Attack on the Cryptographic Scheme NIKS-TAS

Don Coppersmith

IBM Research, T.J. Watson Research Center
Yorktown Heights, NY 10598, USA

Abstract. The NIKS-TAS scheme, proposed by Tsujii, Araki, and Sekine in 1993, is an ID-based cryptographic key sharing scheme. We present an algebraic method for attacking this scheme, requiring the cooperation of a small number of collaborators to discover the key shared by any two parties.

1 Introduction

The NIKS-TAS scheme, proposed by Tsujii, Araki, and Sekine [2] in 1993, is an ID-based cryptographic key sharing scheme. That is, to each user (User A) there corresponds a publicly known quantity ID_A , closely related to his personal identification; for example, this could consist of name, address, and telephone number. User A also owns some secret information, which we temporarily abbreviate X_A . Suppose Users A and B wish to communicate secretly. User A executes the NIKS-TAS protocol, using as inputs his own secret information X_A and User B 's public information ID_B , to prepare a cryptographic key $K_{AB} = f(X_A, ID_B)$. Simultaneously, User B uses X_B and ID_A to compute the key $K_{BA} = f(X_B, ID_A)$. The NIKS-TAS scheme is designed such that $K_{AB} = K_{BA}$. Thus both A and B can compute the same cryptographic key, and use it in private communication with each other.

The scheme shares some aspects with the Diffie and Hellman key exchange scheme [1], but differs in one crucial place. In the Diffie and Hellman scheme, public information includes a large prime p and a generator g of the multiplicative group of integers modulo p , called Z_p^* . The user's public information y_A is a member of the multiplicative group Z_p^* , while the secret information x_A lies in the additive group of exponents or logarithms. The security lies in the fact that computing a group element from its logarithm is easy, but computing the logarithm from the group element seems to be difficult.

$$\begin{array}{ll}
\text{public:} & y_A \in \mathbb{Z}_p^* \\
\text{secret:} & x_A \in \mathbb{Z}_{p-1} \\
& y_A = g^{(x_A)} \in \mathbb{Z}_p^* \\
\text{easy:} & x_A \rightarrow y_A \\
\text{hard:} & y_A \rightarrow x_A
\end{array} \quad (\text{DH})$$

By contrast, in the NIKS-TAS scheme the public information is used to operate in the additive group of exponents, and some of the secret information lies in the multiplicative group $\mathbb{Z}_p^*$.

$$\begin{array}{ll}
\text{public:} & ID_A \rightarrow i_{A,j,1} \in \{0,1\} \text{ operating on } \mathbb{Z}_{p-1} \\
\text{secret:} & X_A \rightarrow G_{Ak} \in \mathbb{Z}_p^*
\end{array} \quad (\text{NIKS-TAS})$$

This reversal comes about because of the requirements of ID-based cryptography. For example, if one tried to make the Diffie-Hellman scheme ID-based, the public information y_A would be predetermined (by the user's name, etc.) and someone (a trusted key distribution center, perhaps) would have to derive the secret information x_A from this public information y_A . But the premise of the Diffie-Hellman scheme is that this task is computationally infeasible, even for the key distribution center. So, in the NIKS-TAS, in order to accommodate the ID-based requirements, the roles are reversed.

When the NIKS-TAS scheme was announced, its inventors offered a reward of \$3,000 (US) for showing "theoretically substantial defects proving that our scheme cannot overcome the collusion problem." The present paper represents a cryptanalysis of this scheme and an attempt to claim the reward.

We show here that a small number of colluding users can pool their secret information and obtain enough knowledge that they can easily find the key shared by any two other users. For a typical value of the parameters of the system, $N = 20, 43$ users can usually conspire and break the system.

The attack on the system is made possible by the reversal of roles from the Diffie-Hellman system: the fact that the public information is involved in the ring $\mathbb{Z}_{p-1}$ instead of the multiplicative group $\mathbb{Z}_p^*$. Where the legitimate user computes a key in $\mathbb{Z}_p^*$, the attacker mirrors this computation with computations in the group of exponents $\mathbb{Z}_{p-1}$, where he has greater flexibility in the computations available to him, and thus is able to carry out an attack not en-

visioned by the system's designers. The attack uses differential techniques: User A considers keys K_{AB} and $K_{AB'}$ which he would share with two users B and B' whose public information ID_B and $ID_{B'}$ differ in only one bit, and uses the difference between K_{AB} and $K_{AB'}$ to probe the secret information that should be inaccessible to him.

In the balance of the paper, we present the NIKS-TAS scheme, condensed from [2], followed by our attack, and a worked example.

2 The NIKS-TAS Scheme

This description is copied from [2], except that indices have been reordered for ease of understanding (particularly on the quantities $i_{A,j,m}$), and background material has been omitted.

Public information common for all entities

The trusted key distribution center publishes a large prime number p such that the discrete logarithm problem (mod p) is inaccessible. It determines the parameters of the system, ℓ , N , and L ; although not spelled out in the proposal, we can imagine $N = 20$, $L = 50$ and $\ell = 200$. The center chooses and publishes a one-way hash function which sends ℓ bits to $K = NL$ bits.

Public information for a single entity

Each entity, User A , publicizes his ℓ -bit IDCODE (which may be name and address) to all other entities, and registers the IDCODE at a trusted key distribution center. This code is processed by the publicly known hash function to obtain the K -bit ID vector of a typical entity A , called ID_A :

$$\begin{aligned}
 ID_A &= \text{Hash}(IDCODE_A) \\
 ID_A &= [i_{A,1,1}, i_{A,1,2}, \dots, i_{A,N,L}] = [i_{A1}, i_{A2}, \dots, i_{AN}] \\
 i_{A,j,m} &\in \{0,1\}, \quad j = 1, \dots, N, \quad m = 1, \dots, L \\
 i_{Aj} &\in \{0,1\}^L, \quad j = 1, \dots, N \\
 i_{Aj} &= [i_{A,j,1}, i_{A,j,2}, \dots, i_{A,j,L}]
 \end{aligned}$$

This data is publicly known, derived from the ℓ -bit IDCODE by the public hash function.

Center Secrets common for all entities

Meanwhile, the key distribution center chooses the following integers as secret parameters of its own:

$$\begin{aligned}
 &g: \text{a generator of } \mathbb{Z}_p^* \\
 &\gamma \in \mathbb{Z}_{p-1} \\
 &\mathbf{x}_j, \mathbf{y}_j, \mathbf{u}_j, \mathbf{v}_j \in \mathbb{Z}_{p-1}^L, \quad j = 1, 2, \dots, N \\
 &X = [x_{1,1}, x_{1,2}, \dots, x_{N,L}] = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N] \\
 &Y = [y_{1,1}, y_{1,2}, \dots, y_{N,L}] = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_N] \\
 &U = [u_{1,1}, u_{1,2}, \dots, u_{N,L}] = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N] \\
 &V = [v_{1,1}, v_{1,2}, \dots, v_{N,L}] = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N]
 \end{aligned}$$

where $\mathbb{Z}_{p-1}^L$ designates the set of L -dimensional vectors whose components belong to $\mathbb{Z}_{p-1}$. $\mathbf{x}_j, \mathbf{y}_j, \mathbf{u}_j$, and $\mathbf{v}_j$ are global parameters with which the individual secret keys are generated.

Individual secrets kept in center

The center produces random numbers for each entity; namely the center produces $\alpha_{Aj\ell}, \beta_{Aj\ell}, c_{Ak}$, and $r_{Aj\ell}$ for entity A .

$$\begin{aligned}
 \alpha_{Aj\ell}, \beta_{Aj\ell} &\in \mathbb{Z}_{p-1}^L, \quad j = 1, 2, \dots, N, \quad \ell = 1, 2, \dots, 2(N+1-j) \\
 c_{Ak} &\in \mathbb{Z}_{p-1}, \quad k = 1, 2, \dots, N+1 \\
 r_{Aj\ell} &\in \mathbb{Z}_{p-1}, \quad j = 1, 2, \dots, N, \quad \ell = 1, 2, \dots, 2(N+1-j)
 \end{aligned}$$

subject to the following relations:

$$\begin{aligned}
 c_{A,i}^{(0)} &= c_{A,i}, \quad i = 1, 2, \dots, N+1 \\
 c_{A,i}^{(k)} &= c_{A,i}^{(k-1)} r_{A,k,2i-1} + c_{A,i+1}^{(k-1)} r_{A,k,2i}, \quad k = 1, 2, \dots, N, \quad i = 1, 2, \dots, N-k+1 \\
 c_{A,i}^{(k-1)} \alpha_{A,k,2i-1} + c_{A,i+1}^{(k-1)} \alpha_{A,k,2i} &= 0, \quad k = 1, 2, \dots, N, \quad i = 1, 2, \dots, N-k+1 \\
 c_{A,i}^{(k-1)} \beta_{A,k,2i-1} + c_{A,i+1}^{(k-1)} \beta_{A,k,2i} &= 0, \quad k = 1, 2, \dots, N, \quad i = 1, 2, \dots, N-k+1 \\
 \gamma &= c_{A1}^{(N)}
 \end{aligned}$$

The center calculates the following parameters for each entity in secret and keeps these parameters out of reach of any entity:

$$\begin{aligned} S_{Aj} &= \mathbf{x}_j \cdot \mathbf{i}_{Aj} + \mathbf{y}_j \cdot \mathbf{i}_{Aj}^c \\ T_{Aj} &= \mathbf{u}_j \cdot \mathbf{i}_{Aj} + \mathbf{v}_j \cdot \mathbf{i}_{Aj}^c \end{aligned}$$

where $\mathbf{i}_{Aj}^c$ is the complement of $\mathbf{i}_{Aj}$, and $\mathbf{x}_j \cdot \mathbf{i}_{Aj}$ is the inner product.

Individual secrets distributed to each entity

The center calculates the following parameters for each entity, then distributes to each entity his own secret parameters through a secure channel:

$$\begin{aligned} \pi_A &\in S_N \quad (\text{a permutation}) \\ G_{Ak} &= g^{c_{Ak}}, \quad k = 1, \dots, N+1 \\ \mathbf{X}_{Aj\ell} &= r_{Aj\ell}(S_{Aj'}\mathbf{x}_{j'} + T_{Aj'}\mathbf{u}_{j'}) + \alpha_{Aj\ell} \\ \mathbf{Y}_{Aj\ell} &= r_{Aj\ell}(S_{Aj'}\mathbf{y}_{j'} + T_{Aj'}\mathbf{v}_{j'}) + \beta_{Aj\ell} \\ j &= 1, 2, \dots, N, \quad \ell = 1, 2, \dots, 2(N+1-j) \end{aligned}$$

Here π_A is a permutation on N elements, and we define $j' = \pi_A(j)$.

Key Sharing Algorithm

Two users A and B each compute a common key. User A sets

$$G_{A,B,i}^{(0)} = G_{A,i}, \quad i = 1, \dots, N+1.$$

For each $k = 1, 2, \dots, N$ in turn, User A calculates:

$$\begin{aligned} Z_{A,k,j}^{(B)} &= \mathbf{X}_{A,k,j} \cdot \mathbf{i}_{Bk'} + \mathbf{Y}_{A,k,j} \cdot \mathbf{i}_{Bk'}^c, \quad j = 1, 2, \dots, 2(N+1-k) \\ &= r_{A,k,j}(S_{Ak'}S_{Bk'} + T_{Ak'}T_{Bk'}) + \alpha_{Akj} \cdot \mathbf{i}_{Bk'} + \beta_{Akj} \cdot \mathbf{i}_{Bk'}^c \\ G_{A,B,i}^{(k)} &= (G_{A,B,i}^{(k-1)})^{(Z_{A,k,2i-1}^{(B)})} \times (G_{A,B,i+1}^{(k-1)})^{(Z_{A,k,2i}^{(B)})} \\ &= g^{c_{A,i} \prod_{j=1}^k (S_{Aj'}S_{Bj'} + T_{Aj'}T_{Bj'})}, \quad i = 1, 2, \dots, (N-k+1). \end{aligned}$$

Verifying these equalities is an exercise in algebra, and depends on the relations among the quantities α , etc.

After the N th step we have

$$K_{AB} = G_{A,B,1}^{(N)} = g^{\{y \prod_{j=1}^N (S_{Aj} S_{Bj} + T_{Aj} T_{Bj})\}},$$

recalling $y = c_{A,1}^{(N)} = c_{B,1}^{(N)}$.

Meanwhile User B performs analogous calculations using his secret quantities and A 's public quantities, to produce K_{BA} . One can check [2] that the two keys computed by these two users are the same:

$$K_{AB} = K_{BA}.$$

User A and User B can use this common quantity $K_{AB} = K_{BA}$ as a cryptographic key for secret communication with each other.

3 The Attack

We show how $2N + 3$ collaborators can usually break the scheme. If $N = 20$, then $M = 43$ collaborators should suffice.

Notation: Users A, B will be collaborators, and users D, E, F will be victims. We will use public and secret information from the collaborators A, B , and we will use only public information from the victims D, E, F ; in fact, we will fabricate public information for D, E, F (invent new users with new public information, but no secret information) when it suits our purpose.

For any collaborator A and user E (who may be victim or collaborator), let A follow the computation of key K_{AE} from $G_{Ak} = g^{c_{Ak}}$, $1 \leq k \leq N + 1$. Instead of computing K_{AE} , A will compute exponents $e_{AE}^{(k)} \in \mathbb{Z}_{p-1}$, $1 \leq k \leq N + 1$, such that

$$K_{AE} = \prod_{k=1}^{N+1} (G_{Ak})^{e_{AE}^{(k)}} = g^{\{\sum_k c_{Ak} e_{AE}^{(k)}\}}.$$

He does this by keeping track of powers of G_{Ak} in the computation of K_{AE} . Each $e_{AE}^{(k)}$ is a sum of products of the quantities $Z_{A,j}^{(E)}$, ℓ .

Of special interest is the case where both A and $B = E$ are collaborators. In this case, from the requirement $K_{AB} = K_{BA}$ we get:

$$\sum_{k=1}^{N+1} c_{Ak} e_{AB}^{(k)} = \sum_{k=1}^{N+1} c_{Bk} e_{BA}^{(k)} \quad (*)$$

in the ring $\mathbb{Z}_{p-1}$. The $e_{AB}^{(k)}$ are known, while c_{Ak} are unknown.

As we let $\{A, B\}$ range over all unordered pairs of collaborators, we obtain $M(M-1)/2$ such linear equations (*), one for each unordered pair $\{A, B\}$. There are $M(N+1)$ unknowns c_{Ak} , $1 \leq k \leq N+1$, $A \in \{\text{collaborators}\}$. Because we chose $M \geq 2N+3$, we get $M(M-1)/2 \geq M(N+1)$, so that there are at least as many equations as unknowns. These are homogeneous linear equations in c_{Ak} . We solve them to get the ratios $c_{Ak} : c_{Bj}$. The example in the next section gives reason to believe that the rank of this system of homogeneous linear equations is $M(N+1) - 1$, allowing for unique solution; and possible remedies for the opposite case.

Fix one collaborator $\underline{B}$ as a reference point, and set

$$\hat{g} = G_{\underline{B}1} = g^{c_{\underline{B}1}}.$$

We can calculate $\hat{g}$, and from now on we use it instead of the center's secret generator g , which we cannot calculate. We have gotten exponents $\hat{c}_{Ak}$ such that

$$G_{Ak} = \hat{g}^{\hat{c}_{Ak}}, \quad \forall A \in \{\text{collaborators}\}, \quad 1 \leq k \leq N+1.$$

Now we can work with these exponents $\hat{c}_{Ak}$ and solve algebraic equations in the exponents.

Set $f_{AB} = \sum_k \hat{c}_{Ak} e_{AB}^{(k)}$ so that $K_{AB} = \hat{g}^{f_{AB}}$, and $f_{AB} = f_{BA}$. We can calculate f_{AB} if we know A 's private information and B 's public information. (B need not be a collaborator; in fact we can fabricate B 's public information.)

Explanation. The data $\hat{c}_{Ak}$ is gotten from the collaboration and from A 's private information. The data $e_{AB}^{(k)}$ is gotten from A 's private information and B 's public information. So if A is a collaborator and E is a victim, A can compute f_{AE} such that $f_{AE} = \sum_k \hat{c}_{Ak} e_{AE}^{(k)}$ and $K_{AE} = \hat{g}^{f_{AE}}$.

We have

$$f_{AE} = \hat{\gamma} \prod_{j=1}^N (S_{A_j} S_{E_j} + T_{A_j} T_{E_j})$$

for $\hat{\gamma}$ unknown, fixed, and independent of A, E . In fact, $\hat{\gamma} = \gamma/c_{B1}$.

Now fix two collaborators, A_1 and A_2 , and a victim, D . Let A_1 compute

$$f_{A_1 D}, f_{A_1 D'_1}, f_{A_1 D'_2}, \dots, f_{A_1 D'_L},$$

where D'_j is a fabricated user with the same public information as D except for bit i_{D1j} . That is,

$$\begin{aligned} i_{D'_j, a, b} &= i_{D, a, b} \text{ if } (a, b) \neq (1, j) \\ i_{D'_j, 1, j} &= 1 - i_{D, 1, j} \end{aligned}$$

Set $d_{A_1 D1} = \hat{\gamma} \pi_i \neq 1 (S_{A_1 i} S_{D i} + T_{A_1 i} T_{D i})$. As it happens, we do not know $d_{A_1 D1}$, but it is the same as $d_{A_1 D'_j 1}$. We can compute

$$\pm (f_{A_1 D'_j} - f_{A_1 D}) = d_{A_1 D1} \{S_{A_1 1} (x_{1j} - y_{1j}) + T_{A_1 1} (u_{1j} - v_{1j})\},$$

where $x_{1j} \in \mathbb{Z}_{p-1}$ is the j th entry of $\mathbf{x}_1$. The sign $\pm$ is determined by the bit $i_{D, 1, j}$. By combining these values for the different j , we get some (unknown) linear combination of the two vectors $(\mathbf{x}_1 - \mathbf{y}_1)$ and $(\mathbf{u}_1 - \mathbf{v}_1)$; call it $\mathbf{q}_1$:

$$\mathbf{q}_1 = (d_{A_1 D1} S_{A_1 1})(\mathbf{x}_1 - \mathbf{y}_1) + (d_{A_1 D1} T_{A_1 1})(\mathbf{u}_1 - \mathbf{v}_1).$$

Repeat for another collaborator A_2 instead of A_1 to get a second (unknown) linear combination of $(\mathbf{x}_1 - \mathbf{y}_1)$ and $(\mathbf{u}_1 - \mathbf{v}_1)$; call this one $\mathbf{r}_1$:

$$\mathbf{r}_1 = (d_{A_2 D1} S_{A_2 1})(\mathbf{x}_1 - \mathbf{y}_1) + (d_{A_2 D1} T_{A_2 1})(\mathbf{u}_1 - \mathbf{v}_1).$$

Note: We can compute the vectors $\mathbf{q}_1$ and $\mathbf{r}_1$, but not the quantities $d_{A_1 E1}$, $S_{A_1 1}$, $\mathbf{x}_1$, $\mathbf{y}_1$, $T_{A_1 1}$, $\mathbf{u}_1$, or $\mathbf{v}_1$.

The vectors $\mathbf{q}_1$, $\mathbf{r}_1$ span the same space as $(\mathbf{x}_1 - \mathbf{y}_1)$ and $(\mathbf{u}_1 - \mathbf{v}_1)$, a 2-dimensional subspace of $\mathbb{Z}_{p-1}^L$. We have computed $\mathbf{q}_1$ and $\mathbf{r}_1$. Put another way, we have computed the subspace spanned by $(\mathbf{x}_1 - \mathbf{y}_1)$ and $(\mathbf{u}_1 - \mathbf{v}_1)$.

Letting A be an arbitrary collaborator and E an arbitrary victim, we have that $(S_{A1} S_{E1} + T_{A1} T_{E1})$ is an inhomogeneous bilinear form in

$\{(\mathbf{x}_1 - \mathbf{y}_1) \cdot \mathbf{i}_{A1}, (\mathbf{u}_1 - \mathbf{v}_1) \cdot \mathbf{i}_{A1}\}$ (on the one hand) and $\{(\mathbf{x}_1 - \mathbf{y}_1) \cdot \mathbf{i}_{E1}, (\mathbf{u}_1 - \mathbf{v}_1) \cdot \mathbf{i}_{E1}\}$ (on the other hand), namely

$$\begin{aligned} (S_{A1}S_{E1} + T_{A1}T_{E1}) &= \\ &= [(\Sigma_j \nu_{1j}) + (\mathbf{x}_1 - \mathbf{y}_1) \cdot \mathbf{i}_{A1}] \times [(\Sigma_j \nu_{1j}) + (\mathbf{x}_1 - \mathbf{y}_1) \cdot \mathbf{i}_{E1}] + \\ &+ [(\Sigma_j \nu_{1j}) + (\mathbf{u}_1 - \mathbf{v}_1) \cdot \mathbf{i}_{A1}] \times [(\Sigma_j \nu_{1j}) + (\mathbf{u}_1 - \mathbf{v}_1) \cdot \mathbf{i}_{E1}] \end{aligned}$$

It is symmetric in the interchange of (A, E) .

Imagine a change of basis from $\{(\mathbf{x}_1 - \mathbf{y}_1) \cdot \mathbf{i}_{A1}, (\mathbf{u}_1 - \mathbf{v}_1) \cdot \mathbf{i}_{A1}\}$ to $\{\mathbf{q}_1 \cdot \mathbf{i}_{A1}, \mathbf{r}_1 \cdot \mathbf{i}_{A1}\}$. (We do not know the coordinates of this basis change.) Then we have that $(S_{A1}S_{E1} + T_{A1}T_{E1})$ is an inhomogeneous bilinear form in $\{\mathbf{q}_1 \cdot \mathbf{i}_{A1}, \mathbf{r}_1 \cdot \mathbf{i}_{A1}\}$ (on one hand) and $\{\mathbf{q}_1 \cdot \mathbf{i}_{E1}, \mathbf{r}_1 \cdot \mathbf{i}_{E1}\}$ (on the other hand), again symmetric with respect to (A, E) . So we may write it as

$$\begin{aligned} (S_{A1}S_{E1} + T_{A1}T_{E1}) &= \alpha_1 + \beta_1(\mathbf{q}_1 \cdot \mathbf{i}_{A1} + \mathbf{q}_1 \cdot \mathbf{i}_{E1}) + \gamma_1(\mathbf{r}_1 \cdot \mathbf{i}_{A1} + \mathbf{r}_1 \cdot \mathbf{i}_{E1}) + \\ &+ \delta_1(\mathbf{q}_1 \cdot \mathbf{i}_{A1})(\mathbf{q}_1 \cdot \mathbf{i}_{E1}) + \varepsilon_1(\mathbf{r}_1 \cdot \mathbf{i}_{A1})(\mathbf{r}_1 \cdot \mathbf{i}_{E1}) + \\ &+ \tau_1[(\mathbf{q}_1 \cdot \mathbf{i}_{A1})(\mathbf{r}_1 \cdot \mathbf{i}_{E1}) + (\mathbf{r}_1 \cdot \mathbf{i}_{A1})(\mathbf{q}_1 \cdot \mathbf{i}_{E1})] \end{aligned}$$

where we do not know $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \tau_1 \in \mathbb{Z}_p - 1$. (We should not confuse α_1 with the previously defined $\alpha_{Aj\ell}$.)

By repeating with E'_j instead of E , and seeing the dependence of $f_{AE'_j}$ on $\mathbf{q}_1 \cdot \mathbf{i}_{E'_j1}$ and $\mathbf{r}_1 \cdot \mathbf{i}_{E'_j1}$, user A can establish the *ratios* of the coefficients of 1, $\mathbf{q}_1 \cdot \mathbf{i}_{E1}$, $\mathbf{r}_1 \cdot \mathbf{i}_{E1}$, respectively:

$$\begin{aligned} [\alpha_1 + \beta_1 \mathbf{q}_1 \cdot \mathbf{i}_{A1} + \gamma_1 \mathbf{r}_1 \cdot \mathbf{i}_{A1}] &: [\beta_1 + \delta_1 \mathbf{q}_1 \cdot \mathbf{i}_{A1} + \tau_1 \mathbf{r}_1 \cdot \mathbf{i}_{A1}], \\ [\alpha_1 + \beta_1 \mathbf{q}_1 \cdot \mathbf{i}_{A1} + \gamma_1 \mathbf{r}_1 \cdot \mathbf{i}_{A1}] &: [\gamma_1 + \tau_1 \mathbf{q}_1 \cdot \mathbf{i}_{A1} + \varepsilon_1 \mathbf{r}_1 \cdot \mathbf{i}_{A1}]. \end{aligned}$$

Call this data $(**)_{A}$. When other collaborators A', A'' repeat the experiment with their different values of $\mathbf{q}_1 \cdot \mathbf{i}_{A'1}, \mathbf{r}_1 \cdot \mathbf{i}_{A'1}, \mathbf{q}_1 \cdot \mathbf{i}_{A''1}, \mathbf{r}_1 \cdot \mathbf{i}_{A''1}$, but using the same basis vectors $\mathbf{q}_1$ and $\mathbf{r}_1$, the data $(**)_{A}$ and $(**)_{A'}$ and $(**)_{A''}$ suffices to calculate the *ratios*

$$\alpha_1 : \beta_1 : \gamma_1 : \delta_1 : \varepsilon_1 : \tau_1.$$

Without loss of generality we can divide through by α_1 (and change $\hat{\gamma}$ accordingly), and so we calculate

$$(\alpha_1 = 1), \quad \beta_1, \quad \gamma_1, \quad \delta_1, \quad \varepsilon_1, \quad \tau_1.$$

Repeat with $k = 2, 3, \dots, N$ replacing 1, to calculate

$$(\alpha_k = 1), \beta_k, \gamma_k, \delta_k, \varepsilon_k, \tau_k, \quad k = 1, 2, \dots, N.$$

Then we have expressed

$$f_{AE} = \tilde{\gamma} \prod_{j=1}^N [\alpha_j + \beta_j(\mathbf{q}_j \cdot \mathbf{i}_{Aj} + \mathbf{q}_j \cdot \mathbf{i}_{Ej}) + \gamma_j(\mathbf{r}_j \cdot \mathbf{i}_{Aj} + \mathbf{r}_j \cdot \mathbf{i}_{Ej}) + \delta_j(\mathbf{q}_j \cdot \mathbf{i}_{Aj})(\mathbf{q}_j \cdot \mathbf{i}_{Ej}) + \varepsilon_j(\mathbf{r}_j \cdot \mathbf{i}_{Aj})(\mathbf{r}_j \cdot \mathbf{i}_{Ej}) + \tau_j\{(\mathbf{q}_j \cdot \mathbf{i}_{Aj})(\mathbf{r}_j \cdot \mathbf{i}_{Ej}) + (\mathbf{r}_j \cdot \mathbf{i}_{Aj})(\mathbf{q}_j \cdot \mathbf{i}_{Ej})\}]$$

with $\tilde{\gamma}$ unknown, and $\alpha_j = 1$, and the rest of the data $\mathbf{q}_j, \beta_j$, etc. known. We can compute $\tilde{\gamma}$ from one known key $f_{A_0B_0}$, so we know the complete expression.

This expression depends only on the public information $\mathbf{i}_{Aj}, \mathbf{i}_{Ej}$ of A and E , so it holds as well for victims F, E :

$$f_{FE} = \tilde{\gamma} \prod_{j=1}^N [\alpha_j + \beta_j(\mathbf{q}_j \cdot \mathbf{i}_{Fj} + \mathbf{q}_j \cdot \mathbf{i}_{Ej}) + \gamma_j(\mathbf{r}_j \cdot \mathbf{i}_{Fj} + \mathbf{r}_j \cdot \mathbf{i}_{Ej}) + \delta_j(\mathbf{q}_j \cdot \mathbf{i}_{Fj})(\mathbf{q}_j \cdot \mathbf{i}_{Ej}) + \varepsilon_j(\mathbf{r}_j \cdot \mathbf{i}_{Fj})(\mathbf{r}_j \cdot \mathbf{i}_{Ej}) + \tau_j\{(\mathbf{q}_j \cdot \mathbf{i}_{Fj})(\mathbf{r}_j \cdot \mathbf{i}_{Ej}) + (\mathbf{r}_j \cdot \mathbf{i}_{Fj})(\mathbf{q}_j \cdot \mathbf{i}_{Ej})\}]$$

and

$$K_{FE} = \hat{g}^{\wedge(f_{FE})}.$$

We have used the private information of $2N + 3$ collaborators and the public information $\mathbf{i}_{Fj}, \mathbf{i}_{Ej}$ of the victims F, E , and obtained the key shared by victims F, E .

Remark: It is interesting to note that the act of collusion may be even easier than the legitimate use of the system, because the colluders use addition and multiplication, respectively, in Z_{p-1} while the legitimate users are using multiplication and exponentiation, respectively, in Z_p^* .

4 Example

Take the simple case $N = 1$. Here we can ignore the permutations π_A , which give just minor complications. We have $M = 2N + 1 = 5$ users, A, B, C, D, E , colluding to break the scheme. They determine the following $M(M - 1)/2 = 10$ equations:

$$e_{AB}^{(1)}c_{A1} + e_{AB}^{(2)}c_{A2} \equiv e_{BA}^{(1)}c_{B1} + e_{BA}^{(2)}c_{B2} \pmod{p-1}$$

$$e_{AC}^{(1)}c_{A1} + e_{AC}^{(2)}c_{A2} \equiv e_{CA}^{(1)}c_{C1} + e_{CA}^{(2)}c_{C2} \pmod{p-1}$$

...

$$e_{DE}^{(1)}c_{D1} + e_{DE}^{(2)}c_{D2} \equiv e_{ED}^{(1)}c_{E1} + e_{ED}^{(2)}c_{E2} \pmod{p-1}$$

Notice that $e_{AB}^{(1)} = Z_{A,1,1}^{(B)}$.

Starting with the $[M(M-1)/2] \times [M(N+1)] = 10 \times 10$ matrix of coefficients $\pm e_{AB}^{(1)}$, delete one column to make it an inhomogeneous system, and delete one row to make it square. We obtain the 9×9 matrix F :

$$\begin{bmatrix} e_{AB}^{(1)} & e_{AB}^{(2)} & -e_{BA}^{(1)} & -e_{BA}^{(2)} & 0 & 0 & 0 & 0 & 0 \\ e_{AC}^{(1)} & e_{AC}^{(2)} & 0 & 0 & -e_{CA}^{(1)} & -e_{CA}^{(2)} & 0 & 0 & 0 \\ 0 & 0 & e_{BC}^{(1)} & e_{BC}^{(2)} & -e_{CB}^{(1)} & -e_{CB}^{(2)} & 0 & 0 & 0 \\ 0 & 0 & e_{BD}^{(1)} & e_{BD}^{(2)} & 0 & 0 & -e_{DB}^{(1)} & -e_{DB}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & e_{CD}^{(1)} & e_{CD}^{(2)} & -e_{DC}^{(1)} & -e_{DC}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & e_{CE}^{(1)} & e_{CE}^{(2)} & 0 & 0 & -e_{EC}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{DE}^{(1)} & e_{DE}^{(2)} & -e_{ED}^{(1)} \\ -e_{AD}^{(1)} & -e_{AD}^{(2)} & 0 & 0 & 0 & 0 & e_{DA}^{(1)} & e_{DA}^{(2)} & 0 \\ -e_{AE}^{(1)} & -e_{AE}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & e_{EA}^{(1)} \end{bmatrix}$$

Define vectors C, e as

$$C = (c_{A1}, c_{A2}, c_{B1}, c_{B2}, c_{C1}, c_{C2}, c_{D1}, c_{D2}, c_{E1})^T$$

$$\mathbf{e} = (0, 0, 0, 0, 0, e_{EC}^{(2)}, e_{ED}^{(2)}, 0, -e_{EA}^{(2)})^T.$$

We are then faced with solving the 9×9 system of inhomogeneous linear equations $FC = \mathbf{e}c_{E2}$. To facilitate the solution of this system, we depend on F having full rank, that is, $\det(F)$ is a unit in Z_{p-1} . To verify that F generally has full rank, we perform some elementary column operations. Letting F_j denote the j th column, we set $F_2 \leftarrow (c_{A1}F_1 + c_{A2}F_2)\hat{\gamma}/\gamma$, $F_4 \leftarrow (c_{B1}F_3 + c_{B2}F_4)\hat{\gamma}/\gamma$, $F_6 \leftarrow (c_{C1}F_5 + c_{C2}F_6)\hat{\gamma}/\gamma$, and $F_8 \leftarrow (c_{D1}F_7 + c_{D2}F_8)\hat{\gamma}/\gamma$. The new entry in the first row, second column, is now

$$\begin{aligned} (c_{A1}e_{AB}^{(1)} + c_{A2}e_{AB}^{(2)})\hat{\gamma}/\gamma &= (c_{A1}Z_{A11}^{(B)} + c_{A2}Z_{A12}^{(B)})\hat{\gamma}/\gamma \\ &= \gamma(S_{A1}S_{B1} + T_{A1}T_{B1})\hat{\gamma}/\gamma = \hat{\gamma}(S_{A1}S_{B1} + T_{A1}T_{B1}) = f_{AB}, \end{aligned}$$

recalling $f_{AB} = \hat{\gamma}(S_{A1}S_{B1} + T_{A1}T_{B1}) = f_{BA}$. The altered matrix is then seen to be

$$\begin{bmatrix} \underline{e_{AB}^{(1)}} & f_{AB} & -e_{BA}^{(1)} & -f_{BA} & 0 & 0 & 0 & 0 & 0 \\ e_{AC}^{(1)} & f_{AC} & 0 & 0 & \underline{-e_{CA}^{(1)}} & -f_{CA} & 0 & 0 & 0 \\ 0 & 0 & \underline{e_{BC}^{(1)}} & f_{BC} & -e_{CB}^{(1)} & -f_{CB} & 0 & 0 & 0 \\ 0 & 0 & e_{BD}^{(1)} & f_{BD} & 0 & 0 & -e_{DB}^{(1)} & -f_{DB} & 0 \\ 0 & 0 & 0 & 0 & e_{CD}^{(1)} & f_{CD} & \underline{-e_{DC}^{(1)}} & -f_{DC} & 0 \\ 0 & 0 & 0 & 0 & e_{CE}^{(1)} & f_{CE} & 0 & 0 & -e_{EC}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{DE}^{(1)} & f_{DE} & \underline{-e_{ED}^{(1)}} \\ -e_{AD}^{(1)} & -f_{AD} & 0 & 0 & 0 & 0 & e_{DA}^{(1)} & f_{DA} & 0 \\ -e_{AE}^{(1)} & -f_{AE} & 0 & 0 & 0 & 0 & 0 & 0 & e_{EA}^{(1)} \end{bmatrix}$$

Consider the five underlined entries. A typical one is

$$e_{BC}^{(1)} = r_{B11}(S_{B1}S_{C1} + T_{B1}T_{C1}) + \alpha_{B11} \cdot i_{C1} + \beta_{B11} \cdot i_{C1}^c$$

It depends on the random vector α_{B11} , as do the other entries $e_{BX}^{(1)}$ in that column. But the dot product $\alpha_{B11} \cdot i_{C1}$ is linearly independent of the other such products $\alpha_{B11} \cdot i_{X1}$. So, by varying α_{B11} , the entry $e_{BC}^{(1)}$ can vary without changing any other matrix entries. (Note: $e_{BC}^{(2)}$ also depended on $\alpha_{B11} \cdot i_{C1}$, through the dependence between α_{B11} and α_{B12} . But k_{BC} is independent of the choice of α_{B11} . This is why we did the elementary column operations, to remove this interdependence between entries.)

The cofactor of the five underlined entries is the 4×4 matrix

$$\begin{bmatrix} 0 & f_{BD} & 0 & -f_{DB} \\ 0 & 0 & f_{CE} & 0 \\ -f_{AD} & 0 & 0 & f_{DA} \\ -f_{AE} & 0 & 0 & 0 \end{bmatrix}$$

whose determinant, $f_{BD}f_{CE}f_{AD}f_{AE}$, is in general nonzero. Thus $\det(F)$ is not identically 0. This gives evidence that the linear equations will usually be solvable.

It may become necessary to solve the equations mod q for each prime q dividing $p-1$, and piece together with the Chinese remainder theorem. In particular, for small primes q the determinant might vanish accidentally and we might have to compute logarithms mod q by brute force.

Consider now the general case $N \geq 1$. Now the general matrix entry $e_{AB}^{(k)}$ is not just $Z_{A,k,j}^{(B)}$ but rather a sum of products of such Z values. Nonetheless there are enough randomly chosen values α_{Aje} that an independence argument can be made similar to the case $N = 1$.

Select $M \geq 2N + 1$ conspirators randomly, and build an $[M(M-1)/2] \times [M(N+1)]$ matrix F corresponding to the example F ; we have again omitted one column to make an inhomogeneous system, but this time we do not drop any rows. Also M may be somewhat larger than $2N + 3$.

If the rank of F is fully $M(N+1) - 1$, we solve as before.

Suppose that, as M grows, the rank of F is always at least $M(N+1) - I$, where I is independent of M . Then, instead of representing all c_{Ai} as multiples of a single c_{B1} and using the corresponding group element $\hat{g}$, we select I different reference points c_{Bi} and use corresponding group elements $\hat{g}_i$. This is more tedious but still possible. One or two more conspirators might be necessary in this case.

Suppose that the rank of F is always bounded by $M(N+1-J)$, $J \geq 1$. This suggests that J linear relations exist among the $N+1$ columns of $e_{AX}^{(i)}$ corresponding to each user A , and that the quantities c_{Ai} could vary with J degrees of freedom without affecting the keys. (This would mean that the quantities described as independent variables were actually being chosen in a dependent fashion.) Detect these linear combinations and describe the system in terms of a reduced set of $N+1-J$ variables c_{Ai} . Then continue as before.

Finally, any systematically low rank of $F \bmod q$ would indicate systematic relations among the keys K_{AB} , which we should be able to exploit directly.

References

1. W. Diffie and M. E. Hellman, "New directions in cryptography," *IEEE Trans. Informat. Theory*, vol. IT-22, pp. 644-654, Nov. 1976.
2. S. Tsujii, K. Araki, and T. Sekine, "A new scheme of non interactive ID-based key sharing with explosively high degree of separability (second version)," Technical report of Department of Computer Science, Tokyo Institute of Technology, 93TR-0020, July 1993.

Note: the NIKS-TAS scheme has since appeared in the 1993 Korea-Japan Joint Workshop on Information Security and Cryptology. A partial attack, in the same workshop, was "An Attack on an ID-Based Key Sharing System," by V. Luchangco and K. Koyama.