A Linear Time Heuristics for Trapezoidation of GIS Polygons

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Abstract. The decomposition of planar polygons into triangles is a well studied area of Computer Graphics with particular relevance to GIS. Trapezoidation is often performed as a first step to triangulation. Though a linear time algorithm [2] for the decomposition of a simple polygon into triangles exists, it is extremely complicated and in practice $O(n \log n)$ algorithms are used. We present a very simple $O(n)$ -time heuristics for the trapezoidation of simple polygons without holes. Such polygons commonly occur in Geographic Information Systems (GIS) databases.

1 Introduction

The decomposition of a simple polygon into triangles is a well studied field of computational geometry and computer graphics [1]. In computer graphics and GIS applications, it is much easier to render a simple shape like a triangle compared to a large polygon. Similarly in computational geometry, a large number of data structures related to polygons are based on triangulations of these polygons. For many triangulation algorithms, a *trapezoidal decomposition* of the polygon is computed as a first step for *triangulation*. A trapezoid is a four-sided polygon with a pair of opposite sides parallel to each other. The main difference between a trapezoidal decomposition and a triangulation is that the vertices of the trapezoids can be additional points on the boundary of the polygon, whereas, in a triangulation it is not allowed to introduce new vertices.

The best known triangulation algorithm is by Chazelle [2] and it runs in $O(n)$ time, where n is the number of vertices of the polygon. However, Chazelle's algorithm is extremely complicated and difficult to implement. Earlier, Garey *et al.* [5] designed an $O(n \log n)$ algorithm for triangulation by dividing the input polygon into monotone pieces. Even now, the algorithm by Garey *et al.* [5] is extensively used for its simplicity and programming ease. Later, Chazelle and Incerpi [3] improved upon this algorithm by introducing the notion of *sinuosity* which is the number of *spiraling* and *anti-spiraling* polygonal chains on the polygon boundary. The algorithm by Chazelle and Incerpi [3] runs in $O(n \log s)$ time, where s is the sinuosity of the polygon. The sinuosity of a polygon is usually a small constant in most practical cases and the algorithm in [3] is an

almost linear-time algorithm. However, the algorithm by Chazelle and Incerpi [3] is based on a divide-and-conquer strategy and still quite complex.

In this paper, we are interested in the trapezoidal decomposition of a simple polygon rather than a triangulation. In computational geometry, a triangulation is performed for building efficient data structures based on this triangulation. Such data structures are used for many applications e.g., for processing shortest path queries or ray shooting. On the other hand, in almost all applications in computer graphics and geographic information systems (GIS), the main requirement is the fast rendering of a polygon. A trapezoid is as simple a shape as a triangle and a trapezoid is easy to render like a triangle in raster graphics devices. In this paper, we present an $O(n)$ -time simple heuristics for computing trapezoidal decomposition of a simple polygon without holes. Our heuristics can be used for fast rendering of large GIS data sets which quite often consist of polygons with thousands or hundreds of thousands of vertices.

The fastest known algorithm for trapezoidation of general GIS data sets is by Lorenzetto *et al.* [6]. Their algorithm improve upon a previously published algorithm by Žalik and Clapworthy [7] and can do trapezoidal decomposition of general GIS polygons with holes in $O(n \log n)$ time, where n is the total number of vertices for the polygon and all holes inside it. However, many GIS polygons do not contain holes inside them and it is interesting to investigate whether it is possible to design a simpler and faster algorithm for this problem when the polygons do not contain holes. Our algorithm is simple and terminates in $O(n)$ time for most polygons encountered in GIS applications, where n is the number of vertices in the polygon. We call the algorithm as a linear-time heuristics since we are unable to prove a tight theoretical upper bound for the running time of the algorithm. However for almost all polygons encountered in GIS data sets, the running time is $O(n)$ with the hidden constant in big-Oh very small. However, our algorithm always produces a correct trapezoidation and this is ensured through an in-built correctness test in the algorithm.

The rest of the paper is organized as follows. We discuss some preliminaries and an algorithm for trapezoidal decomposition of monotone polygons in Section 2. We present our $O(n)$ time heuristics in Section 3 and finally we discuss the complexity of the heuristics and our experimental results in Section 4.

2 Terminology

We consider only *simple polygons* without holes inside them and the edges of the polygons intersect only at end points. A *trapezoid* is a four-sided polygon with a pair of opposite sides parallel to each other. In this paper, all trapezoids have two sides parallel to the x -axis. We assume that no two vertices are on the same line parallel to the x or the y axis. In other words, no two vertices have the same y or x coordinates. This assumption simplifies our presentation considerably but we do not impose this restriction in our implementation. The algorithm processes vertices based on their type. We consider three types of vertices depending on the two end points of the two edges incident on the vertex. In the following, we

use the notion of the *left* and *right* side of the plane for a directed edge $\overrightarrow{uv}$ of the polygon. If the edge is directed from u to v , we assume that an observer is standing at u , facing v and the notion of the left or right side is with respect to this observer.

Fig. 1. (a) The three vertex types used in the algorithm. **(b)** The assignment of sweep lines. v_4 is of type INT, v_2 is of type MAX and v_6 is of type MIN. The other four vertices do not throw sweep lines.

Each vertex v_i is classified as one of the following types.

- **INT** if the neighboring vertices of v_i have a greater and a lower y -coordinate, respectively, than v_i . The interior of the polygon is to the left sides of the directed edges $\overrightarrow{v_{i-1}v_i}$ and $\overrightarrow{v_iv_{i+1}}$.
- **MIN** if both neighboring vertices have a greater y -coordinate than v_i . The interior of the polygon is to the right sides of the directed edges $\overrightarrow{v_{i-1}v_i}$ and $\overrightarrow{v_iv_{i+1}}$.
- **MAX** if both neighboring vertices have a lower y -coordinate than v_i . The interior of the polygon is to the left sides of the directed edges $\overrightarrow{v_{i-1}v_i}$ and $\overrightarrow{v_iv_{i+1}}$.

Note that all the vertices can be classified according to these three categories by traversing the polygon boundary once. In our algorithm, a vertex of type INT may become a corner vertex of a trapezoid. We indicate this by assigning a sweepline to an INT vertex. This sweepline starts at the INT vertex and goes either towards left or right depending on which side the interior of the polygon is. Similarly, a vertex of type MAX supports the bottom edge of a trapezoid and sweeplines may be thrown in both directions to determine the edge. A MIN vertex supports a top edge of a trapezoid and hence sweeplines are thrown in both directions. The different vertex types are shown in Figure 1(a). The shaded parts in Figure 1(a) denote the interior of the polygon. An example polygon with directions for sweeplines for its vertices is shown in Figure 1(b). Note that the directions of the sweeplines can be stored in the same pass of the polygon

boundary while determining the vertex types. This computation takes $O(n)$ time when n is the number of vertices.

2.1 A trapezoidation algorithm for monotone polygons

A monotone polygon is a polygon such that the polygon boundary can be decomposed into two chains by splitting the polygon at the two vertices with maximum and minimum y coordinates. These two chains are monotone with respect to the y -axis. In other words, a monotone polygon can have only vertices of type INT and no MAX or MIN vertices. Note that the vertex with highest (resp. least) y coordinate is not a MAX (resp. MIN) vertex according to our definition. Let v_j be the vertex with highest y -coordinate and v_{j-1} and v_{j+1} are the vertices just before and after v_j during a clockwise traversal of the polygonal boundary. Then the interior of the polygon is towards the right of the two directed edges $\overrightarrow{v_{j-1}v_j}$ and $\overrightarrow{v_jv_{j+1}}$. This violates the definition of a MAX vertex. It is easy to show in a similar way that the vertex with minimum y -coordinate is not a MIN vertex.

Our algorithm is based on a traversal of the polygon boundary and throwing sweeplines from each vertex. We first determine the direction in which each vertex throws sweepline in a preprocessing step as explained before. Our main task is to determine where each of the sweeplines intersect the polygon boundary. To determine this, we traverse the polygon boundary twice starting from v_{min} , the vertex with the minimum y -coordinate. The traversals are first in the clockwise and then in the counter-clockwise direction. We use a stack for holding the vertices which we encounter. In the following, by v_{top} we mean the vertex at the top of the stack. The stack is empty initially. During the clockwise traversal of the polygon boundary, we push a vertex v_k on the stack if v_k throws a sweepline towards right. When we encounter vertex v_i , we execute the following steps.

- We check whether a horizontal sweepline towards right from v_{top} intersects the edge $\overrightarrow{v_i v_{i+1}}$. If there is an intersection, we mark the intersecting point as a vertex of a trapezoid. In this case, v_{top} is removed from the stack.
- If v_i throws sweepline to the right, we push v_i onto the stack.

We execute a similar counter-clockwise traversal for vertices which throw sweeplines towards left. The trapezoids are reported in the following way. We store the intersection of the sweeplines with the edges in a separate data structure. For example, for an edge e_i , all the intersections of sweeplines with e_i are stored as a linked list of intersection points. These intersection points can be kept ordered according to y -coordinate easily. We can report the trapezoids starting from v_{min} and checking these intersection points as we move upwards.

3 An $O(n)$ -time algorithm for general polygons

The approach of the previous section does not work for more general polygons containing vertices of types MAX and MIN. The main problem is that the sweepline-edge intersections are not always correct.

In our algorithm, we use three stacks, one each for MAX, MIN, and INT vertex types. The only difference is that when pushing a vertex onto the stack, we check the vertex type and the vertex is pushed onto the correct stack. Similarly, previously an edge was tested against the top of the stack for an intersection, whereas now the edge must be tested against the top of all three stacks.

Fig. 2. Illustrations for incorrect trapezoidation. **(a)** After a clockwise traversal vertex v_3 has thrown a sweepline right and caused an incorrect intersection with edge e_4 . **(b)** After a counter-clockwise traversal vertex v_6 has thrown a sweepline right and caused an incorrect intersection with e_4 . Both traversals start at v_{min} .

Consider Figure 2(a) and in particular vertex v_3 . During a counter-clockwise traversal v_3 correctly throws a sweepline to the left intersecting edge e_1 . However it does not throw a sweepline to the right as v_3 is seen after edges e_6 , e_5 , and e_4 — the only edges a sweepline right from v_3 would intersect.

Note that in a clockwise traversal edge e_4 is seen before e_6 . For correctness a sweepline right from v_3 must intersect edge e_6 , but as e_4 is seen first, the right intersection from v_3 is calculated with e_4 , and v_3 is then popped from the stack. In situations like this, we need to check whether the sweepline is correctly resolved, i.e., it intersects the correct edge. A similar situation is shown in Figure 2(b) when a counter-clockwise traversal causes a wrong intersection for vertex v_6 . We denote the x and y coordinates of a point p_i by x_i and y_i . Consider a MIN vertex v_i and a MAX vertex v_j . Suppose the sweeplines towards right from v_i and v_j intersect an edge e_k at the points p_i and p_j respectively. Then the following lemma holds.

Lemma 1. *If $y_i < y_j$, then the sweepline towards right (i) either from v_i or (ii) from v_j , cannot be resolved correctly on the polygonal chain from v_i to v_j .*

Proof. First note that the sweepline towards right from the MAX vertex v_j has a greater y coordinate compared to the sweepline from the MIN vertex v_i . Also, v_j must have been encountered after v_i during a clockwise traversal starting from v_{min} . Otherwise, the sweepline from v_j would have intersected either the edge with v_i as an endpoint or some other edge before reaching v_i .

Suppose the sweepline from v_i is resolved on an edge e_k on the polygonal chain from v_i to v_j . Note that v_j has a higher y coordinate compared to v_i . There are two possibilities. In the first case, the sweepline from v_i intersects some edge on the polygonal chain from v_j to v_{min} . This is the case in Figure 2(a). The vertices v_3 and v_6 play the role of v_i and v_j and the edge e_4 plays the role of e_k in this case. The other possibility is that the sweepline from v_j (towards right) intersects some edge on the polygonal chain from v_{min} to v_i . This is the case shown in Figure 2(b). Hence, either the sweepline from v_i or the sweepline from v_j cannot be correctly resolved on the polygonal chain from v_i to v_j .

In our algorithm, we produce a correct trapezoidation iteratively and after each iteration we need to check the correctness of the trapezoidation. This check is based on a generalization of Lemma 1. Consider an edge e_k with two end points v_k and v_{k+1} . We call the polygonal chain from v_{min} to v_k (resp. v_{k+1} to v_{min}) during a clockwise traversal as the chain before e_k (resp. after e_k) and denote by $before(e_k)$ (resp. $after(e_k)$). Suppose in the previous iteration a set of vertices on $before(e_k)$ and $after(e_k)$ had their sweepelines resolved on e_k . We denote the minimum y -coordinate among all the vertices on $before(e_k)$ that are resolved on e_k by b_k and similarly the maximum y -coordinate among all the vertices on $after(e_k)$ that are resolved on e_k by a_k . Then the following lemma allows us to check the correctness of our trapezoidation.

Lemma 2. *A trapezoidation is correct if and only if for every edge $e_k, 1 \leq k \leq n$, $b_k > a_k$.*

Proof. First assume that there is at least one edge e_k such that $b_k < a_k$. From Lemma 1 it is clear that the vertex on $before(e_k)$ that is responsible for b_k has not resolved its sweepline correctly. Hence, the presence of at least one such edge e_k makes the trapezoidation incorrect.

To prove the other direction, assume that for every edge e_k , $b_k > a_k$. In that case, all the sweepelines are resolved correctly and hence the trapezoidation is correct.

In Figure 2, for clockwise traversal, the polygonal chain from v_1 to v_4 is $before(e_4)$ and the polygonal chain from v_5 to v_1 is $after(e_4)$. b_4 and a_4 are the y coordinates of the intersections on the edge e_4 by vertices on $before(e_4)$ and $after(e_4)$. In this case, b_4 is determined by v_3 and a_4 is determined by v_6 . The trapezoidation is incorrect since $y_3 < y_6$.

It is clear that if we identify condition of Lemma 1, we cannot resolve one of the sweepelines, either from v_i or from v_j , on the polygonal chain from v_i to v_j . In the first case, we should try to resolve the sweepline from v_i only after

crossing v_j during the clockwise traversal of the polygon. In the second case, we should try to resolve the sweepline from v_j only after crossing v_i during a counter-clockwise traversal of the polygon. A simple heuristics to overcome the problem described above is to extend the stack approach to *stack-of-stacks of vertices*, in effect restricting which edges see which sweeplines. In the above example from Figure 2 the problem is caused by the overlapping MAX and MIN vertices. This causes the order in which edges are seen to be incorrect.

3.1 Preprocessing

The preprocessing stage classifies each vertex based on type. Once a vertex has been classified it is assigned sweeplines. We assign an initial intersection point for each vertex in the preprocessing stage itself. In general some of these initial intersections are incorrect and they are refined in the next stage of the algorithm. Intersections are calculated for each vertex that throws a sweepline. The first vertex tested is v_{min} . From v_{min} each vertex is tested in order, in both clockwise and counter-clockwise directions. The polygon boundary is traversed twice for the following reason. For a sweepline-edge intersection to occur the vertex must be seen before the edge. This is necessary because the vertex must be on the stack when the edge is encountered. In some cases this will only occur during a clockwise traversal of the polygon boundary, and in others only during a counter-clockwise traversal. During the pre-processing stage only MAX and MIN vertices are considered, all other vertices are ignored. The following deals with MAX vertices, but the case for MIN is identical. As a sweeping MAX vertex is encountered the edge containing the intersection is updated to note that it has been intersected by a MAX vertex. Once both the clockwise and counter-clockwise passes are complete all edges intersected by MAX vertices are marked.

At the end of the preprocessing, each edge potentially stores two pieces of information. An array *min* of y -coordinates for intersections of all the sweeplines thrown by the vertices of type MIN and an array *max* of y -coordinates for intersections of all the sweeplines thrown by vertices of type MAX. These intersections occur according to increasing y coordinates for vertices of type MIN and according to decreasing y -coordinates for vertices of type MAX. Also, overall there are at most $2n$ intersections since each sweepline intersects only one edge. Hence, these two arrays are maintained in linear time during preprocessing.

By y_{min} we denote the minimum y -coordinate in the array *min* and by y_{max} the maximum y -coordinate in the array *max*. This information is used for checking the condition of Lemma 1. Note that not all intersections are correct at this stage. Consider again Figure 2. After pre-processing vertex v_3 still throws a sweepline to the right incorrectly, intersecting edge e_4 .

3.2 Main iteration

We now discuss the main iterations of our algorithm. Each iteration has two parts, first a trapezoidation step and then a testing stage. We compute a trapezoidation first and then test whether the trapezoidation is correct by checking

the condition of Lemma 2. We proceed to the next iteration if the trapezoidation is incorrect otherwise the algorithm terminates after reporting the trapezoids.

The trapezoidation in each iteration proceeds similar to the second part of the preprocessing stage, one clockwise and one counter-clockwise traversal of the polygon boundary. As in Section 2.1, we consider vertices of type INT which throw sweepline towards right during the clockwise traversal and vertices of type INT which throw sweepelines towards left during the counter-clockwise traversal. However, we consider vertices of types MAX and MIN during both the traversals. A vertex of type MIN or MAX is pushed onto the corresponding stack when encountered and popped from the stack when both the sweepelines from such a vertex is resolved. The MAX/MIN intersection information stored in the pre-processing stage is used to control which vertices are tested against each edge. A vertex can cause a sweepline-edge intersection left, or right, during both the clockwise and counter-clockwise traversals. As an edge e_i is encountered, we take the following actions with respect to v_{top} , the top vertices in each of the three stacks for MAX, MIN and INT vertices. We denote the y -coordinate of v_{top} by y_{top} . The following steps are taken when v_{top} is the top vertex in the MIN stack. The actions are similar when v_{top} is the top vertex in the other two stacks.

- We check whether a sweepline from v_{top} intersects e_i . If there is an intersection, we check whether the condition in Lemma 1 is violated. For example, if v_{top} is a MIN vertex, this condition will be violated if $y_{top} < y_{max}$. Suppose, a vertex v_j is responsible for the y_{max} value. We now initialize three new stacks for the three types of vertices. This action can be viewed as maintaining three data structures which are *stack-of-stacks*, one each for vertex types MIN, MAX and INT. We push three new empty stacks on top of each of the three stack-of-stacks. Note that the sweepline from the vertex v_{top} cannot be resolved until we encounter the vertex v_j , and hence we need to maintain these new stacks until we reach v_j .
- We delete y_{top} from the array min and if necessary, update y_{min} .
- If both (in case of MIN or MAX) or one (in case of INT) sweepline(s) of v_{top} is resolved, we pop v_{top} out of the stack-of-stacks. In other words, v_{top} is popped from the top-most stack of one of the three stack-of-stacks depending on its type.
- If the vertex v_i of edge e_i throws sweepelines, we push v_i into the correct stack-of-stacks depending on its type.

Once both the clockwise and counter-clockwise traversals are over, we test the condition of Lemma 2 for each edge of the polygon. This is done by comparing y_{min} and y_{max} for each of the edges. If at least one edge violates the condition, we proceed to the next iteration. Otherwise we report the trapezoids and the algorithm terminates.

Note that the motivation for using a stack-of-stacks data structure is the property discussed in Lemma 1. The current top stack in a stack-of-stacks takes care of the polygonal chain between a MIN vertex and a corresponding MAX

vertex when these two vertices are interleaved, i.e., the y coordinate of the MAX vertex is higher than the y -coordinate of the MIN vertex as in Figure 2. An old stack is removed from the stack-of stacks when we reach a matching MAX vertex. For example, in Figure 2(a) when we are doing a clockwise traversal, we initialize a new stack when we are at edge e_4 and find the overlapping of MIN/MAX pair of v_3 and v_6 . We remove this stack when we reach v_6 and the sweepline from v_3 is correctly resolved after that.

When the algorithm terminates, we report the trapezoids by traversing the arrays *min* and *max* for each of the edges in the polygon. These are the correct intersections of sweeplines and vertices and hence the supports of the trapezoids. We omit the details of the reporting of trapezoids in this version. and refer to Figure 3 for an example.

Fig. 3. An example polygon decomposed into trapezoids after the termination of our heuristics.

4 Experimental Results and conclusion

The preprocessing stage as well as each iteration of the heuristics takes $O(n)$ time. The reporting of the trapezoids also takes $O(n)$ time. However, we are unable to prove a theoretical upper bound for the heuristics at this point. We strongly suspect that there is a close relationship between the number of iterations required for producing a correct trapezoidation and the sinuosity of the polygon as defined by Chazelle and Incerpi [3]. It is quite difficult to generate polygons with high sinuosity and we have tested our algorithm for polygons up to sinuosity 5 and the number of iterations required are always equal to the sinuosity of the polygon up to this point. Hence, we conjecture that the theoret-

ical upper bound for the running time of our heuristics is $O(ns)$, where s is the sinuosity of the polygon.

However, most GIS polygons from the GIS Data Depot [4] are polygonal data for land subdivisions, property boundaries and roadways. It is not common to have a high sinuosity for such polygonal data. For all GIS polygons that we have used, the algorithm terminates in four or five iterations. We used a Celeron 400 PC with 328MB of RAM running Linux Redhat 6 as our experimental platform. We implemented the algorithm by Lorenzetto *et al.* [6] and the present algorithm, both in C++ using the g++ compiler. Some of the results are shown in Table 1. The present algorithm becomes more efficient for polygons with a large number of vertices and we expect that it will be useful in practice when it is necessary to do trapezoidal decomposition of a large number of polygons very fast.

Table 1. The comparison between our algorithm and the $O(n \log n)$ algorithm by Lorenzetto *et al.* [6]. All running times are in seconds.

<i>Number of vertices</i>	<i>$O(n \log n)$ algorithm in [6]</i>	<i>Our $O(n)$ heuristics</i>
5000	0.12	0.11
10000	0.26	0.21
20000	0.50	0.42
30000	0.76	0.64

We plan to investigate the worst-case complexity of our algorithm and construct some worst-case example polygons. This will help us to check whether such worst-case polygons occur in practice in GIS data sets. We also plan to test our algorithm more extensively on GIS data sets in order to optimize our code further.

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