mid n \in \omega\}$ does not belong to $\Sigma(N)$. In a generic pca having all the tools of arithmetic we can still express concepts like “the n -th turing machine applied at input m converges in at least k -steps” however the problem is that it can still be the case that $\{\{n\} \mid n \in \omega\}$ belongs to $\Sigma(N)$. This is not surprising as the operation of application can be completely unrelated to the gödel-numbering. For example consider the pca satisfying in the category of directed complete partial orders and partial continuous maps: $\omega + (D \rightarrow D) \triangleleft D$, where ω are the natural numbers with the flat ordering. Then $N_D \triangleq \{ \{ d \} \mid d \text{ is the image of } \omega \text{ via embedding} \}$ is a nno, however all its subsets are still semi-computable as they are Scott's open. The following theorem shows that if $A \in c_N\Sigma\text{per}_\omega$ then the semi-computable predicates on A are Scott's open.

3.2.3 Theorem ($\Sigma(A) \subseteq \tau(a)$)

If $A \in c_N\Sigma\text{per}_\omega$ and $U \in \Sigma(A)$ then $U \in \tau(A)$.

Proof

The first condition of upward closure follows by the definition of intrinsic order. Take $x:U$ and suppose $x \leq_A y$ then $y:U$ as $x \leq_A y$ iff $\forall U \in \Sigma(A). (x:U \Rightarrow y:U)$.

The proof of the second condition takes advantage of the specific recursion-theoretic character of the structures we have built. In this sense the following proof is the kernel of this approach to the development of the theory. In the following: $N \equiv N_\omega$.

Consider the set $K \triangleq \{\{n\} \mid n \in \omega\} \subseteq D/N$. Observe $K \in \Sigma(N)$ and $K^c \notin \Sigma(N)$, where K^c is K complement w.r.t. N . Consider the predicate $nn \downarrow i$ meaning that the computation nn of the n -th machine applied to the input n will stop in at most i steps. This is a decidable predicate.

Now let us proceed by *contradiction* assuming there is $\chi: AS(A)$ such that:

$\exists \text{lub}_A \chi: U$ and $\forall n: N. \sim(\chi n: U)$.

The crucial idea is to build a map $h: N \rightarrow A$ such that, under the condition $\forall n: N. \sim(\chi n: U)$, maps K in $\{\chi n \mid n: N\}$ and K^c in $\text{lub}_A \chi$. Next one uses the pullback condition to conclude that $h^{-1}(U) = K^c \in \Sigma(N)$.

We now define a family of chains $c(n, i): N \rightarrow (N \rightarrow A)$. Let $\mu k \leq i. nn \downarrow k$ be the least element k less than i s.t. nn converges in k steps. Define

$c(n, i) \triangleq \text{if } \sim(nn \downarrow i) \text{ then } \chi_i \text{ else } \chi(\mu k \leq i. nn \downarrow k)$

Observe that for any given n if $n \in K$ then $c(n, i)$ coincides with the ascending sequence χ up to the first k s.t. $nn \downarrow k$ and then becomes definitely constant; on the other hand if $n \in K^c$ then $c(n, i)$ coincides with χ .

We use the existence of a map that uniformly realizes the lub of ascending sequences to define the map $h: N \rightarrow A$ as follows:

$h(n) \triangleq \sigma_A(\lambda i. c(n, i))$.

As we have just observed $h(n) = \text{lub}_A \chi$ if $n \in K^c$ and $h(n) \in \{\chi n \mid n: N\}$ o.w. .

From this we can conclude $h^{-1}(U) = K^c \in \Sigma(N)$ obtaining the desired contradiction.

For the reader logically inclined let us mention how this proof by contradiction can be turned into a "constructive proof" via Markov Principle that is stated as follows (see Troelstra & van Dalen [88], vol. I):

$\forall n: N. (Rn \vee \neg Rn) \wedge \neg \neg \exists n: N. (Rn) \rightarrow \exists n: N. (Rn)$.

This is often rephrased as follows: if we have an effective way of testing the predicate R and if the non-existence of an element satisfying R leads to an absurd then there is an element satisfying R (that can be effectively found by an enumeration technique). If the predicate is just semi-decidable like in the case $\chi n: U$ then the principle can be easily adapted. In our case we proved $\neg \neg \exists n: N. (\chi n: U)$ from which we can conclude $\exists n: N. (\chi n: U)$. $\square$

3.2.4 Definition (*N-chains preservation*)

Let $A, B \in \Sigma_{\text{per}}$. Then we say that $f: A \rightarrow B$ preserves *N-chains* iff

$\forall \chi: AS(A). (\exists \text{lub}_A \chi: A \Rightarrow (\exists \text{lub}_B f\chi: B \wedge f(\text{lub}_A \chi) = \text{lub}_B f\chi))$.

3.2.5 Proposition (*Realized maps are Scott continuous*)

Suppose $A, B \in c_N \Sigma_{\text{per}} \omega$.

(1) Any map $f: A \rightarrow B$ preserves *N-chains*.

(2) If $f: A \rightarrow B$ preserves *N-chains* then it is (*N*-)Scott continuous.

Proof

(1) Consider $\chi: AS(A)$ and assume $\exists \text{lub}_A \chi: A$. In order to show $\exists \text{lub}_B f\chi$ we prove that for any majorant $y: B$ of $f\chi: AS(B)$ we have $f(\text{lub}_A \chi) \leq_B y$.

We recall that $f(\text{lub}_A \chi) \leq_B y$ iff $\forall U \in \Sigma(B). (f(\text{lub}_A \chi): U \Rightarrow y: U)$. Now $U \in \Sigma(B)$ implies by the pullback condition of admissible domains: $f^{-1}(U) \in \Sigma(A)$ i.e. by the previous theorem $f^{-1}(U) \in \tau(A)$. Since $f(\text{lub}_A \chi): U$ we have $\text{lub}_A \chi: f^{-1}(U)$ that implies by the definition of open set $\exists n: N. (\chi n: f^{-1}(U))$. Therefore $\exists n: N. (f\chi n: U)$ and this implies $y: U$.

(2) As usual take $U \in \tau(B)$ and consider $f^{-1}(U)$. This is upward closed by the fact that f is monotone. Moreover consider $\chi: AS(A)$ and suppose $\exists \text{lub}_A \chi: f^{-1}(U)$. Then by hypothesis $f(\text{lub}_A \chi) = \text{lub}_B f\chi: U$. Therefore $\exists n: N. (f\chi n: U)$ i.e. $\chi n: f^{-1}(U)$. $\square$

3.2.6 Notes

(1) *On Myhill and Shepherdson theorem.* Suppose $\{E_n\}_{n \in \omega}$ is an enumeration of finite subsets of ω and $<, >: \omega \times \omega \rightarrow \omega$ is a pairing map. Given $X, Y \in 2^\omega$ define:

$X \cdot Y \triangleq \{m \mid \exists n. (<n, m> \in X \wedge E_n \subseteq Y)\}$

The reader may recognize in this definition the notion of application that arises in graph

models. Denote with RE the collection of recursively enumerable sets. Every $W \in RE$ determines a functional $F_W: RE \rightarrow RE$ that is defined as:

$$F_W(X) \triangleq W \cdot X$$

Such functionals are called *enumeration operators* (or also recursive functionals). It is immediate to check that these functionals are continuous w.r.t. the order given by set-containment.

Next observe that RE can be seen as a totally enumerated set and therefore as a total equivalence relation. In particular it can be identified with $Id_\omega \rightarrow 1$ where: $nId_\omega m \Leftrightarrow n=m$ and $\forall n, m. n1m$. Note that the intrinsic order on $Id_\omega \rightarrow 1$ coincides with the one given by set-containment.

Myhill-Shepherdson theorem can then be stated as follows:

$$F \in per_\omega[RE, RE] \Leftrightarrow F \text{ is an enumeration operator}$$

This was intended as a characterization of certain “type-2” functionals which are called *effective operations* and which correspond exactly to the morphisms in $per_\omega[RE, RE]$. The interesting part of the proof consists in showing that the effective operations are enumeration operators and, therefore, are continuous. This is the sense that is generalized in 3.2.5.(2).

(2) *On the Effectivity of the Interpretation.* Consider the definition of the exponential: $fA^A g \Leftrightarrow \forall n, m. nAm \Rightarrow fnAgm$. Since A occurs both positively and negatively it is immediate to observe that the logical complexity of A^A is greater than that of A . In fact it is easy to build examples in which starting with a decidable relation, say Id_ω , and iterating the exponentiation operation one can reach any level of the arithmetic hierarchy. Therefore the problem arises of the “effectivity” of this class of models.¹³ It may be the case that the category considered here is too large and that further conditions of “algebraicity” have to be imposed.

4. Σ -connection and Extensionality

In 4.1, following Phoa[90], we introduce the notion of Σ -connected object. It turns out that on Σ -connected objects the intrinsic order on a product coincides with the pointwise order. Moreover every $c_N \Sigma per$ over $(\omega, \cdot)$ is Σ -connected. Hence we can regard the intrinsic order of products existing in $c_N \Sigma per_\omega$ as pointwise.

The quest for a realizer that, uniformly, for every object and for every ascending sequence on that object, computes the lub can be seen as the principal motivation for the invention of the full subcategory of extensional pers ($exper$), proposed in Freyd&al.[90] (4.2). It turns out that $exper_D$ is reflective in Σper_D and that, moreover, the category of N -complete extensional pers over $(\omega, \cdot)$ ($c_N exper_\omega$) is reflective in $exper_\omega$ (4.3). Finally we show that there is an $exper_\omega$ that is not isomorphic to any Σper_ω (4.4).

4.1 Intrinsic Ordering vs. Pointwise Ordering

Let $x, y: \prod_{i \in I} A_i$ be two points of an indexed product in per_D .

We say that x is *pointwise-less* than y iff $\forall i \in I. \pi_i x \leq_{A_i} \pi_i y$.

Observe that if $x \leq y$ in the *intrinsic* order then, by the monotonicity of the projections, x is *pointwise-less* than y . Under which conditions does the point-wise order imply the intrinsic order?

Let us start by re-examining the properties of Σ . We recall (see 2.2.4) that if we take $1 = D \times D$ then $\Sigma 1 \rightarrow 1$ is composed of two equivalence classes that of total elements and that of always divergent elements. We write $D/\Sigma = \{1, \top\}$. It is easy to see that $\neg(\perp \leq_\Sigma \top)$ (hence Σ is separated). In the following we assume to work over $(\omega, \cdot)$. Then it is also possible to show:

¹³ Compare the situation with what happens in standard domain theory. There the interpretations can be effectively enumerated and the notions of approximation and equivalence of two terms are in Π_2^0 .

$\perp \leq_{\Sigma} \top$ (see 4.4.1 for a recursion-theoretic argument).

Observe that given $x, y: A$, if there is $f: \Sigma \rightarrow A$ such that $f\perp = x$ and $f\top = y$ then we can conclude $x \leq_A y$ by the monotonicity of f . Moreover such map f , if it exists, is unique. The idea is to show that for $x, y: \prod_{i \in I} A_i$ in the relation given by the pointwise order we can construct a map f that proves that they are in the relation of the intrinsic order.

4.1.1 Definition (Σ -connection)¹⁴

We say that A is Σ -connected iff $\forall x, y: A. (x \leq_A y \Rightarrow \exists f: \Sigma \rightarrow A. (f\perp = x \wedge f\top = y))$.

It has been observed by W. Phoa that Σ -connection is a property that enjoys good closure properties and forces the coincidence of pointwise and intrinsic orders on products.

4.1.2 Proposition (Σ -connection is preserved by arbitrary products)

If $\forall i \in I A_i$ is Σ -connected and if it exists the product, $\prod_{i \in I} A_i$, then such product is Σ -connected. Moreover if $D \equiv (\omega, \cdot)$ then the intrinsic order coincides with the pointwise order.

Proof

First an elementary remark: if $x, y: B \rightarrow \prod_{i \in I} A_i$ and $\forall i \in I. \pi_i x = \pi_i y$ then $x = y$. Next consider $x, y: \prod_{i \in I} A_i$ such that $\forall i \in I. \pi_i x \leq_{A_i} \pi_i y$. By hyp. $\forall i \in I. \exists f_i: \Sigma \rightarrow A_i. (f_i\perp = \pi_i x \wedge f_i\top = \pi_i y)$. By def. of product we have a map $\langle f_i \rangle: \Sigma \rightarrow \prod_{i \in I} A_i$ such that $\pi_i \langle f_i \rangle = f_i$. But this forces: $\forall i \in I. (\pi_i \langle f_i \rangle \perp = f_i\perp = \pi_i x \wedge \pi_i \langle f_i \rangle \top = f_i\top = \pi_i y)$. By the initial remark: $\langle f_i \rangle \perp = x \wedge \langle f_i \rangle \top = y$. Hence the product is Σ -connected. Finally observe that in this proof we only used the fact that x is pointwise less than y , hence the map $\langle f_i \rangle$ proves that $x \leq_A y$. $\square$

4.1.3 Corollary

Let A, B be pers over a pca D , and Σ the dominance.

(1) Σ and Σ^B are Σ -connected.

(2) If B is Σ -connected and $D/A \subseteq D/B$ then A is Σ -connected. Moreover the intrinsic order on A coincides with the restriction of the intrinsic order on B .

Proof

(1) That Σ is connected is obvious. Σ^B can be seen as a degenerate case of product, therefore we can apply 4.1.2.

(2) In general if $x \leq_A y$ then $x \leq_{B^y}$, indeed every map $f: B \rightarrow \Sigma$ can be restricted to a map $f': A \rightarrow \Sigma$. Vice versa take $x, y: A$. If $x \leq_{B^y}$ then we can consider the connecting map $f: \Sigma \rightarrow B$ such that $f\perp = x \wedge f\top = y$. Now the related connecting map $f': \Sigma \rightarrow A$ obtained by restricting the codomain of f proves that $x \leq_A y$. $\square$

In the case $D \equiv (\omega, \cdot)$ one can prove that every N -complete Σ -per is Σ -connected.

4.1.4 Proposition (every $c_N \Sigma \text{per}_{\omega}$ is Σ -connected)

Let $D \equiv (\omega, \cdot)$. If A is in $c_N \Sigma \text{per}_{\omega}$ then A is Σ -connected.

Proof

The proof does not use at all the fact that we have lubs of arbitrary chains but rather the less obvious fact that lubs can be effectively computed. Let $x \leq_A y$ and consider the following family of chains:

$$c(n, i) \triangleq \text{if } n \downarrow i \text{ then } y \text{ else } x$$

Fixed n the ascending sequence $\lambda i. c(n, i)$ is apparently innocuous because it can take at most two values, however in general we have no tool to compute the lub as we do not know a priori how far in the sequence to look.

We use the existence of the map $\sigma_A: AS(A) \rightarrow A$ to define a map $h: \{K^c, K\} \rightarrow A$ such that:

¹⁴This is the "external analogous" of Phoa's notion of linked Σ -space. Warning: there is no direct relation to the topological notion of connection.

$h(K^c) = x, h(K) = y$ (K and K^c are defined as in 3.2.3). Next observe that it is possible to define an iso realized by the identity: $h: \Sigma \rightarrow \{K^c, K\}$. Hence by composition we obtain the desired map. $\square$

4.1.5 Corollary (Closure properties)

$c_N \Sigma_{\text{per}}_\omega$ is cartesian closed and it has all finite limits.

Proof

Use the fact that the intrinsic order is pointwise to define the lub of a chain of maps. One can verify that equalizers preserve N-completeness. $\square$

4.2 Extensional PER (exper)

In this section, following work by Freyd&al.[90], we introduce the notion of *extensional per* (exper). Extensional pers are Σ -connected and they form a full reflective subcategory of Σpers .

4.2.1 Intuition

What is a per? The question may seem rethoric in that our intuition so far has been that pers are data types. The data in the per A are preordered by a collection of legal observations or tests that are represented by the maps from A into the dominance Σ . We would like to emphasize this point of view by setting:

$$\text{Test}(A) \triangleq A \rightarrow \Sigma.$$

On the other hand we could ask if a per can be seen as a collection of tests on a data type (i.e. a per). We will call such a per *extensional*. Towards this purpose define a per of *invariants* for A , say $\text{Inv}(A)$, as follows:

$$d\text{Inv}(A)e \Leftrightarrow \forall f, g. fAg \Rightarrow fd \cong ge$$

The intuition is that $\text{Inv}(A)$ is the greatest per for which A can be seen as a collection of functions on it. The following proposition supports this idea. We will take the assumption of 4.2.2.(4) as definition of exper (4.2.3).

4.2.2 Proposition (on Test and Inv)

Let A be a per over a pca D . Then:

- (1) $A \subseteq \text{Inv}(\text{Test}(A))$.
- (2) $A \subseteq \text{Test}(\text{Inv}(A))$.
- (3) If $D/A \subseteq D/\text{Inv}(\text{Test}(A))$ then A is separated.
- (4) If $D/A \subseteq D/(B \rightarrow \Sigma)$ then $D/A \subseteq D/\text{Test}(\text{Inv}(A))$.

Proof

(1) Note: $d\text{Inv}(B)e \Leftrightarrow \lambda u. ud (B \rightarrow \Sigma) \lambda u. ue$. Moreover:

$dAe \Rightarrow \lambda u. ud (A \rightarrow \Sigma) \rightarrow \Sigma \lambda u. ue$, hence by instantiating B with $A \rightarrow \Sigma$ we get $A \subseteq \text{Inv}(\text{Test}(A))$.

(2) By definition $f \text{Test}(\text{Inv}(A)) g$ iff $f \text{Inv}(A) \rightarrow \Sigma g$. But suppose fAg and $d\text{Inv}(A)e$ then, by definition of $\text{Inv}(A)$, $fd \cong ge$.

(3) In general if $A \subseteq B$ then $D/A \subseteq D/B$ iff $\forall d \in |A|. (dBe \Rightarrow dAe)$. $A \in \Sigma\text{per}$ iff $\forall d, ee \in |A|. (\lambda u. ud (A \rightarrow \Sigma) \rightarrow \Sigma \lambda u. ue \Rightarrow dAe)$. But by hyp.

$\forall d \in |A|. d\text{Inv}(\text{Test}(A))e \Rightarrow dAe$, and $d\text{Inv}(\text{Test}(A))e \Leftrightarrow \lambda u. ud (A \rightarrow \Sigma) \rightarrow \Sigma \lambda u. ue$.

(4) First observe that if $D/A \subseteq D/(B \rightarrow \Sigma)$ then $B \subseteq \text{Inv}(A)$. From which follows: $\text{Test}(\text{Inv}(A)) \equiv \text{Inv}(A) \rightarrow \Sigma \subseteq (B \rightarrow \Sigma)$. Together with (2) we obtain: $A \subseteq \text{Test}(\text{Inv}(A)) \subseteq (B \rightarrow \Sigma)$. Finally observe: $\forall d \in |A|. d\text{Test}(\text{Inv}(A))e \Rightarrow d(B \rightarrow \Sigma)e \Rightarrow dAe$. $\square$

4.2.3 Definition (exper)

A per is *extensional* iff there is a per B such that: $D/A \subseteq D/(B \rightarrow \Sigma)$.

4.2.4 Proposition (basic properties of *exper*)

In the following let A, B be pers over a pca D , and Σ the dominance.

- (1) If A is extensional then A is separated and Σ -connected.
- (2) If A is extensional, say $D/A \subseteq D/(B \rightarrow \Sigma)$, then the intrinsic order on A coincides with the restriction of the intrinsic order on $B \rightarrow \Sigma$ which, if $D \equiv (\omega, \cdot)$, coincides with the pointwise order. That is, given A *exper*, $f, g: A$ we have:

$$f \leq_A g \Leftrightarrow f \leq_{B \rightarrow \Sigma} g \Leftrightarrow \forall b: B. fb \leq_{\Sigma} gb.$$
- (3) If B is extensional and $D/A \subseteq D/B$ then A is extensional.
- (4) *exper* is closed under arbitrary intersections.

Proof

(1) From 1.2.4 we know that Σ^B is separated and Σ per is closed under subobjects. From 4.1.3(1) we know that Σ^B is Σ -connected and that the Σ -connected objects are closed under subobjects obtained by selecting a subset of the quotient space.

(2) $f \leq_A g \Leftrightarrow f \leq_{B \rightarrow \Sigma} g$ follows from 4.1.3.(2). $f \leq_{B \rightarrow \Sigma} g \Leftrightarrow \forall b: B. fb \leq_{\Sigma} gb$ follows from 4.1.2.

(3) Immediate.

(4) Suppose $D/A_i \subseteq D/(B_i \rightarrow \Sigma)$ for $i \in I$. Then $D/\bigcap_{i \in I} A_i \subseteq D/(\bigcup_{i \in I} B_i \rightarrow \Sigma)$. $\square$

4.2.5 Remark (comparison with Freyd&al.[90])

The notion of extensional per in Freyd&al.[90], say *ex'*, can be stated as follows: A is an *ex'*per iff there is a set $X \subseteq D$ such that:

$$D/A \subseteq D/(\text{Diag}(X) \rightarrow \Sigma) \quad \text{where } d \text{Diag}(X)e \text{ iff } d = e \in X.$$

Notice that $(\text{Diag}(X) \rightarrow \Sigma)$ is always a *total* equivalence relation. However the two definitions are equivalent. Obviously any *ex'*per is an *exper*. For the vice versa observe that $D/(B \rightarrow 1) \subseteq D/(\text{Diag}(|B|) \rightarrow 1)$.

4.2.6 Theorem ($\text{exper}_D \subsetneq \Sigma\text{per}_D$)

Let D be a pca, then exper_D is a full and reflective subcategory of Σper_D .

Proof (after Freyd&al.[90])

The existence of an inclusion functor follows from 4.2.4.(1). We define the reflector

$L_{\text{ex}}: (\Sigma)\text{per} \rightarrow \text{exper}$ as follows: $D/L_{\text{ex}}(A) \triangleq \{[\lambda u. \text{ud}]_{(A \rightarrow \Sigma) \rightarrow \Sigma} / d \in |A|\}$.

This is an *exper* as $D/L_{\text{ex}}(A) \subseteq (A \rightarrow \Sigma) \rightarrow \Sigma$. The universal map $e_A: A \rightarrow L_{\text{ex}}(A)$ is the one realized by $\lambda u. \text{ud}$. Intuitively it takes an element d to the collection of its neighborhoods $\lambda u. \text{ud}$. By construction e_A is an *epi*, moreover it is also a *mono* if A is separated. In general L_{ex} can also work as a reflection from *per* to *exper*.

Next we intend to show that if B is extensional then $e_B: B \rightarrow L_{\text{ex}}(B)$ is an *iso*. Suppose: $D/B \subseteq D/(C \rightarrow \Sigma)$. We will take $e_B^{-1}: L_{\text{ex}}(B) \rightarrow B$ as the map realized by: $\lambda i. \lambda c. i(\lambda u. \text{uc})$. Let us try to assign a type to this term. First check that we can write:

$$\lambda i (B \rightarrow \Sigma) \rightarrow \Sigma. \lambda c C. i(\lambda u B. \text{uc}) : ((B \rightarrow \Sigma) \rightarrow \Sigma) \rightarrow (C \rightarrow \Sigma).$$

Since $B \subseteq (C \rightarrow \Sigma)$ in the application we can “coerce” u from B to $C \rightarrow \Sigma$. Since $L_{\text{ex}}(B) \subseteq (B \rightarrow \Sigma) \rightarrow \Sigma$ we can also assign to the same term the type $L_{\text{ex}}(B) \rightarrow (C \rightarrow \Sigma)$. Finally we derive the type-assignment: $\lambda i. \lambda c. i(\lambda u. \text{uc}): L_{\text{ex}}(B) \rightarrow B$. Recalling the definition of $L_{\text{ex}}(B)$ we know for $f, f' \in |B|$:

$$fbf \Leftrightarrow \lambda v. vf \ L_{\text{ex}}(B) \ \lambda v. vf'.$$

Assume: $\lambda v. vf \ L_{\text{ex}}(B) \ \lambda v. vf'$ and cCc' . Since $B \subseteq (C \rightarrow \Sigma)$ we derive from the former: $\lambda v. vf \ ((C \rightarrow \Sigma) \rightarrow \Sigma) \rightarrow \Sigma \ \lambda v. vf'$. From the latter follows:

$\lambda u. \text{uc} \ (C \rightarrow \Sigma) \rightarrow \Sigma \ \lambda u. \text{uc}'$. Hence after application we reduce the problem to $fc \equiv f'c'$, that holds by the assumption fbf .

As for showing *inversions*:

$(\lambda i.\lambda c.i(\lambda u.uc))((\lambda dw.wd) f) = (\lambda i.\lambda c.i(\lambda u.uc))(\lambda w.wf) = \lambda c.(\lambda w.wf)(\lambda u.uc) = \lambda c.fc.$
 Vice versa: $(\lambda dw.wd)((\lambda i.\lambda c.i(\lambda u.uc))(\lambda v.vf)) = (\lambda dw.wd)(\lambda c.(\lambda v.vf)(\lambda u.uc)) =$
 $= (\lambda dw.wd)(\lambda c.fc) = (\lambda w.w(\lambda c.fc)).$ And one observes that $\lambda c.fc \subseteq \Sigma$ f.

It is usual to define a functor from $(A \rightarrow \Sigma) \rightarrow \Sigma$ to $(B \rightarrow \Sigma) \rightarrow \Sigma$. Given $\phi \in |A \rightarrow B|$ follow this pattern to define $\phi' \in |L_{ex}(A) \rightarrow L_{ex}(B)|$ as follows: $\phi' \triangleq \lambda i.\lambda u.i(\lambda a.u(\phi a))$. If $f = [\phi]_A \rightarrow_B$ set $L_{ex}(f) \triangleq [\phi']_{L_{ex}(A) \rightarrow L_{ex}(B)}$. $\square$

4.3 Reflection of N-complete exper ($c_N\text{exper}$)

Suppose A extensional and f, g: A. From 4.2.4.(4) we know that we can look at f and g as functions on some domain so that $f \leq_A g$ iff the domain of definition of f is smaller than that of g. This resembles very much the usual order on recursively enumerable sets seen as domains of partial recursive functions. We may then remember that on such orders we know how to compute effectively the lub of an effectively given chain; namely we return the function that on a given input converges exactly when one of the elements of the chain converges. This analogy leads to the following

4.3.1 Theorem ($c_N\text{exper}_\omega \subseteq \text{exper}_\omega$)

(1) There is a realizer that uniformly for every N-complete extensional per computes the lub of ascending sequences:

$$\exists \sigma \in \omega. \forall A \in c_N\text{exper}_\omega. \forall \chi: AS(A). \forall \phi \vdash_A \chi. \sigma \phi \vdash_A \text{lub}_A \chi$$

(2) $c_N\text{exper}_\omega$ is closed under arbitrary intersections.

(3) $c_N\text{exper}_\omega$ is *reflective* in exper .

Proof (after Freyd&al.[90])

(1) Suppose A extensional i.e. $D/A \subseteq D/B \rightarrow \Sigma$, for some B. Let $\chi: AS(A)$, then we have: $\forall n: N. \chi(n) \leq_A \chi(n+1)$ that is: $\forall n: N. \forall b: B. \chi(n)(b) \leq \chi(n+1)(b)$.

Now consider the map: $Y_A \triangleq \lambda \chi: AS(A). \lambda b: B. \text{if } (\exists n: N. \chi(n)(b) \downarrow) \text{ then } \downarrow$.

It is fairly obvious how to find a realizer for this map, namely:

$$\sigma \triangleq \lambda \gamma. \lambda d. \text{if } (\exists n. \gamma(n)(d) \downarrow) \text{ then } \downarrow$$

Given the existence in the model of Y_A it is easy to check that $Y_A(\chi) = \text{lub}_A \chi$. Moreover one observes that σ does not depend on A.

(2) In 4.2.4.(4) we showed that exper is closed under intersections. In particular suppose: $D/A_i \subseteq D/(B_i \rightarrow \Sigma)$ for $i \in I$. Then: $D/\bigcap_{i \in I} A_i \subseteq D/(\bigcup_{i \in I} B_i \rightarrow \Sigma)$. Next consider $\chi: AS(\bigcap_{i \in I} A_i)$, we have: $\forall i \in I. \forall n: N. \forall b: B_i. \chi(n)(b) \leq \chi(n+1)(b)$. Suppose $\gamma \vdash \chi$ then $\sigma \gamma \vdash \text{lub}_{A_i} \chi$. From this one concludes: $\sigma \gamma \vdash \text{lub}_A \chi$.

(3) One defines a reflector $L_{cN}: \text{exper} \rightarrow c_N\text{exper}$. First a remark:
 $\forall i \in I. A_i \in c_N\text{exper} \Rightarrow \exists \prod_i A_i \in \text{exper} \Rightarrow \prod_i A_i$ has pointwise order $\Rightarrow$
 $\Rightarrow$ using σ we can find the lub of any ascending sequence on $\prod_i A_i \Rightarrow \prod_i A_i \in c_N\text{exper}$.
 Hence $c_N\text{exper}$ is closed under products.

Suppose $D/A \subseteq D/(B \rightarrow \Sigma)$. Define $L_{cN}(A)$ as the least $c_N\text{exper}$ such that:

$$D/A \subseteq D/L_{cN}(A) \subseteq D/(B \rightarrow \Sigma).$$

This is well defined because $B \rightarrow \Sigma$ is an N-complete exper . In other terms A is a partial order that is the restriction of the complete order $B \rightarrow \Sigma$. If we have a chain in A whose lub is not in A, but only in $B \rightarrow \Sigma$, then we add such point. The behavior of a map on such points is determined by A hence given $\phi \vdash f: A \rightarrow B$, A exper and B $c_N\text{exper}$ one takes:

$$L_{cN}(f) \triangleq [\phi]_{L_{cN}(A) \rightarrow B} \quad \square$$

4.4 Separation vs. Extensionality

In Phoa[90] it is observed that there is a Σper_ω that is not Σ -connected. Since the property of being Σ -connected is closed under isomorphism and since every extensional per is Σ -connected we conclude that there is a Σper_ω that is not isomorphic to any exper . In

particular this implies that the inclusion functor from exper_ω to Σper_ω is not an equivalence. The proof points-out (yet another) surprising property of the structure: there can be two separated pers $A = \{K^c, K\}$ and $B = \{X^c, X\}$ each one with two points so that $K^c \leq_A K$, $X^c \leq_B X$ and the only maps from A to B are the constant ones. That is there can be two domains with the same order as Sierpinski space that are not isomorphic !

We recall some basic notions of recursion theory (see, e.g., Soare[87]). Let us denote with X, Y subsets of ω , and with X^c, Y^c their complements. X is *m-reducible* to Y , written, $X \leq_m Y$ iff $\exists f$ total recursive. $(x \in X \Leftrightarrow f(x) \in Y)$. The set $K \triangleq \{n \mid n \downarrow\}$ is $\leq_m$ -complete. Moreover there is a set X that is not m-complete (see Soare[87], V.1.3). From this one proves the following list of elementary facts:

4.4.1 Proposition

- (1) If X is r.e., non-recursive, then $A \triangleq \{X^c, X\}$ is a Σper_ω where $X^c \leq_A X$.
- (2) $\{K^c, K\} \cong \{\perp, \top\} \cong \Sigma$.
- (3) Let $X, Y \subseteq \omega$. $X \leq_m Y$ iff $\exists h: \{X^c, X\} \rightarrow \{Y^c, Y\}$ mono such that $h(X) = Y$.
- (4) For any X r.e. there is $h: \{X^c, X\} \rightarrow \{K^c, K\}$ mono such that $h(X) = K$.
- (5) There is an X r.e. such that $\{X^c, X\}$ is not Σ -connected.
- (6) There is a Σper_ω that is not isomorphic to any exper_ω .

Proof

- (1) A partial map from $\{X^c, X\}$ to the terminal object cannot converge on X^c and diverge on X without contradicting the fact that X^c is not r.e. .
- (2) From $\{K^c, K\}$ to $\{\perp, \top\}$ consider the map realized by $\lambda n. \lambda m. n \cdot n$. In the other direction consider the map realized by the identity.
- (3) Immediate from the definitions.
- (4) From the fact that K is m-complete and (3).
- (5) From the existence, mentioned above, of an r.e. non recursive set to which K is not m-reducible.
- (6) Recall every exper is Σ -connected, Σ -connection is invariant under isomorphisms. $\square$

5. D-completeness and Uniformity

The deeper results obtained so far, namely the fact that computable subsets are Scott's opens (3.4) and the existence of a uniform realizer for the lub of ascending sequences on extensional pers (4.3.1) are based on the realizability structure $(\omega, \cdot)$.

On one hand $(\omega, \cdot)$ is a very fundamental structure as it embodies the classical notion of effective computation. Moreover it is possible to build from it a very interesting model, namely the effective topos, for higher order logic enriched with principles like Church Thesis, Markov Principle, Countable Axiom of Choice, Uniformity Principle, etcetera.

On the other hand $(\omega, \cdot)$ is a highly unstructured model of computation in which we heavily use all the "coding tricks" of elementary recursion theory as exemplified by the proof of 3.4. One may wonder if it is not more reasonable towards the aim of modelling the high level constructs of programming languages to consider a more structured pca in which the basic properties of computability are abstracted in terms of order-theoretic notions and the (first order) data types have a direct and explicit correspondence at the level of the realizability structure. Before going into a formalization of this approach we should point-out that it is not clear if, in this case, an internal approach could be carried out successfully.

5.1 D-complete, Σper ($c_D\Sigma\text{per}$)

We introduce the notion of D-completeness which refers to the refinement pre-order of the pca (2.1.3.(2)) rather than to the intrinsic pre-order.

5.1.1 Assumption on the pca: D_{dcpo}

Assume that $D \rightarrow D \triangleleft D$ in the category of directed complete partial orders and partial continuous maps (dcpo). Then $(D, \leq_D)$ is a directed complete partial order. In the following, if confusion may arise, we refer to any structure built in this way as D_{dcpo} . As observed in 2.1.4.(2) $\Sigma(D)$ is the collection of Scott opens on D . Given a per A the intrinsic preorder, $\leq_A$, induces a preorder on $|A|$, $\leq_A$, defined as follows: $\forall d, e \in |A|. d \leq_A e \Leftrightarrow [d]_A \leq_A [e]_A$. Equivalently we have: $\forall d, e \in |A|. (d \leq_A e \Leftrightarrow \forall \phi \in |A \rightarrow 1|. \phi d \downarrow \Rightarrow \phi e \downarrow)$.

5.1.2 Definition (D -completeness)

Given a, say, binary relation R on D_{dcpo} , we say that R is D -complete iff $X \subseteq R, X \text{ directed} \Rightarrow \cup_{D \times D} X \in R$.¹⁵

5.1.3 Theorem ($c_D(\Sigma)\text{per}_{D_{\text{dcpo}}} \hookrightarrow (\Sigma)\text{per}_{D_{\text{dcpo}}}$)

The full subcategory of D -complete $(\Sigma)\text{pers}$ over D_{dcpo} is reflective in $(\Sigma)\text{per}$.

Proof

One shows that if $A \in \text{per}$ and $B \in c_D \text{per}$ then $B^A = \underline{B}^A$ where $\underline{A}$ is the least $c_D \text{per}$ containing A . First one gives an inductive definition of $\underline{A}$ and then one can prove $B^A = \underline{B}^A$ by induction (see Amadio[89]). Finally observe that if A is D -complete then $L_\Sigma(A)$ is D -complete (see 2.3.1 for L_Σ definition). $\square$

5.1.4 Remark

We can exploit the continuity properties of the realizability structure in order to interpret a *fixed point combinator on terms*. Suppose $(D, \leq_D)$ has a least element denoted with $\perp_D$. As usual one can define a fixed point combinator on D as: $\text{Fix}_D \triangleq \lambda f. \cup_{n < \omega} f^n(\perp_D)$.

Say that A , per, is *pointed* iff $\perp_D A \perp_D$. Suppose that A is a pointed, D -complete per. Define: $\text{Fix}_A \triangleq [\text{Fix}_D]_A A \rightarrow A$. It is easy to show (Amadio[89]) that Fix_A is a fixed point combinator on A .

Actually if A is separated then Fix_A returns the *least fixed point* w.r.t. the intrinsic order. The proof of this fact follows the standard argument adapted to the intrinsic order.

Also observe that given a per A the preorder $\leq_A$ is D -complete. The proof is immediate from the characterization in 5.1.1. On the other hand we still lack a result stating some form of *continuity* of the realized maps w.r.t. the intrinsic order, a solution to this problem will be given in 5.3.

5.2 Uniform $c_D \Sigma \text{per}$ ($c_D u \Sigma \text{per}$)

Suppose that D_{dcpo} is built out of an inverse limit technique. The domain D is then naturally stratified in levels. Given a per A we can define its approximation at the n -th level $A|_n$. It is then natural to consider a condition of uniformity that guarantees:

$$\forall n. A|_n = B|_n \Rightarrow A = B.$$

To the best of our knowledge this condition first appears, with the name of “completeness under approximation” and quite different purposes, in Coppo[85].

5.2.1 Assumption on the pca: D_ω

Let D be a dcpo for which there is a sequence $\{p_n \mid p_n: D \rightarrow D\}_{n \in \omega}$ such that:

- p_n is a projection: $p_n \circ p_n = p_n \leq \text{id}_D$.
- $\{p_n\}_{n \in \omega}$ is a growing sequence with id as lub: $p_0 = \lambda d. \perp_D$, $p_n \leq p_{n+1}$, $\cup_{n \in \omega} \{p_n\} = \text{id}_D$.

We also assume that the following dcpos can be embedded in the dcpo D by means of the

¹⁵ In the following to avoid confusion we denote with “lub” the lower upper bound in the quotient space and with “ $\cup$ ” the lower upper bound in D .

following *embedding projection pairs* that commute conveniently with the sequence of projections:

$$\begin{array}{lll} \omega & (i_\omega, j_\omega) & p_{n+1} \circ i_\omega = i_\omega \quad (\text{Numbers with flat order}) \\ D \rightarrow D: & (i_\omega, j_\omega) & p_{n+1}(i_\omega(f)) = i_\omega(p_n \circ f \circ p_n) \quad (\text{Partial Functions}) \\ D \times D: & (i_\omega, j_\omega) & p_{n+1} \circ i_\omega = i_\omega \circ \langle p_n, p_n \rangle \quad (\text{Product}) \end{array}$$

In practice such a D can be exhibited by means of an inverse limit construction (Scott[72]), therefore we refer to a D_{dcpo} satisfying all the previous conditions with D_∞ .

5.2.2 Definition (uniformity)

Given a binary relation R on D_∞ , we say that R is *uniform* iff

$$R \neq \emptyset \wedge dRe \Rightarrow \forall n \in \omega. d_n R e_n$$

Convention

We write: $D_n \equiv p_n(D)$, $A|_n \equiv A \cap (D_n \times D_n)$. Note that D_n is uniform and if A is uniform then $A|_n$ is uniform.

5.2.3 Definition (metric on $c_{\text{Duper}} D_\infty$)

Let $A, B \in c_{\text{Duper}} D_\infty$. Define $c(A, B) \triangleq \max\{n \mid A|_n = B|_n\}$, $d(A, B) \triangleq 2^{-c(A, B)}$, where conventionally: $\max \emptyset \triangleq \infty$, and $2^{-\infty} \triangleq 0$.

5.2.4 Facts (on $c_{\text{Du}}(\Sigma)_{\text{per}}$, see Amadio[89])

(1) $(c_{\text{Duper}} D, d)$ is a complete ultra-metric space. (2) The operators of exponentiation and product are contractive. (3) $c_{\text{Duper}} D$ is closed under arbitrary intersections. (4) The full subcategory of c_{Duper} is cartesian closed.

5.2.5 Remark (limits of the uniformity condition)

The category c_{Duper} lacks equalizers and closure under retraction. In particular for equalizers observe that a per composed by the equivalence classes $\{\{d\}, \{e\}\}$ where d and e are incomparable is not isomorphic to any cuper. In fact any cuper containing two incomparable equivalence classes would also include a least equivalence class. The problem of the lack of equalizers appears more serious than the one arising in categories of cpos and continuous maps.

Trying to fix this problem one may consider a notion of uniform map where $[\phi]: A \rightarrow B$ is uniform iff $\phi d_n = (\phi d)_n$. It is easy to check that equalizers of uniform maps are uniform. However uniform maps are strict and therefore do not behave well w.r.t. exponentiation. Another idea is needed.

5.3 $c_{\text{Du}} \Sigma_{\text{per}}$ as domains

Following Abadi&Plotkin[90] we show that by requiring that the range of the projections p_n is finite one can prove that if $A \in c_{\text{Du}} \Sigma_{\text{per}}$ then the quotient space D/A ordered with the intrinsic order is an sfp. Moreover all realized maps are continuous.

5.3.1 Assumption on the pca: D_{sfp}

We impose a further condition on D_∞ , namely we assume that for each n $\text{rg}(p_n)$ is finite. Since this implies that D_∞ is an sfp we refer to it with D_{sfp} . In this section (5.3) D is an abbreviation for D_{sfp} .

After Abadi&Plotkin[90], we characterize the induced preorder, $\preceq_A$ (5.1.1) as the *least D-complete preorder containing* $A \cup \leq_{D|A|}$ (5.3.10). In other terms the intrinsic pre-order is as little as it could possibly be. Moreover $\preceq_A$ turns out to be uniform (5.3.9).

5.3.2 Definition

Let $\prec_A \triangleq \text{TC}(A \cup \leq_{D \mid |A|})$, where $\text{TC} \equiv \text{Transitive Closure}$.

5.3.3 Fact

$\prec_A$ is a uniform preorder on $|A|$.

Proof

Observe that $A \cup \leq_{D \mid |A|}$ is uniform as A is uniform by assumption. Next remark that the operator of transitive closure, TC , preserves uniformity. $\square$

5.3.4 Proposition

Let $d \in D_0$, where D_0 is the collection of compact elements. Then

$W(d) \triangleq \{e \in D \mid \exists e' \in D \cap |A|. d \prec_A e' \leq_D e\} \in \Sigma(D)$ that is a Scott open in D .

Proof

If $d \notin |A|$ then $W(d) = \emptyset$. $W(d)$ is clearly upward closed. Suppose $\{e_i \mid i \in I\}$ directed and such that $\cup\{e_i \mid i \in I\} \in W(d)$, that is: $d \prec_A e' \leq_D \cup\{e_i \mid i \in I\}$. By the uniformity of $\prec_A$ and the compactness of d : $\exists n. d_n = d \prec_A (e')_n \leq_D (\cup\{e_i \mid i \in I\})_n$. By the fact that p_n has finite image: $\exists j. d \prec_A (e')_n \leq_D (e_j)_n \leq_D e_j$. Hence $e_j \in W(d)$. $\square$

5.3.5 Note

Given $d \in D_0$ define $U(d) \triangleq \{[e]_A \mid d \prec_A e\}$. Then $U(d) \in \Sigma(A)$ as $W(d) \cap |A| = |U(d)|$ (see 2.2.1 for def. $\Sigma(A)$).

5.3.6 Corollary

Given d compact then $d \prec_A e$ iff $d \approx_A e$.

Proof

In general $\approx_A$ contains $\prec_A$. Vice versa, *by contradiction*, suppose $d \approx_A e$ and $\neg(d \prec_A e)$. Build the Scott open $W(d) \in \Sigma(D)$ as in prop. 5.3.4 and the semi-computable subset $U(d)$ as in note 5.3.5. Then $[d]_A \in U(d)$ and $[e]_A \notin U(d)$. Contradiction. $\square$

5.3.7 Definition

Let $\prec_{Ac}$ be the least D -complete preorder containing $\prec_A$.

5.3.8 Proposition

$\prec_{Ac}$ coincides with $\approx_A$.

Proof

By 5.1.4 $\approx_A$ is D -complete moreover it contains $\prec_A$ and therefore $\prec_{Ac}$. Vice versa suppose $d \approx_A e$. Then $\forall n. d_n \leq_D d \approx_A e$. Since $\approx_A$ contains $\leq_{D \mid |A|}$ we have $\forall n. d_n \approx_A e$. By 5.3.6 $\forall n. d_n \prec_A e$ and by completeness $d = \cup\{d_n \mid n \in \omega\} \prec_{Ac} e$. $\square$

5.3.9 Proposition

$\prec_{Ac}$ is uniform.

Proof

Give a transfinite inductive definition of the completion process and observe that uniformity is preserved at each step (see also Amadio[89]). $\square$

5.3.10 Corollary

The preorder, $\approx_A$, induced by the intrinsic preorder on $|A|$ is the least D -complete preorder on $|A|$ containing $A \cup \leq_{D \mid |A|}$.

Proof

Immediate from 5.3.8, 5.3.9. $\square$

The interesting fall-out of the characterization of the intrinsic preorder is the following theorem that explicits the domain structure of the quotient space. The proof shows how to extract from a directed set $\{[d_i]_A \mid i \in I\}$ in $(D/A, \leq_A)$ a chain $\{e_n \mid n \in \omega\}$ in D such that $e_n \in D_n \cap |A|$ and $\text{lub}\{[d_i]_A \mid i \in I\} = [\cup\{e_n \mid n \in \omega\}]_A$.

5.3.11 Theorem (The quotient space is a cpo)

Let A be a $c_{Du}\Sigma\text{per}$ then $(D/A, \leq_A)$ is a cpo.

Proof (after Abadi&Plotkin[90])

Clearly under the hypothesis $A \in \Sigma\text{per}$ $(D/A, \leq_A)$ is a partial order with least element $[\perp_D]_A$. Given a directed set, say $\{x_i \mid i \in I\} \triangleq \{[d_i]_A \mid i \in I\}$, we show that it has a lub in $(D/A, \leq_A)$. Given $J' \subseteq J \subseteq I$ we say that J' is *cofinal* with J if

$$\forall i \in J. \exists j \in J'. (d_i \leq_A d_j).$$

Let $X_n \triangleq \{e \mid \forall i \in I. \exists j \in I. (d_i \leq_A d_j \wedge e = (d_j)_n)\}$. In other words $e \in X_n$ if there is a subset J of I , cofinal with I , and such that $\forall j \in J. e = (d_j)_n$.

a. $X_n \subseteq D_n \cap |A|$ is finite since $\text{rg}(p_n)$ is finite.

Assuming the directed set infinite (o.w. there is nothing to prove), again from the finiteness of $\text{rg}(p_n)$, follows that each X_n is non-empty as at least an element will be hit infinitely often.

b. $\forall e \in X_n. \exists e' \in X_{n+1}. e \leq_D e'$.

Proceed by induction on n . If $n=0$ then $e = \perp$ and every e' will do. If $e \in X_n$ then there is a J , cofinal with I , such that $J \subseteq I$ and $\forall j \in J. (d_j)_n = e$. Since $\text{rg}(p_n)$ is finite there is a $J' \subseteq J$, cofinal with J , such that $\forall j \in J'. (d_j)_{n+1} = e'$, for some e' . Observe for $j \in J'$: $e = (d_j)_n \leq_D (d_j)_{n+1} = e'$. Hence pick up e' in X_{n+1} . The result of this process is that we can *choose* a sequence $\{e_n \mid n \in \omega\}$ such that $e_n \in X_n$ and $e_n \leq_D e_{n+1}$.

By D -completeness we have that $\cup\{e_n \mid n \in \omega\} \in |A|$. The claim is:

$$\exists \text{lub}\{[d_i]_A \mid i \in I\} = [\cup\{e_n \mid n \in \omega\}]_A.$$

In the first place let us show: $\forall i \in I. d_i \leq_A \cup\{e_n \mid n \in \omega\}$. By completeness and uniformity it is enough to show that: $\forall i \in I. \forall m \in \omega. (d_i)_m \leq_A \cup\{e_n \mid n \in \omega\}$. Now:

$$\forall i \in I. \forall m \in \omega. \exists j \in I. (d_i \leq_A d_j \wedge (d_j)_m = e_m \wedge e_m \leq_D \cup\{e_n \mid n \in \omega\}),$$

hence by the characterization of $\leq_A$ we can conclude: $(d_i)_m \leq_A (d_j)_m \leq_A \cup\{e_n \mid n \in \omega\}$ that gives the desired assertion.

Finally we prove that: $\forall n \in \omega. \exists i \in I. e_n \leq_A d_i$. For some i , $e_n = (d_i)_n \leq_D d_i$ that implies $e_n \leq_A d_i$. Hence given $[d]_A$ majorant for A it is immediate to show: $\cup\{e_n \mid n \in \omega\} \leq_A d$. $\square$

5.3.12 Corollary (realized maps are continuous)

Let A, B $c_{Du}\Sigma\text{pers}$ and $f: A \rightarrow B$ a morphism in the category. Then f is a continuous map w.r.t. the domain structure of the quotient spaces.

Proof

Take $f: A \rightarrow B$, $\{x_i \mid i \in I\} \triangleq \{[d_i]_A \mid i \in I\}$ directed in D/A . The existence of $\text{lub}\{f(x_i) \mid i \in I\}$ is guaranteed by 5.3.11 and the monotonicity of f . We just have to show that:

$$f(\text{lub}\{x_i \mid i \in I\}) \leq_B \text{lub}\{f(x_i) \mid i \in I\}.$$

Suppose ϕ realizes f and consider the chain $\{e_n \mid n \in \omega\}$ built in 5.3.11. Then we have: $\phi(\cup\{e_n \mid n \in \omega\}) = \cup\{\phi e_n \mid n \in \omega\}$ and we observe that: $\forall n \in \omega. \exists i \in I. \phi e_n \leq_B \phi d_i$, hence we can conclude $[\cup\{\phi e_n \mid n \in \omega\}]_B \leq_B \text{lub}\{f(x_i) \mid i \in I\}$. $\square$

5.3.13 Corollary (the quotient space is an sfp)

Let A be a $c_{Du}\Sigma\text{per}$ then $(D/A, \leq_A)$ is an sfp.

Proof

Consider the sequence $\{\pi_n \mid n \in \omega\}$ where: $\pi_n: A \rightarrow A$ is the map realized by p_n as specified in 5.2.1 and 5.3.1. All the checks follow immediately by the properties of $\{p_n \mid n \in \omega\}$. $\square$

6. Conclusion

Much work remains to be done. In the first place there is a strong need for the development and refinement of the *theory*. For example the categories of pers over $(\omega, \cdot)$ built in sections 3 and 4 do not seem to satisfy enough *algebraicity* and *effectivity* properties (see 3.2.6.(2)). More research is needed in this direction. Taking another approach it has been shown in section 5 that it is possible to define a category of pers that enjoys many properties of standard categories of domains. This result is achieved by imposing a condition of *uniformity* whose categorical properties are not completely satisfying. This should stimulate further investigation.

On the other hand there is a wide spectrum of possible *applications*. So far the main one has been in the area of *typed functional programming languages*. $\text{cu}\Sigma\text{pers}$ can be considered as a model for a functional, typed language, based on Girard's system F extended with a fixed-point combinator on types and subtyping.

Adequacy results with respect to some standard "operational semantics" can be obtained rather easily. The basic property is roughly the following: let M be a term of type α and let $\text{er}(M)$ be the type free lambda term underlying M (for simplicity assume there are no free variables). Let $\llbracket \cdot \rrbracket$ denote the interpretation of, respectively, α in a $\text{cu}\Sigma\text{per}$, $\text{er}(M)$ in the realizability structure D , and M in an equivalence class of $\llbracket \alpha \rrbracket$. It can be proved that:

$\llbracket M \rrbracket = \llbracket \llbracket \text{er}(M) \rrbracket \rrbracket_{\llbracket \alpha \rrbracket}$. From this one can apply classical results relating the typed calculus to the type-free calculus as well as the type-free calculus to the realizability structure D (see, e.g., Amadio[89]).

More original insights can be obtained in the modelling of subtyping. See, for example, the proposal for the "Quest" programming language in Cardelli&Longo[90]. Amadio&Cardelli[90] illustrates the role of cupers in modelling the subtyping of recursive types. Another potential field of application is the interpretation of *extensions of calculi of constructions with recursive definitions*. The structures introduced are then relevant in proving the consistency of such extensions and in optimizing the process of programs' extraction. See Paulin-Mohring[89] for a proposal related to Coquand-Huet calculus of constructions. Finally let us mention that Streicher[89] exploits extensional pers to study consistency and inter-relations of various category-theoretic axiomatizations of calculi of dependent types.

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¹⁶While this paper goes to the press new results are announced by many people including Hyland, Phoa, Rosolini, and Taylor. Unfortunately it is impossible for us to report on them as well as to give exact references.

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