Strong Linear Dependence and Unbiased Distribution of Non-propagative Vectors

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Abstract. This paper proves (i) in any $(n - 1)$ -dimensional linear subspace, the non-propagative vectors of a function with n variables are linearly dependent, (ii) for this function, there exists a non-propagative vector in any $(n - 2)$ -dimensional linear subspace and there exist three non-propagative vectors in any $(n - 1)$ -dimensional linear subspace, except for those functions whose nonlinearity takes special values.

Keywords: Cryptography, Boolean Function, Propagation, Nonlinearity.

1 Introduction

In examining the nonlinearity properties of a function f with n variables, it is important to understand $\mathfrak{R}_f$, the set of so-called non-propagative vectors where f does not satisfy the propagation criterion. In this work, we are concerned with both $\#\mathfrak{R}_f$ (the number of non-propagative vectors in $\mathfrak{R}_f$) and the distribution of $\mathfrak{R}_f$. More specifically, we prove two properties of $\mathfrak{R}$. One is called the strong linear dependence and the other the unbiased distribution, of $\mathfrak{R}$.

The strong linear dependence property states that if W is a $(n - 1)$ -dimensional linear subspace satisfying $\#(\mathfrak{R} \cap W) \geq 4$, then the non-zero vectors in $\mathfrak{R} \cap W$ are linearly dependent. This improves a previously known result. The unbiased distribution property says that any function f with n variables, except for those whose nonlinearity takes the special value of $2^{n-1} - 2^{\frac{1}{2}(n-1)}$, $2^{n-1} - 2^{\frac{1}{2}n}$ or $2^{n-1} - 2^{\frac{1}{2}n-1}$, fulfills the condition that every $(n - 2)$ -dimensional linear subspace contains a non-zero vector in $\mathfrak{R}_f$ and every $(n - 1)$ -dimensional linear subspace contains at least three non-zero vectors in $\mathfrak{R}_f$. In special cases, $\#(\mathfrak{R} \cap W)$ may significantly effect other cryptographic properties of a function. The strong linear dependence and the unbiased distribution are helpful for the design of cryptographic functions as these conclusions provide more information on the number and the status of non-propagative vectors in any $(n - 1)$ -dimensional linear subspace.

2 Cryptographic Criteria of Boolean Functions

We consider functions from V_n to $GF(2)$ (or simply functions on V_n), V_n is the vector space of n tuples of elements from $GF(2)$. The *truth table* of a function f on V_n is a $(0, 1)$ -sequence defined by $(f(\alpha_0), f(\alpha_1), \dots, f(\alpha_{2^n-1}))$, and the *sequence* of f is a $(1, -1)$ -sequence defined by $((-1)^{f(\alpha_0)}, (-1)^{f(\alpha_1)}, \dots, (-1)^{f(\alpha_{2^n-1})})$, where $\alpha_0 = (0, \dots, 0, 0)$, $\alpha_1 = (0, \dots, 0, 1)$, $\dots$, $\alpha_{2^n-1} = (1, \dots, 1, 1)$. The *matrix* of f is a $(1, -1)$ -matrix of order 2^n defined by $M = ((-1)^{f(\alpha_i \oplus \alpha_j)})$ where $\oplus$ denotes the addition in $GF(2)$. f is said to be *balanced* if its truth table contains an equal number of ones and zeros.

Given two sequences $\tilde{a} = (a_1, \dots, a_m)$ and $\tilde{b} = (b_1, \dots, b_m)$, their *component-wise product* is defined by $\tilde{a} * \tilde{b} = (a_1 b_1, \dots, a_m b_m)$. In particular, if $m = 2^n$ and $\tilde{a}, \tilde{b}$ are the sequences of functions f and g on V_n respectively, then $\tilde{a} * \tilde{b}$ is the sequence of $f \oplus g$ where $\oplus$ denotes the addition in $GF(2)$.

Let $\tilde{a} = (a_1, \dots, a_m)$ and $\tilde{b} = (b_1, \dots, b_m)$ be two sequences or vectors, the *scalar product* of $\tilde{a}$ and $\tilde{b}$, denoted by $\langle \tilde{a}, \tilde{b} \rangle$, is defined as the sum of the component-wise multiplications. In particular, when $\tilde{a}$ and $\tilde{b}$ are from V_m , $\langle \tilde{a}, \tilde{b} \rangle = a_1 b_1 \oplus \dots \oplus a_m b_m$, where the addition and multiplication are over $GF(2)$, and when $\tilde{a}$ and $\tilde{b}$ are $(1, -1)$ -sequences, $\langle \tilde{a}, \tilde{b} \rangle = \sum_{i=1}^m a_i b_i$, where the addition and multiplication are over the reals.

An *affine* function f on V_n is a function that takes the form of $f(x_1, \dots, x_n) = a_1 x_1 \oplus \dots \oplus a_n x_n \oplus c$, where $a_j, c \in GF(2)$, $j = 1, 2, \dots, n$. Furthermore f is called a *linear* function if $c = 0$.

A $(1, -1)$ -matrix N of order n is called a *Hadamard* matrix if $NN^T = nI_n$, where N^T is the transpose of N and I_n is the identity matrix of order n . A Sylvester-Hadamard matrix of order 2^n , denoted by H_n , is generated by the following recursive relation

$$H_0 = 1, H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}, n = 1, 2, \dots$$

Let ℓ_i , $0 \leq i \leq 2^n - 1$, be the i row of H_n . It is known that ℓ_i is the sequence of a linear function $\varphi_i(x)$ defined by the scalar product $\varphi_i(x) = \langle \alpha_i, x \rangle$, where α_i is the i th vector in V_n according to the ascending alphabetical order.

The *Hamming weight* of a $(0, 1)$ -sequence ξ , denoted by $W(\xi)$, is the number of ones in the sequence. Given two functions f and g on V_n , the *Hamming distance* $d(f, g)$ between them is defined as the Hamming weight of the truth table of $f(x) \oplus g(x)$, where $x = (x_1, \dots, x_n)$.

Definition 1. The nonlinearity of a function f on V_n , denoted by N_f , is the minimal Hamming distance between f and all affine functions on V_n , i.e., $N_f = \min_{i=1,2,\dots,2^{n+1}} d(f, \varphi_i)$ where $\varphi_1, \varphi_2, \dots, \varphi_{2^{n+1}}$ are all the affine functions on V_n .

The following characterisations of nonlinearity will be useful (for a proof see for instance [2]).

Lemma 1. *The nonlinearity of f on V_n can be expressed by*

$$N_f = 2^{n-1} - \frac{1}{2} \max\{|\langle \xi, \ell_i \rangle|, 0 \leq i \leq 2^n - 1\}$$

where ξ is the sequence of f and $\ell_0, \dots, \ell_{2^n-1}$ are the rows of H_n , namely, the sequences of linear functions on V_n .

Definition 2. *Let f be a function on V_n . For a vector $\alpha \in V_n$, denote by $\xi(\alpha)$ the sequence of $f(x \oplus \alpha)$. Thus $\xi(0)$ is the sequence of f itself and $\xi(0) * \xi(\alpha)$ is the sequence of $f(x) \oplus f(x \oplus \alpha)$. Set*

$$\Delta_f(\alpha) = \langle \xi(0), \xi(\alpha) \rangle,$$

the scalar product of $\xi(0)$ and $\xi(\alpha)$. $\Delta(\alpha)$ is called the auto-correlation of f with a shift α . Write

$$\Delta_M = \max\{|\Delta(\alpha)| \mid \alpha \in V_n, \alpha \neq 0\}$$

We omit the subscript of $\Delta_f(\alpha)$ if no confusion occurs.

Definition 3. *Let f be a function on V_n . We say that f satisfies the propagation criterion with respect to α if $f(x) \oplus f(x \oplus \alpha)$ is a balanced function, where $x = (x_1, \dots, x_n)$ and α is a vector in V_n . Furthermore f is said to satisfy the propagation criterion of degree k if it satisfies the propagation criterion with respect to every non-zero vector α whose Hamming weight is not larger than k (see [3]).*

The strict avalanche criterion (SAC) [5] is the same as the propagation criterion of degree one.

Obviously, $\Delta(\alpha) = 0$ if and only if $f(x) \oplus f(x \oplus \alpha)$ is balanced, i.e., f satisfies the propagation criterion with respect to α .

Definition 4. *Let f be a function on V_n . $\alpha \in V_n$ is called a linear structure of f if $|\Delta(\alpha)| = 2^n$ (i.e., $f(x) \oplus f(x \oplus \alpha)$ is a constant).*

For any function f , $\Delta(\alpha_0) = 2^n$, where α_0 is the zero vector on V_n . It is easy to verify that the set of all linear structures of a function f form a linear subspace of V_n , whose dimension is called the *linearity* of f . It is also well-known that if f has non-zero linear structures, then there exists a nonsingular $n \times n$ matrix B over $GF(2)$ such that $f(xB) = g(y) \oplus h(z)$, where $x = (y, z)$, $y \in V_p$, $z \in V_q$, g is a function on V_p that has no non-zero linear structures, and h is an affine function on V_q .

The following lemma is the re-statement of a relation proved in Section 2 of [1].

Lemma 2. *For every function f on V_n , we have*

$$(\Delta(\alpha_0), \Delta(\alpha_1), \dots, \Delta(\alpha_{2^n-1}))H_n = (\langle \xi, \ell_0 \rangle^2, \langle \xi, \ell_1 \rangle^2, \dots, \langle \xi, \ell_{2^n-1} \rangle^2).$$

where ξ denotes the sequence of f and ℓ_i is the i th row of H_n , $i = 0, 1, \dots, 2^n - 1$.

The balance and the nonlinearity are necessary in most cases. The propagation or especially the SAC, is an important cryptographic criterion.

3 Introduction to $\mathfrak{R}$

Notation 1. Let f be a function on V_n . Set $\mathfrak{R}_f = \{\alpha \mid \Delta(\alpha) \neq 0, \alpha \in V_n\}$, $\Delta_M = \max\{|\Delta(\alpha)| \mid \alpha \in V_n, \alpha \neq 0\}$.

We simply write $\mathfrak{R}_f$ as $\mathfrak{R}$ if no confusion occurs. It is easy to verify that $\#\mathfrak{R}$ and Δ_M are invariant under any nonsingular linear transformation on the variables, where $\#$ denotes the cardinal number of a set.

$\#\mathfrak{R}$ and the distribution of $\mathfrak{R}$ reflects the propagation characteristics, while Δ_M forecasts the avalanche property of the function. Therefore information on $\mathfrak{R}$ and Δ_M is useful in determining important cryptographic characteristics of f . Usually, small $\#\mathfrak{R}$ and Δ_M are desirable.

Definition 5. A function f on V_n is called a bent function [4] if $\langle \xi, \ell_i \rangle^2 = 2^n$ for every $i = 0, 1, \dots, 2^n - 1$, where ℓ_i is the i th row of H_n .

A bent function on V_n exists only when n is even, and it achieves the highest possible nonlinearity $2^{n-1} - 2^{\frac{1}{2}n-1}$. The algebraic degree of bent functions on V_n is at most $\frac{1}{2}n$ [4]. From [4] and Parseval's equation, we have the following:

Theorem 1. Let f be a function on V_n . Then the following statements are equivalent: (i) f is bent, (ii) $\#\mathfrak{R} = 1$, (iii) $\Delta_M = 0$, (iv) the nonlinearity of f , N_f , satisfies $N_f = 2^{n-1} - 2^{\frac{1}{2}n-1}$, (v) the matrix of f is an Hadamard matrix.

The following result is called the *linear dependence theorem* that can be found in [7]

Theorem 2. Let f be a function on V_n that satisfies the propagation criterion with respect to all but $k + 1$ vectors $0, \beta_1, \dots, \beta_k$ in V_n , where $k \geq 2$. Then $\beta_1, \dots, \beta_k$ are linearly dependent, namely, there exist k constants $c_1, \dots, c_k \in GF(2)$, not all of which are zeros, such that $c_1\beta_1 \oplus \dots \oplus c_k\beta_k = 0$.

Note that $n + 1$ non-zero vectors in V_n must be linearly dependent. Hence if $\#\mathfrak{R} \geq n + 2$ (i.e., $\#(\mathfrak{R} - \{0\}) \geq n + 1$) then Theorem 2 is trivial. For this reason, we improve Theorem 2 in this paper. We prove two properties of $\mathfrak{R}$: the strong linear dependence and the unbiased distribution of $\mathfrak{R}$.

4 The Strong Linear Dependence Theorem

Note the i th (i.e., the α_i th) row of H_n , where $\alpha_i \in V_n$ is the binary representation of integer j , $j = 0, 1, \dots, 2^n - 1$, is the sequence of linear function $\varphi_i(x) = \langle \alpha_i, x \rangle$. Lemma 4 of [7] can be restated as follows:

Lemma 3. Let Q be the $2^n \times q$ that consists of the α_{j_1} th, $\dots$, the α_{j_q} th rows of H_n , where each $\alpha_j \in V_n$ is the binary representation of integer j , $0 \leq j \leq 2^n - 1$. If $\alpha_{j_1}, \dots, \alpha_{j_q}$ are linearly independent then each $(a_1, \dots, a_q)^T$, where each $a_j = \pm 1$, appears as a column in Q precisely 2^{n-q} times.

The following Lemma can be found in [7].

Lemma 4. *Let $n \geq 3$ be a positive integer and $2^n = \sum_{j=1}^4 a_j^2$ where $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$ and each a_j is an integer. We have the following statements:*

- (i) *if n is odd, then $a_1^2 = a_2^2 = 2^{n-1}$, $a_3 = a_4 = 0$,*
- (ii) *if n is even, then $a_1^2 = 2^n$, $a_2 = a_3 = a_4 = 0$ or $a_1^2 = a_2^2 = a_3^2 = a_4^2 = 2^{n-2}$.*

Lemma 5. *For every function f on V_n , we have*

$$2(\Delta(\alpha_0), \Delta(\alpha_2), \dots, \Delta(\alpha_{2^n-2}))H_{n-1} \\ = (\langle \xi, \ell_0 \rangle^2 + \langle \xi, \ell_1 \rangle^2, \langle \xi, \ell_2 \rangle^2 + \langle \xi, \ell_3 \rangle^2, \dots, \langle \xi, \ell_{2^n-2} \rangle^2 + \langle \xi, \ell_{2^n-1} \rangle^2)$$

where ξ denotes the sequence of f and ℓ_i is the i th row of H_n , $i = 0, 1, \dots, 2^n - 1$.

Proof. From Lemma 2,

$$2^n(\Delta(\alpha_0), \Delta(\alpha_1), \dots, \Delta(\alpha_{2^n-1})) = (\langle \xi, \ell_0 \rangle^2, \langle \xi, \ell_1 \rangle^2, \dots, \langle \xi, \ell_{2^n-1} \rangle^2)H_n \quad (1)$$

Comparing the 0th, the 2nd, $\dots$, the $(2^n - 2)$ th terms in the two sides of equality (1), we obtain

$$2^n(\Delta(\alpha_0), \Delta(\alpha_2), \dots, \Delta(\alpha_{2^n-2})) \\ = (\langle \xi, \ell_0 \rangle^2 + \langle \xi, \ell_1 \rangle^2, (\langle \xi, \ell_2 \rangle^2 + \langle \xi, \ell_3 \rangle^2, \dots, \langle \xi, \ell_{2^n-2} \rangle^2 + \langle \xi, \ell_{2^n-1} \rangle^2)H_{n-1}$$

This proves the lemma. $\square$

The following theorem is called *the strong linearly dependence theorem* which is an improvement on Theorem 2 (the linearly dependence theorem).

Theorem 3. *Let f be a function on V_n , and W be a $(n - 1)$ -dimensional linear subspace satisfying $\Re \cap W = \{0, \beta_1, \dots, \beta_k\}$ ($k \geq 3$). Then $\beta_1, \dots, \beta_k$ are linearly dependent, namely, there exist k constants $c_1, \dots, c_k \in GF(2)$ with $(c_1, \dots, c_k) \neq (0, \dots, 0)$, such that $c_1\beta_1 \oplus \dots \oplus c_k\beta_k = 0$.*

Proof. The theorem is obviously true if $k > n$. Now we prove the theorem for $k \leq n$. We only need to prove the lemma in the special case when W is composed of $\alpha_0, \alpha_2, \dots, \alpha_{2^n-2}$, where $\alpha_{2j} \in V_n$ is the binary representation of an even number $2j$, $j = 0, 1, \dots, 2^{n-1} - 1$. In other words, W is composed of all the vectors in V_n , that can be expressed in the form $(a_1, \dots, a_{n-1}, 0)$, where each $a_j \in GF(2)$. In the general case, we can use a nonsingular linear transformation on the variables so as to change W into the special case. Let ξ be the sequence of f .

Since $\beta_j \in W$, $j = 1, \dots, k$, β_j can be expressed as $\beta_j = (\gamma_j, 0)$ where $\gamma_j \in V_{n-1}$, $j = 1, \dots, k$, and $0 \in GF(2)$.

Let P be a $(k+1) \times 2^{n-1}$ matrix composed of the 0th, the γ_1 th, $\dots$, the γ_k th rows of H_{n-1} . Set $a_j^2 = \langle \xi, \ell_j \rangle^2$, $j = 0, 1, \dots, 2^n - 1$. Note that $\Delta(\alpha) = 0$ if $\alpha \notin \{0, \beta_1, \dots, \beta_k\}$. Hence the equality in Lemma 5 can be specialized as

$$2(\Delta(0), \Delta(\beta_1), \dots, \Delta(\beta_k))P = (a_0^2 + a_1^2, a_2^2 + a_3^2, \dots, a_{2^{n-2}}^2 + a_{2^{n-1}}^2) \quad (2)$$

where $\Delta(0)$ is identical to $\Delta(\alpha_0)$ where $\alpha_0 = 0$.

Write $P = (p_{ij})$, $i = 0, 1, \dots, k$, $j = 0, 1, \dots, 2^{n-1} - 1$. As the top row of P is $(1, 1, \dots, 1)$, from (2),

$$2(\Delta(0) + \sum_{i=1}^k p_{ij} \Delta(\beta_i)) = a_{2j}^2 + a_{2j+1}^2 \quad (3)$$

$j = 0, 1, \dots, 2^{n-1} - 1$. Let P^* be the submatrix of P obtained by removing the top row from P .

We now prove the theorem by contradiction. Suppose k vectors in V_n , $\beta_1, \dots, \beta_k$, are linearly independent. Hence k vectors in V_{n-1} , $\gamma_1, \dots, \gamma_k$, are also linearly independent and hence $k \leq n - 1$.

Applying Lemma 3 to matrix P^* , we conclude that each k -dimensional $(1, -1)$ -vector appears in P^* , as a column vector of P^* precisely 2^{n-1-k} times. Thus for each fixed j there exists a number j_0 , $0 \leq j_0 \leq 2^{n-1} - 1$, such that $(p_{1j_0}, \dots, p_{kj_0}) = -(p_{1j}, \dots, p_{kj})$ and hence

$$2(\Delta(0) - \sum_{i=1}^k p_{ij_0} \Delta(\beta_i)) = a_{j_0}^2 + a_{2j_0+1}^2 \quad (4)$$

Adding (3) and (4) together, we have $4\Delta(0) = a_j^2 + a_{2j+1}^2 + a_{j_0}^2 + a_{2j_0+1}^2$. Hence $a_j^2 + a_{2j+1}^2 + a_{j_0}^2 + a_{2j_0+1}^2 = 2^{n+2}$. There are two cases to be considered: even n and odd n .

Case 1: n is odd. By using Lemma 4,

$$\{a_j^2, a_{2j+1}^2, a_{j_0}^2, a_{2j_0+1}^2\} = \{2^{n+1}, 2^{n+1}, 0, 0\}, j = 0, 1, \dots, 2^{n-1} \quad (5)$$

Hence from (3), we have $\Delta(0) + \sum_{i=1}^k p_{ij} \Delta(\beta_i) = 2^{n+1}, 2^n, 0$ and hence

$$\sum_{i=1}^k p_{ij} \Delta(\beta_i) = 2^n, 0, -2^n, j = 0, 1, \dots, 2^n - 1 \quad (6)$$

For each fixed j , rewrite (6) as

$$p_{1j} \Delta(\beta_1) + \sum_{i=2}^k p_{ij} \Delta(\beta_i) = 2^n, 0, -2^n \quad (7)$$

By using Lemma 3, there exists a number j_1 , $0 \leq j_1 \leq 2^{n-1} - 1$, such that $(p_{1j_1}, p_{2j_1}, \dots, p_{kj_1}) = (p_{1j}, -p_{2j}, \dots, -p_{kj})$.

Hence

$$p_{1j_1}\Delta(\beta_1) - \sum_{i=2}^k p_{ij_1}\Delta(\beta_i) = 2^n, 0, -2^n \quad (8)$$

Adding (7) and (8) together, we have

$$p_{1j}\Delta(\beta_1) = \pm 2^n, \pm 2^{n-1}, 0$$

Since $\Delta(\beta_1) \neq 0$, we conclude $\Delta(\beta_1) = \pm 2^n, \pm 2^{n-1}$. By the same reasoning we can prove

$$\Delta(\beta_j) = \pm 2^n, \pm 2^{n-1}, j = 1, 2, \dots, k \quad (9)$$

Thus we can write

$$(\Delta(\beta_1), \dots, \Delta(\beta_k)) = 2^{n-1}(b_1, \dots, b_k) \quad (10)$$

where each $b_j = \pm 1, \pm 2$. By using Lemma 3, there exists a number s , $0 \leq s \leq 2^{n-1} - 1$, such that

$$(p_{1s}, \dots, p_{ks}) = \left(\frac{b_1}{|b_1|}, \dots, \frac{b_k}{|b_k|} \right). \quad (11)$$

Due to (10) and (11),

$$\sum_{i=1}^k p_{is}\Delta(\beta_i) = \sum_{i=1}^k \frac{b_i}{|b_i|}\Delta(\beta_i) = \sum_{i=1}^k \frac{b_i^2}{|b_i|}2^{n-1} = 2^{n-1} \sum_{i=1}^k |b_i| \geq k2^{n-1}. \quad (12)$$

Since $k \geq 3$, (12) contradicts (6).

Case 2: n is even. By using Lemma 4,

$$\begin{aligned} \{a_j^2, a_{2j+1}^2, a_{j_0}^2, a_{2j_0+1}^2\} &= \{2^{n+2}, 0, 0, 0\} \text{ or} \\ \{a_j^2, a_{2j+1}^2, a_{j_0}^2, a_{2j_0+1}^2\} &= \{2^n, 2^n, 2^n, 2^n\}, j = 0, 1, \dots, 2^{n-1} \end{aligned} \quad (13)$$

Hence from (3), we have $\Delta(0) + \sum_{i=1}^k p_{ij}\Delta(\beta_i) = 2^{n+1}, 2^n, 0$, and hence

$$\sum_{i=1}^k p_{ij}\Delta(\beta_i) = 2^n, 0, -2^n$$

Repeating the same deduction as in Case 1, we obtain a contradiction in Case 2.

Summarizing Cases 1 and 2, we conclude that the assumption that $\beta_1, \dots, \beta_k$ are linearly independent is wrong. This proves the theorem. $\square$

Theorem 3 shows that $\mathfrak{R}$ is subject to crucial restrictions. We now compare Theorem 3 with Theorem 2. Since $n+1$ non-zero vectors in V_n must be linearly dependent, Theorem 2 is trivial when $\#\mathfrak{R} \geq n+2$ (i.e., $\#(\mathfrak{R} - \{0\}) \geq n+1$). In contrast, in Theorem 3 the linear dependence of vectors takes place in each $\mathfrak{R} \cap W$ not only in $\mathfrak{R}$.

We notice that there exist $n-1$ $(n-1)$ -dimensional linear subspaces. Hence Theorem 3 is more profound than Theorem 2.

5 The Unbiased Distribution of $\mathfrak{R}$

In this section we focus on the distribution of $\mathfrak{R}$ for the functions on V_n , whose nonlinearity does not take the special value $2^{n-1} - 2^{\frac{1}{2}(n-1)}$ or $2^{n-1} - 2^{\frac{1}{2}n}$ or $2^{n-1} - 2^{\frac{1}{2}n-1}$.

The next result is from [6] (Theorem 18).

Lemma 6. *Let f be a function on V_n ($n \geq 2$), ξ be the sequence of f , and p is an integer, $2 \leq p \leq n$. If $\langle \xi, \ell_j \rangle \equiv 0 \pmod{2^{n-p+2}}$, where ℓ_j is the j th row of H_n , $j = 0, 1, \dots, 2^n - 1$, then the algebraic degree of f is at most $p - 1$.*

Lemma 7. *For every function f on V_n , we have*

$$\begin{aligned} & 4(\Delta(\alpha_0), \Delta(\alpha_4), \dots, \Delta(\alpha_{2^{n-4}}))H_{n-2} \\ &= \left(\sum_{j=0}^3 \langle \xi, \ell_j \rangle^2, \sum_{j=4}^7 \langle \xi, \ell_j \rangle^2, \dots, \sum_{j=2^{n-4}}^{2^n-1} \langle \xi, \ell_j \rangle^2 \right) \end{aligned}$$

Where ξ denotes the sequence of f and ℓ_i is the i th row of H_n , $i = 0, 1, \dots, 2^n - 1$.

Proof. Comparing the $4j$ th terms, $j = 0, 1, \dots, 2^{n-2} - 1$, in the two sides of equality (1), we obtain

$$\begin{aligned} & 2^n(\Delta(\alpha_0), \Delta(\alpha_4), \dots, \Delta(\alpha_{2^{n-4}})) \\ &= \left(\sum_{j=0}^3 \langle \xi, \ell_j \rangle^2, \sum_{j=4}^7 \langle \xi, \ell_j \rangle^2, \dots, \sum_{j=2^{n-4}}^{2^n-1} \langle \xi, \ell_j \rangle^2 \right) H_{n-2} \end{aligned}$$

This proves the lemma. $\square$

Theorem 4. *Let f be a function on V_n , and U be a $(n-2)$ -dimensional linear subspace satisfying $\#(\mathfrak{R} \cap U) = 1$ (i.e., $\mathfrak{R} \cap U = \{0\}$). Then we have*

- (i) *if n is odd, then the nonlinearity of f satisfies $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$ and the algebraic degree of f is at most $2^{\frac{1}{2}(n+1)}$,*
- (ii) *if n is even, then f is bent or the nonlinearity of f satisfies $N_f = 2^{n-1} - 2^{\frac{1}{2}n}$ and the algebraic degree of f is at most $2^{\frac{1}{2}n+1}$.*

Proof. We only need to prove the theorem in the special case when U is composed of $\alpha_0, \alpha_4, \alpha_8, \dots, \alpha_{2^{n-4}}$, where $\alpha_{4j} \in V_n$ is the binary representation of even number $4j$, $j = 0, 1, 2, \dots, 2^{n-2} - 1$. In other words, U is composed of all the vectors in V_n , that can be expressed in the form $(a_1, \dots, a_{n-2}, 0, 0)$, where each $a_j \in GF(2)$. For U in general case, we can use a nonsingular linear transformation on the variables so as to change U into the special case. Let ξ be the sequence of f . Set $a_j^2 = \langle \xi, \ell_j \rangle^2$, $j = 0, 1, \dots, 2^n - 1$.

Since $\Delta(0) = 2^n$ and $\Delta(\alpha_{4j}) = 0$, $j = 1, 2, \dots, 2^{n-2} - 1$, the equality in Lemma 7 is specialized as

$$2^{n+2}(1, \dots, 1) = \left(\sum_{j=0}^3 a_j^2, \sum_{j=4}^7 a_j^2, \dots, \sum_{j=2^{n-4}}^{2^n-1} a_j^2 \right) \quad (14)$$

$j = 0, 1, \dots, 2^{n-2} - 1$.

(i) When n is odd, by using Lemma 4,

$$\{a_{4j}^2, a_{4j+1}^2, a_{4j+3}^2, a_{4j+3}^2\} = \{2^{n+1}, 2^{n+1}, 0, 0\}, j = 0, 1, \dots, 2^{n-2}$$

By using Lemma 1, we have proved the nonlinearity of f satisfies $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$, and by using Lemma 6, we have proved that the algebraic degree of f is at most $2^{\frac{1}{2}(n+1)}$.

(ii) When n is even. By using Lemma 4,

$$\{a_{4j}^2, a_{4j+1}^2, a_{4j+3}^2, a_{4j+3}^2\} = \{2^n, 2^n, 2^n, 2^n\} \text{ or } \{2^{n+2}, 0, 0, 0\},$$

$j = 0, 1, \dots, 2^{n-2} - 1$.

If there exists a number j_0 , $0 \leq j_0 \leq 2^{n-2} - 1$, such that

$$\{a_{4j_0}^2, a_{4j_0+1}^2, a_{4j_0+2}^2, a_{4j_0+3}^2\} = \{2^{n+2}, 0, 0, 0\}$$

then by using Lemma 1, we have proved that the nonlinearity of f satisfies $N_f = 2^{n-1} - 2^{\frac{1}{2}n}$, and by using Lemma 6, we have proved that the algebraic degree of f is at most $2^{\frac{1}{2}(n+1)}$.

If there exists no such j_0 , mentioned as above, i.e., $\{a_{4j}^2, a_{4j+1}^2, a_{4j+3}^2, a_{4j+3}^2\} = \{2^n, 2^n, 2^n, 2^n\}$, $j = 0, 1, \dots, 2^{n-2} - 1$. Then f is bent. □

To emphasise the distribution of $\mathfrak{R}$ we modify Theorem 4 as follows:

Theorem 5. *Let f be a function on V_n . If the nonlinearity of f does not take the special value $2^{n-1} - 2^{\frac{1}{2}(n-1)}$ or $2^{n-1} - 2^{\frac{1}{2}n}$ or $2^{n-1} - 2^{\frac{1}{2}n-1}$, then $\#(\mathfrak{R} \cap U) \geq 2$ where U is any $(n-2)$ -dimensional linear subspace, in other words, every $(n-2)$ -dimensional linear subspace U contains a non-zero vector in $\mathfrak{R}$.*

There exist many methods to locate all the $(n-1)$ -dimensional linear subspaces and all the $(n-2)$ -dimensional linear subspaces in V_n . For example, let φ_α denote the linear function on V_n , where $\alpha \in V_n$, such that $\varphi_\alpha(x) = \langle \alpha, x \rangle$. Hence $W = \{\gamma | \alpha \in V_n, \varphi_\alpha(\gamma) = 0\}$ is a $(n-1)$ -dimensional linear subspace and each $(n-1)$ -dimensional linear subspace can be expressed in this form.

Also for any $\alpha, \alpha' \in V_n$ with $\alpha \neq \alpha'$, $U = \{\gamma | \alpha \in V_n, \varphi_\alpha(\gamma) = 0, \varphi_{\alpha'}(\gamma) = 0\}$ is a $(n-2)$ -dimensional linear subspace and each $(n-2)$ -dimensional linear subspace can be expressed in this form.

Lemma 8. *Let Ω be a subset of V_k with $0 \notin \Omega$. If there exists a positive integer p such that $\#(\Omega \cap U) \geq p$ holds for every $(k-1)$ -dimensional linear subspace U , then $\#\Omega \geq 2p + 1$.*

Proof. Note that each non-zero vector is included in precisely $2^{k-1} - 1$ $(k-1)$ -dimensional linear subspaces, on the other hand, there exist exactly $2^k - 1$ $(k-1)$ -dimensional linear subspaces. Hence $(2^{k-1} - 1)\#\Omega = \sum_U \#(\Omega \cap U)$. From $\#(\Omega \cap U) \geq p$, we conclude that $(2^{k-1} - 1)\#\Omega \geq (2^k - 1)p$. Since $\frac{2^k - 1}{2^{k-1} - 1} > 2$, $\#\Omega > 2p$ or $\#\Omega \geq 2p + 1$. $\square$

Theorem 6. *Let f be a function on V_n . If the nonlinearity of f does not take the special values $2^{n-1} - 2^{\frac{1}{2}(n-1)}$ or $2^{n-1} - 2^{\frac{1}{2}n}$ or $2^{n-1} - 2^{\frac{1}{2}n-1}$, then $\#(\mathfrak{R} \cap W) \geq 4$ for every $(n-1)$ -dimensional linear subspace W , in other words, every $(n-1)$ -dimensional linear subspace W contains at least three non-zero vectors in $\mathfrak{R}$.*

Proof. Let W be an arbitrary $(n-1)$ -dimensional linear subspace and U be an arbitrary $(n-2)$ -dimensional linear subspace with $U \subset W$. Note that the inequality in Theorem 5 can be rewritten as

$$\#((\mathfrak{R} - \{0\}) \cap U) \geq 1 \quad (15)$$

and $((\mathfrak{R} - \{0\}) \cap W) \cap U = (\mathfrak{R} - \{0\}) \cap U$. Applying Lemma 8, we have proved $\#((\mathfrak{R} - \{0\}) \cap W) \geq 3$. Since $0 \in \mathfrak{R} \cap W$, $\#(\mathfrak{R} \cap W) \geq 4$. $\square$

Theorems 5 and 6 are helpful to locate the non-propagative vectors.

The properties mentioned together in Theorems 5 and 6 are called *the unbiased distribution of $\mathfrak{R}$* , with respect to every $(n-2)$ -dimensional linear subspace and every $(n-1)$ -dimensional linear subspace.

6 Distribution of $\mathfrak{R}$ in Special Cases

We now turn to the case $\#(\mathfrak{R}_f \cap W) \leq 3$ where W is an $(n-1)$ -dimensional linear subspace. The following Lemma can be found in [7]:

Lemma 9. *Let $n \geq 2$ be a positive integer and $2^n = a^2 + b^2$ where $a \geq b \geq 0$ and both a and b are integers. Then $a^2 = 2^n$ and $b = 0$ when n is even, and $a^2 = b^2 = 2^{n-1}$ when n is odd.*

Theorem 7. *Let f be a function on V_n , and W be an $(n-1)$ -dimensional linear subspace satisfying $\#(\mathfrak{R} \cap W) = 1$ (i.e., $\mathfrak{R} \cap W = \{0\}$). We have*

- (i) *f has at most one non-zero linear structure,*
- (ii) *if n is odd, then the nonlinearity of f satisfies $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$ and the algebraic degree of f is at most $2^{\frac{1}{2}(n+1)}$,*
- (iii) *if n is even, then f is bent.*

Proof. (i) Let $\alpha^* \in V_n$ and $\alpha^* \notin W$, From linear algebra, $V_n = W \cup (\alpha^* \oplus W)$, where $\alpha^* \oplus W = \{\alpha^* \oplus \alpha \mid \alpha \in W\}$, W and $\alpha^* \oplus W$ are disjoint. We now prove that f has at most one non-zero linear structure by contradiction. Suppose f has two

non-zero linear structures, β_1 and β_2 with $\beta_1 \neq \beta_2$. Since all linear structures of f form a linear subspace of V_n , $\beta_1 \oplus \beta_2$ is also a non-zero linear structure of f and hence $\beta_1 \oplus \beta_2 \in \mathfrak{R}$. Since $\mathfrak{R} \cap W = \{0\}$, $\beta_1, \beta_2 \in \alpha^* \oplus W$. Obviously $\beta_1 \oplus \beta_2 \in W$ and hence $\beta_1 \oplus \beta_2 \in \mathfrak{R} \cap W$. This contradicts the condition $\mathfrak{R} \cap W = \{0\}$. The contradiction proves that f has at most one non-zero linear structure.

Recall the proof of Theorem 3, (3) can be specialized as $2\Delta(0) = a_{2j}^2 + a_{2j+1}^2$ and hence $a_{2j}^2 + a_{2j+1}^2 = 2^{n+1}$, where $j = 0, 1, \dots, 2^{n-1} - 1$.

(ii) If n be odd, from Lemma 9, $\{a_{2j}^2, a_{2j+1}^2\} = \{2^{n+1}, 0\}$, where $j = 0, 1, \dots, 2^{n-1} - 1$. From Lemma 1, the nonlinearity of f satisfies $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$. By using Lemma 6 we conclude that the algebraic degree of f is at most $2^{\frac{1}{2}(n+1)}$.

(iii) If n is even, due to Lemma 9, $a_{2j}^2 = a_{2j+1}^2 = 2^n$, where $j = 0, 1, \dots, 2^{n-1} - 1$. This proves that f is bent. □

Example 1. Let n be a positive odd number and $f(x_1, \dots, x_n) = x_1 \oplus g(x_2, \dots, x_n)$ where g is a bent function in V_{n-1} . Let W be an $(n-1)$ -dimensional linear subspace of V_n , composed of all the vectors in V_n , that can be expressed in the form $(0, a_2, \dots, a_n)$, where each $a_j \in GF(2)$. It is easy to see $\alpha^* = (1, 0, \dots, 0) \in V_n$ is a non-zero linear structure of f and $\mathfrak{R} \cap W = \{0\}$. Due to (ii) of Theorem 7, $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$.

We can restate (iii) of Theorem 7 as follows:

Proposition 1. *Let f be a function on V_n where n is even. If there exists an $(n-1)$ -dimensional linear subspace W_0 satisfying $\#(\mathfrak{R} \cap W_0) = 1$ (i.e., $\mathfrak{R} \cap W_0 = \{0\}$), then f satisfies $\mathfrak{R} \cap W = \{0\}$, for every $(n-1)$ -dimensional linear subspace W .*

Next we examine the case of $\#(\mathfrak{R} \cap W) = 2$.

Theorem 8. *Let f be a function on V_n . If there exists a $(n-1)$ -dimensional linear subspace W satisfying $\mathfrak{R} \cap W = \{0, \beta_1\}$, then we have*

- (i) β_1 is a non-zero linear structure of f ,
- (ii) if n is odd, then the nonlinearity of f satisfies $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$ and the algebraic degree of f is at most $2^{\frac{1}{2}(n+1)}$,
- (iii) if n is even, then $N_f = 2^{n-1} - 2^{\frac{1}{2}n}$ and the algebraic degree of f is at most $2^{\frac{1}{2}n+1}$.

Proof. Since any single non-zero vector is linearly independent, we can keep the deduction in the proof of Theorem 3 until inequality (12) where we need the condition $k \geq 3$.

(i) Recall the proof of Theorem 3, (6) can be specialized as $p_{1j}\Delta(\beta_1) = 2^n, 0, -2^n, j = 0, 1, \dots, 2^n - 1$. Since $\beta_1 \in \mathfrak{R}$, $\Delta(\beta_1) \neq 0$. Hence $\Delta(\beta_1) = \pm 2^n$. This proves that β_1 is a non-zero linear structure.

(ii) If n is odd, from (5) we conclude that $\langle \xi, \ell_i \rangle^2 = 2^{n+1}, 0, i = 0, 1, \dots, 2^n - 1$, and hence by using Lemma 1, we have proved $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$. By using Lemma 6 we conclude that the algebraic degree of f is at most $2^{\frac{1}{2}(n+1)}$.

(iii) If n is even, from (13), $\langle \xi, \ell_i \rangle^2 = 2^{n+2}, 0, 2^n$. Since $\#\mathfrak{R} > 1$, f is not bent. Hence $\langle \xi, \ell_i \rangle^2 = 2^n$ cannot hold for all i and hence there exists a number $i_0, 0 \leq i_0 \leq 2^n - 1$, such that $\langle \xi, \ell_{i_0} \rangle^2 = 2^{n+2}$. By using Lemma 1, we have proved $N_f = 2^{n-1} - 2^{\frac{1}{2}n}$, if n is even. By using Lemma 6 we conclude that the algebraic degree of f is at most $2^{\frac{1}{2}n+1}$. □

Example 2. Let n be a positive odd number and $f(x_1, \dots, x_n)$ be the same with that in Example 1. Let W be an $(n-1)$ -dimensional linear subspace of V_n , composed of all the vectors in V_n , that can be expressed in the form $(a_1, \dots, a_{n-1}, 0)$, where each $a_j \in GF(2)$. It is easy to see $\alpha^* = (1, 0, \dots, 0) \in V_n$ is a non-zero linear structure of f and $\mathfrak{R} \cap W = \{0, \alpha^*\}$. Due to (ii) of Theorem 8, $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$.

Let k be a positive even number with $k \geq 4$ and $h(x_1, \dots, x_k) = x_1 \oplus x_2 \oplus q(x_3, \dots, x_k)$ where q is a bent function on V_{k-2} . Let U be an $(n-1)$ -dimensional linear subspace of V_n , composed of all the vectors in V_n , that can be expressed in the form $(0, a_2, \dots, a_k)$, where each $a_j \in GF(2)$. It is easy to see $\alpha_1^* = (0, 1, 0, \dots, 0)$ is a non-zero linear structures of h and $\mathfrak{R} \cap U = \{0, \alpha_1^*\}$. Due to (iii) of Theorem 8, $N_h = 2^{k-1} - 2^{\frac{1}{2}k}$.

It is interesting that by using Theorem 8, we have determined N_h only from the condition $\#(\mathfrak{R} \cap U) = 2$ for an $(n-1)$ -dimensional linear subspace U although we do not search other vectors in $\mathfrak{R}$.

Finally, we consider the case when $\#(\mathfrak{R} \cap W) = 3$.

Theorem 9. *Let f be a function on V_n . If there exists a $(n-1)$ -dimensional linear subspace W satisfying $\mathfrak{R} \cap W = \{0, \beta_1, \beta_2\}$, then the following statements hold:*

- (i) $\Delta(\beta_j) = \pm 2^{n-1}, j = 1, 2$,
- (ii) if n is odd, then the nonlinearity of f satisfies $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$ and the algebraic degree of f is at most $2^{\frac{1}{2}(n+1)}$,
- (iii) if n is even, then $N_f = 2^{n-1} - 2^{\frac{1}{2}n}$ and the algebraic degree of f is at most $2^{\frac{1}{2}n+1}$.

Proof. Since any two non-zero vectors are linearly independent, we can keep the deduction in the proof of Theorem 3 until inequality (12) where we need the condition $k \geq 3$.

Recall the proof of Theorem 3, (9) can be specialized as $\Delta(\beta_j) = \pm 2^n, \pm 2^{n-1}, j = 1, 2$.

On the other hand, (10), (11) and (12) can be rewritten as $(\Delta(\beta_1), \Delta(\beta_2)) = 2^{n-1}(b_1, b_2)$ where each $b_j = \pm 1, \pm 2$, $(p_{1s}, p_{2s}) = (\frac{b_1}{|b_1|}, \frac{b_2}{|b_2|})$. and

$$p_{1s}\Delta(\beta_1) + p_{2s}\Delta(\beta_2) = (|b_1| + |b_2|)2^{n-1} \quad (16)$$

respectively. It is easy to prove $b_1, b_2 = \pm 1$. Otherwise, for example, $b_1 = \pm 2$, from (16), $p_{1s}\Delta(\beta_1) + p_{2s}\Delta(\beta_2) \geq 3 \cdot 2^{n-1}$. This contradicts (6). Since $b_1, b_2 = \pm 1$, $\Delta(\beta_1), \Delta(\beta_2) = \pm 2^{n-1}$. This proves (i).

The rest proof is the same with the proof of Theorem 8. $\square$

Example 3. Let n be a positive odd number with $n \geq 7$, $h(x_1, x_2, x_3, x_4, x_5) = (x_1 \oplus x_2 \oplus x_3)x_4x_5 \oplus x_1x_5 \oplus x_2x_4 \oplus x_1 \oplus x_2 \oplus x_3$ and $g(x_6, \dots, x_n)$ be a bent function on V_{n-5} . Set $f(x_1, \dots, x_n) = h(x_1, x_2, x_3, x_4, x_5) \oplus g(x_6, \dots, x_n)$.

Let W be an $(n-1)$ -dimensional linear subspace of V_n , composed of all the vectors in V_n , that can be expressed in the form $(0, a_2, \dots, a_n)$, where each $a_j \in GF(2)$. Write $\alpha_1^* = (0, 0, 1, 0, \dots, 0)$, $\alpha_2^* = (0, 1, 0, \dots, 0) \in V_n$. It is easy to verify $\alpha_1^*, \alpha_2^* \in \mathfrak{R}$ and $\mathfrak{R} \cap W = \{0, \alpha_1^*, \alpha_2^*\}$. Due to (i) and (ii) of Theorem 9, we conclude $\Delta(\alpha_1^*) = \pm 2^{n-1}$, $\Delta(\alpha_2^*) = \pm 2^{n-1}$ and $N_f = 2^{n-1} - 2^{\frac{1}{2}(n-1)}$.

We notice that by using Theorem 9, we have determined N_h , $\Delta(\alpha_1^*)$ and $\Delta(\alpha_2^*)$ only from the information about $\#(\mathfrak{R} \cap W)$ for an $(n-1)$ -dimensional linear subspace W although we do not search other the vectors in $\mathfrak{R}$.

We can also find an example corresponding to (iii) of Theorem 9. All Theorems 7, 8 and 9 and Examples 1, 2 and 3 show that we can determine the nonlinearity of a function only from some information about $\#(\mathfrak{R} \cap W)$, where W is an $(n-1)$ -dimensional linear subspace. It is interesting that [7] has proved that there exists no a function with $\#\mathfrak{R} = 3$ while Example 3 gives a function satisfying $\#(\mathfrak{R} \cap W) = 3$ for an $(n-1)$ -dimensional linear subspace W .

7 Conclusions

The strong linear dependence is an improvement on a previously known result. The unbiased distribution of non-propagation vectors is valid for most functions. These results provide more information on the non-propagative vectors in any $(n-1)$ -dimensional linear subspace of V_n , and hence they are helpful for designing cryptographic functions.

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