The Discrete Moments of the Circles

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Abstract. The moment of (p, q)-order, $m_{p,q}(C)$, of a circle C given by $(x-a)^2 + (y-b)^2 \leq r^2$, is defined to be $\iint\limits_C x^p y^q dxdy$. It is naturally to assume that the discrete moments $dm_{p,q}(C)$, defined as

$$dm_{p,q}(C) = \sum_{\substack{i,j \text{ are integers} \\ (i-a)^2+(j-b)^2 \leq r^2}} i^p j^q \quad ,$$

can be a good approximation for $m_{p,q}(C)$. This paper gives an answer what is the order of magnitude for the difference between a real moment $m_{p,q}(C)$ and its approximation $dm_{p,q}(C)$, calculated from the corresponding digital picture. Namely, we estimate

$$m_{p,q}(C) - dm_{p,q}(C) = \iint\limits_C x^p y^q dxdy \quad - \sum_{\substack{i,j \text{ are integers} \\ (i-a)^2+(j-b)^2 \leq r^2}} i^p j^q$$

in function of the size of the considered circle C and its center position if p and q are assumed to be integers. These differences are upper bounded with $\mathcal{O}\left(a^p \cdot b^q \cdot r^{\frac{7}{11}+\varepsilon}\right)$, where ε is an arbitrary small positive number. The established upper bound can be understood as very sharp.
The result has a practical importance, especially in the area of image processing and pattern recognition, because it shows what the picture resolution should be used in order to obtain a required precision in the parameter estimation from the digital data taken from the corresponded binary picture.[1]

Key words – Digital geometry, discrete shapes, parameter estimation.

1 Introduction

In this paper asymptotic expressions for the discrete moments of the circles in the xy-plane will be derived. The basic motivation for this paper comes from

[1] Research supported by Mathematical Institute, SANU, Beograd, under the project 04M02

G. Bertrand, M. Couprie, L. Perroton (Eds.): DGCI'99, LNCS 1568, pp. 41–49, 1999.

the area of image processing and pattern recognition. Namely, the mostly used mathematical model for computer picture (binary picture) of a considered shape is the integer grid whose points (so called digital points) take value 1 for points which belong to the given shape and 0 otherwise.

That is a reason that we can assume (through the paper) that all appearing digital (lattice) points have positive coordinates – i.e. the origin is placed into the left-lower corner of the observed digital picture. Since our attention is focused on the binary pictures of the circles: $(x-a)^2+(y-b)^2 \leq r^2$, the inequalities $r < a$ and $r < b$ will be assumed, if not mentioned.

Definition 1. *The moment of (p,q)-order, $m_{p,q}(S)$, of a planar shape S is defined (as usual) to be*

$$m_{p,q}(S) = \iint_S x^p y^q dxdy \ .$$

Remark. The moment $m_{0,0}(S)$ is the area of a planar shape S and will be denoted as $P(S)$.

Definition 2. *The discrete moments $dm_{p,q}(S)$ are defined as*

$$dm_{p,q}(S) = \sum_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} i^p j^q \ .$$

Definition 3. *The three-dimensional $(0,0,0)$-discrete moment, $dm_{0,0,0}(B)$, of a $3D$ body B expresses the number of digital points belonging to B, i.e.*

$$dm_{0,0,0}(B) = \sum_{\substack{i,j,k \text{ are integers} \\ (i,j,k)\in B}} 1 \ .$$

The basic mathematical tool for our derivation is the following result from number theory ([1]).

Theorem 1. *If $\mathcal{B}$ is a convex body in R^2, with C^3 boundary and positive curvature at every point of the boundary, then the number of lattice (i.e. digital) points belonging to $r\cdot\mathcal{B}$ is:*

$$R(r\cdot\mathcal{B}) = r^2\cdot P(\mathcal{B}) + \mathcal{O}\left(r^{\frac{7}{11}}\cdot(\log r)^{\frac{47}{22}}\right) \ , \qquad (1)$$

where $P(\mathcal{B})$ denotes the area of $\mathcal{B}$, while $r\cdot\mathcal{B}$ is the dilatation of $\mathcal{B}$ by factor r.

What we will use is weaker:

$$R(r\cdot\mathcal{B}) = r^2\cdot P(\mathcal{B}) + \mathcal{O}\left(r^{\frac{7}{11}+\epsilon}\right) \qquad \text{for every} \qquad \varepsilon > 0. \qquad (2)$$

The asymptotical expression for $dm_{0,0}(C)$ (where C is a circle of the radius r) is a direct consequence of (2):

$$dm_{0,0}(C) = \sum_{\substack{i,j \text{ are integers} \\ (i-a)^2+(j-b)^2\leq r^2}} 1 = \pi\cdot r^2 + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right) \ . \qquad (3)$$

2 Case $\quad p \cdot q = 0$

In thsi section we estimate the difference $m_{p,q}(C) - dm_{p,q}(C)$ in the case when one of the integers p and q is equal to zero. We need the following lemma.

Lemma 1. *Let the circle C be given by $(x-a)^2 + (y-b)^2 \le r^2$, and $C_1(i)$ and $C_2(i)$ be determined by*

$$C_1(i) \;=\; \left\{ \,(x,y) \;:\; (x-a)^2 + (y-b)^2 \le r^2 \quad and \quad x \le i \,\right\}$$

and

$$C_2(i) \;=\; \left\{ \,(x,y) \;:\; (x-a)^2 + (y-b)^2 \le r^2 \quad and \quad x \ge i \,\right\} \quad ,$$

where $\;i\;$ is an integer satisfying $\;a - r \le i \le a + r$. Then, the following (equivalent) relations

$$dm_{0,0}(C_1(i)) = P(C_1(i)) + \sqrt{r^2 + (i-a)^2} + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)$$

and

$$dm_{0,0}(C_2(i)) = P(C_2(i)) + \sqrt{r^2 + (i-a)^2} + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)$$

are satisfied.

Proof. The conditions of Theorem 1 can be relaxed to allow $\mathcal{B}$ having a finite number of corners. So (1) can be applied to the intersection of the interiors of two convex curves or, by subtraction, to the region within one convex curve and outside another (for details, see [1]). Let us mention here that the boundedness of the radius of curvature is essential; if a long section of the curve is straight, the error term can be as large as the length of the "straight" section.

Let us consider the circle $\tilde{C}$, given by $(x - (2i-a))^2 + (y-b)^2 \le r^2$ ($\tilde{C}$ is the circle symmetric to C with respect to the line $x = i$). The convex set $C \cap \tilde{C}$ satisfies the conditions of Theorem 1. So, the number of digital points belonging to $C \cap \tilde{C}$ can be determined as $dm_{0,0}(C \cap \tilde{C}) = P(C \cap \tilde{C}) + \mathcal{O}(r^{\frac{7}{11}+\varepsilon})$. The statement follows since the set $C \cap \tilde{C}$ is symmetrical with respect to the line $x = i$. Obviously, the term $\sqrt{r^2 + (i-a)^2}$ expresses a half of the number of digital points, inside of the circle C, lying on the line $x = i$, with the error bounded by 2. $\quad$ *QED*

Now, the asymptotical expressions for the discrete moments $dm_{p,0}(C)$ and $dm_{0,q}(C)$ can be given.

Theorem 2. *Let a, b and r be real numbers satisfying $r < a$ and $r < b$. Then the following asymptotical expressions hold:*

$$dm_{p,0}(C) = \sum_{\substack{i,j \text{ are integers} \\ (i-a)^2+(j-b)^2 \le r^2}} i^p = \iint_C x^p dxdy \quad + \quad \mathcal{O}\left(a^p \cdot r^{\frac{7}{11}+\varepsilon}\right) \quad ,$$

$$dm_{0,q}(C) = \sum_{\substack{i,j \text{ are integers} \\ (i-a)^2+(j-b)^2 \le r^2}} j^q = \iint_C y^q dxdy \quad + \quad \mathcal{O}\left(b^q \cdot r^{\frac{7}{11}+\varepsilon}\right) \quad .$$

Proof. Let's notice that $dm_{p,0}(C)$ is equal to the number of digital points belonging to the $3D$ body

$$\mathcal{G} = \left\{(x,y,z) \quad | \quad (x-a)^2 + (y-b)^2 \leq r^2, \quad 0 < z \leq x^p\right\} = \mathcal{G}' \cup \mathcal{G}'' \ ,$$

where

$$\mathcal{G}' = \left\{(x,y,z) \quad | \quad (x-a)^2 + (y-b)^2 \leq r^2, \qquad (a-r)^p < z \leq x^p\right\}$$

and

$$\mathcal{G}'' = \left\{(x,y,z) \quad | \quad (x-a)^2 + (y-b)^2 \leq r^2, \qquad 0 < z \leq (a-r)^p\right\} \ .$$

First, consider the number of digital points belonging to the body $\mathcal{G}'$. If W_i, for $i = \lceil a-r \rceil, \lceil a-r \rceil + 1, \ldots, \lfloor a+r \rfloor - 1$, denotes the body

$$W_i = \left\{(x,y,z) \mid (x-a)^2 + (y-b)^2 \leq r^2,\ x \geq i,\ i^p < z \leq \min\{x^p, (i+1)^p\}\right\} ,$$

then for the volume of $\mathcal{G}'$ we have

$$vol(\mathcal{G}') = \sum_{i=\lceil a-r \rceil}^{\lfloor a+r \rfloor - 1} vol(W_i) + ((\lceil a-r \rceil)^p - (a-r)^p) \cdot r^2 \cdot \pi + \mathcal{O}(a^{p-1} \cdot \sqrt{r}) \quad .$$

Also,

$$vol(W_i) = ((i+1)^p - i^p) \cdot P(C_2(i)) - vol(W_i') \ ,$$

where

$$W_i' = \left\{(x,y,z) \quad | \quad (x-a)^2 + (y-b)^2 \leq r^2, \quad x \geq i, \quad x^p < z \leq (i+1)^p\right\} \quad .$$

It is easy to derive

$$vol(W_i') = ((i+1)^p - i^p) \cdot \sqrt{r^2 - (i-a)^2} \ + \ ((i+1)^p - i^p) \cdot \mathcal{O}(r^{\frac{1}{2}}) \quad .$$

Now, we have, according to Lemma 1, that

$$\sum_{i=\lceil a-r \rceil}^{\lfloor a+r \rfloor - 1} vol(W_i) = \sum_{i=\lceil a-r \rceil}^{\lfloor a+r \rfloor - 1} ((i+1)^p - i^p) \cdot \left(dm_{0,0}(C_2(i)) - \sqrt{r^2 - (i-a)^2}\right) +$$

$$+ \sum_{i=\lceil a-r \rceil}^{\lfloor a+r \rfloor - 1} ((i+1)^p - i^p) \cdot \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right) - \sum_{i=\lceil a-r \rceil}^{\lfloor a+r \rfloor - 1} vol(W_i') =$$

$$= \sum_{i=\lceil a-r \rceil}^{\lfloor a+r \rfloor - 1} ((i+1)^p - i^p) \cdot (dm_{0,0}(C_2(i))) - \sum_{i=\lceil a-r \rceil}^{\lfloor a+r \rfloor - 1} ((i+1)^p - i^p)) \cdot \sqrt{r^2 - (i-a)^2} +$$

$$+\,\mathcal{O}\left(a^{p-1}\cdot r^{\frac{18}{11}+\varepsilon}\right)-\sum_{i=\lceil a-r\rceil}^{\lfloor a+r\rfloor-1} vol(W_i')=$$

$$=\sum_{i=\lceil a-r\rceil}^{\lfloor a+r\rfloor-1}((i+1)^p-i^p)\cdot(dm_{0,0}(C_2(i)))-$$

$$-\sum_{i=\lceil a-r\rceil}^{\lfloor a+r\rfloor-1}2\cdot((i+1)^p-i^p))\cdot\sqrt{r^2-(i-a)^2}\;+\;\mathcal{O}\left(a^{p-1}\cdot r^{\frac{18}{11}+\varepsilon}\right)\quad .$$

So, the volume of $\mathcal{G}'$ can be expressed as

$$vol(\mathcal{G}')=\sum_{i=\lceil a-r\rceil}^{\lfloor a+r\rfloor-1}((i+1)^p-i^p)\cdot\left(dm_{0,0}(C_2(i))-2\cdot\sqrt{r^2-(i-a)^2}\right)+$$

$$+\quad(\lceil a-r\rceil)^p\cdot\pi\cdot r^2-(a-r)^p\cdot\pi\cdot r^2+\mathcal{O}(a^{p-1}\cdot r^{\frac{18}{11}+\varepsilon})).$$

The next, $\quad vol(\mathcal{G}'')=(a-r)^p\cdot\pi\cdot r^2\quad$ gives:

$$vol(\mathcal{G})\;=\;vol(\mathcal{G}')\;+\;vol(\mathcal{G}'')\;=$$

$$=\sum_{i=\lceil a-r\rceil}^{\lfloor a+r\rfloor-1}((i+1)^p-i^p)\cdot\left(dm_{0,0}(C_2(i)-2\cdot\sqrt{r^2-(i-a)^2}\right)+$$

$$+\,(\lceil a-r\rceil)^p\cdot\pi\cdot r^2+\mathcal{O}(a^{p-1}\cdot r^{\frac{18}{11}+\varepsilon}))=$$

$$=dm_{0,0}(\mathcal{G}')+(\lceil a-r\rceil)^p\cdot\pi\cdot r^2+\mathcal{O}(a^{p-1}\cdot r^{\frac{18}{11}+\varepsilon})).$$

Notice that

$$\sum_{i=\lceil a-r\rceil}^{\lfloor a+r\rfloor-1}((i+1)^p-i^p)\cdot\left(dm_{0,0}(C_2(i))-2\cdot\sqrt{r^2-(i-a)^2}\right)$$

expresses the number of digital points in $\mathcal{G}'$ (with the error bounded by $\mathcal{O}(a^{p-1}\cdot r)$), excluding the points belonging to the plane $z=(\lceil a-r\rceil)^p$.

Since the term

$$(\lceil a-r\rceil)^p\cdot\pi\cdot r^2+\mathcal{O}(a^p\cdot r^{\frac{7}{11}+\varepsilon})\quad ,$$

equals the number of digital points belonging to the body $\mathcal{G}''$, adding the number of digital points belonging to the plane $z=(\lceil a-r\rceil)^p$ (if $a-r$ is not an integer), we have

$$m_{p,0}(C)=vol(\mathcal{G})=dm_{0,0,0}(\mathcal{G})+\mathcal{O}(a^p\cdot r^{\frac{7}{11}+\varepsilon})=dm_{p,0}(C)+\mathcal{O}(a^p\cdot r^{\frac{7}{11}+\varepsilon})\;,$$

which completes the proof. $\quad QED$

3 Case $p \cdot q > 0$

In this section asymptotical expressions for the differences $m_{p,q}(C) - dm_{p,q}(C)$, where both p and q are strictly positive integers, will be given.

Theorem 3. *Let a, b and r be real numbers satisfying $r < a$ and $r < b$. Then the following asymptotical expression holds:*

$$dm_{p,q}(C) = \sum_{\substack{i,j \text{ are integers} \\ (i-a)^2+(j-b)^2 \le r^2}} i^p \cdot j^q = \iint_C x^p \cdot y^q \cdot dx \cdot dy \; + \; \mathcal{O}\left(a^p \cdot b^q \cdot r^{\frac{7}{11}+\varepsilon}\right).$$

Proof. Let's notice that $dm_{p,q}(C)$ is equal to the number of digital points belonging to the $3D$ body

$$\mathcal{H} = \left\{(x,y,z) \quad | \quad (x-a)^2 + (y-b)^2 \le r^2, \quad 0 < z \le x^p \cdot y^q\right\} = \mathcal{H}' \cup \mathcal{H}'' \,,$$

where

$$\mathcal{H}' = \left\{(x,y,z) \quad | \quad (x-a)^2 + (y-b)^2 \le r^2, \qquad z_{min} < z \le x^p \cdot y^q\right\}$$

and

$$\mathcal{H}'' = \left\{(x,y,z) \quad | \quad (x-a)^2 + (y-b)^2 \le r^2, \qquad 0 < z \le z_{min}\right\} .$$

z_{min} is the minimal z-value on the curve which is the intersection of the surfaces $(x-a)^2 + (y-b)^2 = r^2$ and $z = x^p \cdot y^q$. In other words:

$$z_{min} = min\{z \quad | \quad z = x^p \cdot y^q \quad \text{and} \quad (x-a)^2 + (y-b)^2 = r^2\}$$

Analogously, z_{max} is defined to be the maximal z value on the same curve:

$$z_{max} = max\{z \quad | \quad z = x^p \cdot y^q \quad \text{and} \quad (x-a)^2 + (y-b)^2 = r^2\} .$$

First, consider the number of digital points belonging to the body $\mathcal{H}'$. If V_i denotes the body

$$V_i = \left\{(x,y,z) | \; (x-a)^2 + (y-b)^2 \le r^2, \; x^p \cdot y^q \ge i, \; i < z \le \min\{x^p \cdot y^q, i+1\}\right\},$$

then for the volume of $\mathcal{H}'$ we have

$$vol(\mathcal{H}') \quad = \sum_{i=\lceil z_{min} \rceil}^{\lfloor z_{max} \rfloor - 1} vol(V_i) \quad + \quad (\lceil z_{min} \rceil - z_{min}) \cdot r^2 \cdot \pi \quad + \quad \mathcal{O}(r^2) =$$

$$= \sum_{i=\lceil z_{min} \rceil}^{\lfloor z_{max} \rfloor - 1} vol(V_i) \quad + \quad \mathcal{O}(r^2) \quad .$$

Since

$$vol(V_i) = P(S_2(i)) \quad - \quad vol(V_i'),$$

where $S_2(i)$ and V_i' are defined as below

$$S_2(i) = \{(x,y) \quad | \quad (x-a)^2 + (y-b)^2 \le r^2, \quad x^p \cdot y^q \ge i\}$$

and

$$V_i' = \left\{(x,y,z) \quad | \quad (x-a)^2 + (y-b)^2 \le r^2, \quad x \ge i, \quad x^p \cdot y^q < z \le i+1\right\} \quad .$$

Because the projections (on xy-plane) of V_i' do not overlap and belong to the the circle $(x-a)^2 + (y-b)^2 \le r^2$, ve have:

$$\sum_{i=\lceil z_{min} \rceil}^{\lfloor z_{max} \rfloor - 1} vol(V_i') = \mathcal{O}(r^2) \quad .$$

Further,

$$\sum_{i=\lceil z_{min} \rceil}^{\lfloor z_{max} \rfloor - 1} vol(V_i) = \sum_{i=\lceil z_{min} \rceil}^{\lfloor z_{max} \rfloor - 1} \left(dm_{0,0}(S_2(i)) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) - \sum_{i=\lceil z_{min} \rceil}^{\lfloor z_{max} \rfloor - 1} vol(V_i') =$$

$$= \sum_{i=\lceil z_{min} \rceil}^{\lfloor z_{max} \rfloor - 1} dm_{0,0}(S_2(i)) \quad + \quad \mathcal{O}\left(a^p \cdot b^q \cdot r^{\frac{7}{11}+\varepsilon}\right) \ .$$

$$(z_{max} - z_{min} = \mathcal{O}(a^p \cdot b^q) \quad \text{is used.})$$

So, the volume of $\mathcal{H}'$ can be expressed as

$$vol(\mathcal{H}') \quad = \quad \sum_{i=\lceil z_{min} \rceil}^{\lfloor z_{max} \rfloor - 1} dm_{0,0}(S_2(i)) \quad + \quad \mathcal{O}(a^p \cdot b^q \cdot r^{\frac{7}{11}+\varepsilon})) \quad .$$

The equality $\quad vol(\mathcal{H}'') = z_{min} \cdot \pi \cdot r^2 \quad$ gives:

$$vol(\mathcal{H}) \quad = \quad vol(\mathcal{H}') \quad + \quad vol(\mathcal{H}'') \quad =$$

$$= \sum_{i=\lceil z_{min} \rceil}^{\lfloor z_{max} \rfloor - 1} dm_{0,0}(S_2(i) \quad + \quad \mathcal{O}(a^p \cdot b^q \cdot r^{\frac{7}{11}+\varepsilon})) \quad + \quad z_{min} \cdot \pi \cdot r^2 =$$

$$= \sum_{i=1}^{\lfloor z_{max} \rfloor - 1} dm_{0,0}(S_2(i)) \quad + \quad \mathcal{O}(a^p \cdot b^q \cdot r^{\frac{7}{11}+\varepsilon}) \quad .$$

Notice that

$$\sum_{i=1}^{\lfloor z_{max} \rfloor - 1} dm_{0,0}(S_2(i))$$

expresses the number of digital points in $\mathcal{G}$ excluding the (possible) points belonging to the plane $z = \lceil z_{max} \rceil$. So, the set of the following equalities

$$m_{p,q}(C) = vol(\mathcal{G}) = dm_{0,0,0}(\mathcal{G}) + \mathcal{O}\left(a^p \cdot b^q \cdot r^{\frac{7}{11}+\varepsilon}\right) = dm_{p,q}(C) + \mathcal{O}(a^p \cdot b^q \cdot r^{\frac{7}{11}+\varepsilon})$$

is proved. *QED*

4 An application and conclusion

We illustrate a possible application of the obtained results by the solution of a problem which is (in details) considered in [5]. The problem is:

How efficiently an original circle can be reconstructed from the corresponded digital data ?

The answer is that the the center position and the diameter of the original circle C can be reconstructed with an error upper bounded with $\mathcal{O}\left(r^{-\frac{7}{11}+\varepsilon}\right)$, if the radius is estimated as $r_{est} = \sqrt{\frac{dm_{0,0}(C)}{\pi}}$, while $(a_{est}, b_{est}) = \left(\frac{dm_{1,0}(C)}{dm_{0,0}(C)}, \frac{dm_{0,1}(C)}{dm_{0,0}(C)}\right)$ estimates the center position. Numerical data strongly confirm the obtained result. A few examples are given in the next table ([5]).

Table 1. Absolute errors between the original values of the radius and coordinates of the center (r, a and b, respectively) and their estimations (a_{est}, b_{est} and c_{est}) from the digitization of the circle $C: \ (x-a)^2 + (y-b)^2 \leq r^2$. The discrete moments: $dm_{0,0}(C)$, $dm_{1,0}(C)$ and $dm_{1,0}(C)$ are used.

r	a	b	$r_{est} - r$	$a_{est} - a$	$b_{est} - b$
2.5	9.4	6.2	-0.04075	-0.03157	0.01052
	282.8	33.3	0.02313	-0.05001	-0.04999
	2444422.4	33222.2	-0.04075	-0.03157	0.01052
23.4	44.7	31.3	-0.00823	0.00041	-0.00041
	29992.3	313.2	0.00536	0.01144	-0.05415
	3111331.3	229992.9	0.01216	-0.00325	0.00708
234.3	2282.4	339.9	-0.00299	0.00751	-0.00053
	229992.9	323.2	0.00855	-0.00413	0.00511
	2882288.8	333322.2	0.00991	-0.00391	0.00391
2345.6	4424.2	3888.8	0.00182	-0.00217	0.00217
	299222.9	6464.4	-0.00068	0.00392	-0.00594
	7444774.7	333311.3	0.00061	0.00436	-0.00436

Extensions of the result to other shapes (different from circles) are also possible. Let us mention here that the moments have been widely used in shape recognition and identification. That is a reason for a large number of papers related to the (discrete) moments calculation from a picture (for example [3]). This paper gives an apriory knowledge about the precision in the estimation of (real) moments from the corresponding (digital) picture. On the other side, the digital circles and the digital circular arcs are on of the basic objects considered in image processing and shape analys. For more results we refer to [4].

The results of this paper are based on the Huxley's result. This is a strong result which is related to the number of integer points inside a smooth enough convex curve γ. This is an old mathematical problem. Gauss and Dirichlet knew that the area of the shape bounded with γ estimate this number within order $\mathcal{O}(s)$, where s is the length of γ. The situation when γ is a circle is mostly studied. The Huxley's result (1) improves the previously best known upper bound ([2]) even in this case. ¿From the given proofs of Theorem 2 and Theorem 3 it can be concluded that the derived result is sharp up to Huxley's result.

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