Bézier surfaces of minimal area

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Abstract. There are minimal surfaces admitting a Bézier form. We study the properties that the associated net of control points must satisfy. We show that in the bicubical case all minimal surfaces are, up to an affine transformation, pieces of the Enneper's surface.

1 Introduction

The study of surfaces minimizing area with prescribed border (the so called *Plateau problem*) has been and still is a main topic in differential geometry. Such kind of surfaces, characterized by the vanishing of the mean curvature, are called minimal surfaces. It is a part of the differential geometry where a lot of research has been done from its very beginning with J. L. Lagrange in 1762.
The construction of curves and surfaces subject to certain constraints (to minimize length, area, curvature or other geometric properties) has been studied from the point of view of Graphics (see [4], [5], [6] or [7]). In the case of the area of the surface, the interest comes from the fact that in some real problems, minimal area means minimal cost of the material used to build the surface. Moreover, the minimization of functionals related with the mean curvature provides a method of efficient fairing.
In this paper we try to give a little account of the minimal surfaces that admit a Bézier form. Up to our knowledge, the study of minimal Bézier surfaces has not yet been done and there are some interesting questions to be raised. Among them let us mention two:

- Can the control net of a minimal Bézier surface be characterized by some minimal property related with areas?
- Is it possible to characterize which control nets are associated to minimal Bézier surfaces?

Our attempts to answer the first question point out that the area of any polyhedron having as vertices the control points do not minimize area among all polihedra with the same border. We have followed here the approach of [9] to study discrete minimal surfaces.
This note deals mainly with the second question. The two main results are the following: First, we have characterized control nets of harmonic

P.M.A. Sloot et al. (Eds.): ICCS 2002, LNCS 2330, pp. 72–81, 2002.

Bézier surfaces, and second, we have proved that any bicubical polynomial minimal surface is, up to an affine transformation, a piece of a well known minimal surface: the Enneper's surface.
The consequence of our results is that minimal surfaces are too rigid to be useful as candidates for blendings between arbitrary surfaces. Only for some configurations of the border control points we can assure that a Bézier surface exists with minimal area.
The connection between the two topics, Bézier and minimal surfaces, is not new. Let us recall some of them. First, Sergei Bernstein, who defined the now called Bernstein polynomials, was a prolific researcher in the realm of minimal surfaces at the beginning of the twentieth century. See for instance [1] and [2]. One of its most celebrated results was to prove that if a minimal surface is the graph of a differentiable function defined on the whole $\mathbb{R}^2$, i.e, $\vec{x}(u,v) = (u, v, f(u,v))$, then it is a plane.
Second, the solutions to some Plateau problems, for example, the Gergonne surface, resemble Bézier surfaces. (Look at Figure I)

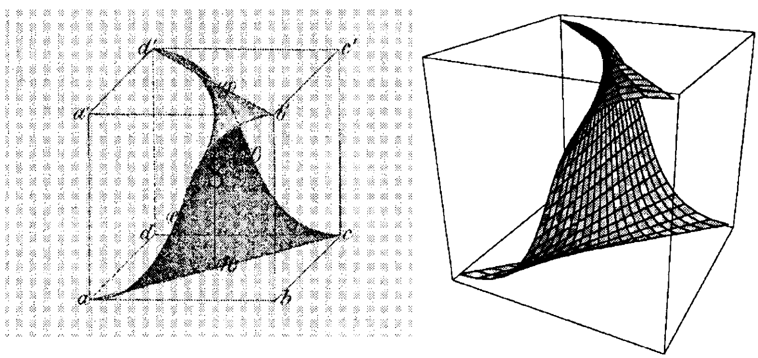

Figure I. Left. The Schwarz's solution (1865) to the Gergonne's problem (1816): find the minimal surface dividing the cube into two equal parts and joining the inverse diagonals of two opposed faces. Right. A Bézier surface with a similar shape.

And third, both kind of surfaces share some crucial properties: A Bézier surface is always included in the convex hull of its net of control points. Analogously, a minimal surface is always included in the convex hull of its border.

2 Definitions

Definition 1. *Given a net of control points in* $\mathbb{R}^3$, $\{P_{ij}\}_{i,j=0}^{n,m}$, *the associated Bézier surface,* $\vec{x} : [0,1] \times [0,1] \to \mathbb{R}^3$, *is given by*

$$\vec{x}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_i^n(u) B_j^m(v) P_{ij}. \tag{1}$$

Definition 2. *A surface S is minimal if its mean curvature vanishes.*

Equivalently, S is a minimal surface iff for each point $p \in S$ one can choose a small neighbourhood, U_p which has minimal area among other patches V having the same boundary as U.

Example 1. The first non trivial example of minimal surface with polynomial coordinate functions is the Enneper's surface (Figure II): $\vec{x} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\vec{x}(u,v) := (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).$$

The control net for the portion of the Enneper's surface defined by $u, v \in [-1,1]$, is given by

$$\begin{array}{llll}
(-\frac{5}{3},-\frac{5}{3},0) & (-1,-\frac{1}{3},-\frac{4}{3}) & (1,-\frac{1}{3},-\frac{4}{3}) & (\frac{5}{3},-\frac{5}{3},0) \\
(-\frac{1}{3},-1,\frac{4}{3}) & (-\frac{5}{9},-\frac{5}{9},0) & (\frac{5}{9},-\frac{5}{9},0) & (\frac{1}{3},-1,\frac{4}{3}) \\
(-\frac{1}{3},1,\frac{4}{3}) & (-\frac{5}{9},\frac{5}{9},0) & (\frac{5}{9},\frac{5}{9},0) & (\frac{1}{3},1,\frac{4}{3}) \\
(-\frac{5}{3},\frac{5}{3},0) & (-1,\frac{1}{3},-\frac{4}{3}) & (1,\frac{1}{3},-\frac{4}{3}) & (\frac{5}{3},\frac{5}{3},0)
\end{array}$$

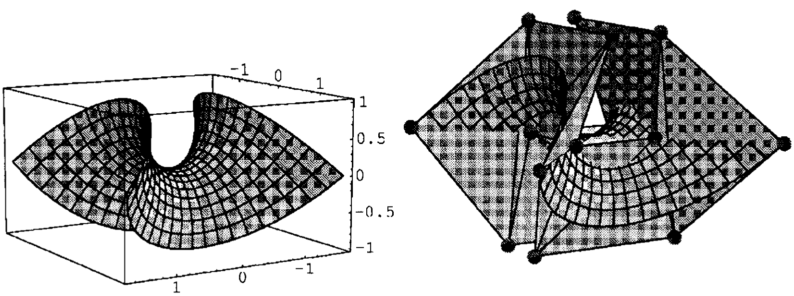

Figure II. Left: a piece of the Enneper's surface ($u, v \in [-1,1]$). Right: Its control net as a Bézier surface.

3 Minimal surfaces with isothermal coordinates

Let us recall that a chart $\vec{x} : U \to S$ on a surface, S, is said to be isothermal the map $\vec{x}$ is a conformal map, i.e, if angles between curves in the surface are equal to the angles between the corresponding curves in the coordinate open subset U. It is easy to check that for an isothermal chart the coefficients, E, F, G, of the first fundamental form satisfy $E = G$ and $F = 0$.
Note that this implies that the two families of coordinate curves of the chat $\vec{x}$ are orthogonal because $F = 0$, and that the length of the coordinate curve from $\vec{x}(u_0, v_0)$ to $\vec{x}(u_0, v_0 + h)$ is equal to the length of the coordinate curve from $\vec{x}(u_0, v_0)$ to $\vec{x}(u_0 + h, v_0)$.
A well known result of the theory of minimal surfaces is the following (see [10]): if $\vec{x}$ is an isothermal map then $\vec{x}$ is minimal iff $\Delta\vec{x} = 0$,

where Δ is the usual Laplacian operator. The relation between the mean curvature and the chart is due to the fact that for an isothermal map

$$\vec{x}_{uu} + \vec{x}_{vv} = 2\lambda^2 HN,$$

where $\lambda = E = G$ and N is the unitary normal vector of the surface associated to the chart.
The conditions that a net of control points must satisfy in order to have an isothermal associated Bézier surface are more difficult to handle (they can be expressed as a system of quadratic equations) than the conditions in order to be harmonic (in this case, the equations are linear). So, let us study first that second condition.
We will compute the Laplacian of a Bézier surface (1).

$$\begin{aligned}\Delta\vec{x}(u,v) &= (\tfrac{\partial^2}{\partial u^2} + \tfrac{\partial^2}{\partial v^2})\vec{x}(u,v)\\ &= n(n-1)\textstyle\sum_{i=0}^{n-2}\sum_{j=0}^{m} B_i^{n-2}(u)B_j^m(v)\Delta^{2,0}P_{ij}\\ &\quad +m(m-1)\textstyle\sum_{i=0}^{n}\sum_{j=0}^{m-2} B_i^n(u)B_j^{m-2}(v)\Delta^{0,2}P_{ij},\end{aligned}$$

where $\Delta^{2,0}P_{ij} = P_{i+2,j} - 2P_{i+1,j} + P_{ij}$, $\Delta^{0,2}P_{ij} = P_{i,j+2} - 2P_{i,j+1} + P_{ij}$.
We shall rewrite the last expression again as a Bézier surface of degrees n and m. In order to do this, we will need the following relation

$$\begin{aligned}B_i^{n-2}(t) = \tfrac{1}{n(n-1)}\ &((n-i)(n-i-1)B_i^n(t)\\ &+2(i+1)(n-i-1)B_{i+1}^n(t) + (i+1)(i+2)B_{i+2}^n(t)).\end{aligned}$$

Let us define, for $i \in \{0,\dots,n-2\}$

$$a_{in} = (n-i)(n-i-1), \quad b_{in} = 2(i+1)(n-i-1), \quad c_{in} = (i+1)(i+2),$$

and $a_{in} = b_{in} = c_{in} = 0$ otherwise.
Therefore,

$$\begin{aligned}\Delta\vec{x}(u,v) = &\textstyle\sum_{i=0}^{n}\sum_{j=0}^{m} B_i^n(u)B_j^m(v)\\ &(n(n-1)(a_{in}\Delta^{2,0}P_{ij} + b_{i-1,n}\Delta^{2,0}P_{i-1,j} + c_{i-2,n}\Delta^{2,0}P_{i-2,j})\\ &+ m(m-1)(a_{jm}\Delta^{0,2}P_{ij} + b_{j-1,m}\Delta^{0,2}P_{i,j-1} + c_{j-2,m}\Delta^{0,2}P_{i,j-2})).\end{aligned}$$

This expression can be seen as the Bézier surface associated to a net of control points $\{Q_{ij}\}_{i,j=0}^{n,m}$. Thus, due to the fact that $\{B_i^n(u)B_j^m(v)\}_{i,j=0}^{nm}$ is a basis of polynomials, we get that $\vec{x}$ is harmonic iff $Q_{ij} = 0$ for all i,j.
Substituting the discrete operators $\Delta^{2,0}$ and $\Delta^{0,2}$ by its definitions and sorting terms we get that for any i,j the following expression vanish:

$$\begin{aligned}&n(n-1)(P_{i+2,j}a_{in} + P_{i+1,j}(b_{i-1,n} - 2a_{in})\\ &\qquad + P_{i-1,j}(b_{i-1,n} - 2c_{i-2,n}) + P_{i-2,j}c_{i-2,n})\\ &+m(m-1)(P_{i,j+2}a_{jm} + P_{i,j+1}(b_{j-1,m} - 2a_{jm})\\ &\qquad + P_{i,j-1}(b_{j-1,m} - 2c_{j-2,m}) + P_{i,j-2}c_{j-2,m})\\ &+P_{ij}((a_{in} - 2b_{i-1,n} + c_{i-2,n})n(n-1)\\ &\qquad + (a_{jm} - 2b_{j-1,m} + c_{j-2,m})m(m-1)).\end{aligned}$$

In the case of a quadratic net ($n = m$) we can state the following theorem

Theorem 1. *Given a quadratic net of points in* $\mathbb{R}^3$, $\{P_{ij}\}_{i,j=0}^{n}$, *the associated Bézier surface,* $\vec{x} : [0,1]\times[0,1] \to \mathbb{R}^3$, *is harmonic, i.e,* $\Delta\vec{x} = 0$ *iff*

$$\begin{aligned} 0 = {} & P_{i+2,j}a_{in} + P_{i+1,j}(b_{i-1,n} - 2a_{in}) + P_{i-1,j}(b_{i-1,n} - 2c_{i-2,n}) \\ & + P_{i-2,j}c_{i-2,n} + P_{i,j+2}a_{jm} + P_{i,j+1}(b_{j-1,m} - 2a_{jm}) \\ & + P_{i,j-1}(b_{j-1,m} - 2c_{j-2,m}) + P_{i,j-2}c_{j-2,m} \\ & + P_{ij}(a_{in} - 2b_{i-1,n} + c_{i-2,n} + a_{jm} - 2b_{j-1,m} + c_{j-2,m}). \end{aligned} \tag{2}$$

Let us study Equation (2) in the simplest cases: biquadratic and bicubical Bézier patches.

3.1 Biquadratic harmonic Bézier patches

In the case $n = m = 2$ from the equations in (2) it is possible to find an expression of four of the control points in terms of the other five. In fact, using Mathematica, we have obtained that the null space of the coefficient matrix of (2) is of dimension four. Moreover, it is possible to choose as free variables points in the first and last column of the control net.

Corollary 1. *A biquadratic Bézier surface is harmonic iff*

$$\begin{aligned} P_{01} &= \tfrac{1}{2}(2P_{00} + P_{02} - 2P_{10} + P_{20}), \\ P_{11} &= \tfrac{1}{4}(P_{00} + P_{02} + P_{20} + P_{22}), \\ P_{21} &= \tfrac{1}{2}(P_{00} + -2P_{10} + 2P_{20} + P_{22}), \\ P_{12} &= \tfrac{1}{2}(-P_{00} + P_{02} + 2P_{10} - P_{20} + P_{22}). \end{aligned} \tag{3}$$

A way of writing for example the equation involving the inner control point, P_{11}, is using a mask

$$P_{11} = \frac{1}{4} \times \begin{matrix} 1 & 0 & 1 \\ 0 & \bullet & 0 \\ 1 & 0 & 1 \end{matrix} \tag{4}$$

Remark 1. In [6], the author presents a method to improve an initial blending, F_0, through a sequence of blending surfaces minimizing some fairing functionals. In section 3.3, the author suggests the following modification: instead of using the initial blending surface, to use a modified surface obtained by averaging the inner control points. The averaging method suggested there, after an analysis of its implementation, is given precisely by the mask (4). Therefore, the use of this mask can be now justified from Equations (3): the inner point of a quadratic harmonic Bézier surface must verify such a mask.

Remark 2. Note that mask (4) is a kind of dual of the mask associated to the Laplace operator. It can be found in [4] that the mask

$$\frac{1}{4} \times \begin{matrix} 0 & 1 & 0 \\ 1 & \bullet & 1 \\ 0 & 1 & 0 \end{matrix} \tag{5}$$

is the discrete form of the Laplacian operator. Such a mask is used in the cited reference to obtain control nets resembling minimal surfaces that fit between given boundary polygons.
In general, the authors define in [4] the notion of permanence patches to be those generated by masks of the form

$$\frac{1}{4} \times \begin{matrix} \alpha & \beta & \alpha \\ \beta & \bullet & \beta \\ \alpha & \beta & \alpha \end{matrix} \tag{6}$$

with $4\alpha + 4\beta = 1$. Therefore, mask (4) is a particular case with $\alpha = 0.25$, whereas mask (5) corresponds to $\alpha = 0$. Anyway, as it is said there, any of such masks do not produce control nets of minimal surfaces.
In fact, let us recall that we are not trying to produce Coons nets. We try to characterize control nets of minimal surfaces. We have found that in the biquadratic case Eqs. (3) must be satisfied. But in order to obtain a minimal patch, we have to impose also the isothermal conditions. It is just a matter of computation to show that any control net verifying Eqs. (3) and the isothermal conditions is a piece of a plane.

3.2 Bicubical harmonic Bézier patches

In the case $n = m = 3$ from the equations in (2) it is possible to put half of the control points in terms of the other eight. In fact, using Mathematica, we have obtained that the null space of the coefficient matrix of (2) is of dimension eight. Moreover, it is possible to choose as free variables exactly the eight points in the first and last column of the control net.

Corollary 2. *A bicubic Bézier surface is harmonic iff*

$$\begin{aligned}
P_{11} &= \tfrac{1}{9}(4P_{00} + 2P_{03} + 2P_{30} + P_{33}), \\
P_{21} &= \tfrac{1}{9}(2P_{00} + P_{03} + 4P_{30} + 2P_{33}), \\
P_{12} &= \tfrac{1}{9}(2P_{00} + 4P_{03} + P_{30} + 2P_{33}), \\
P_{22} &= \tfrac{1}{9}(P_{00} + 2P_{03} + 2P_{30} + 4P_{33}), \\
P_{10} &= \tfrac{1}{3}(4P_{00} - 4P_{01} + 2P_{02} + 2P_{30} - 2P_{31} + P_{32}), \\
P_{20} &= \tfrac{1}{3}(2P_{00} - 2P_{01} + P_{02} + 4P_{30} - 4P_{31} + 2P_{32}), \\
P_{13} &= \tfrac{1}{3}(2P_{01} - 4P_{02} + 4P_{03} + P_{31} - 2P_{32} + 2P_{33}), \\
P_{23} &= \tfrac{1}{3}(P_{01} - 2P_{02} + 2P_{03} + 2P_{31} - 4P_{32} + 4P_{33}).
\end{aligned} \tag{7}$$

Remark 3. This means that given the first and last columns of the control net (eight control points in total), the other eight control points are fully determined by the harmonic condition. In other words, any pair of two opposed borders of a harmonic Bézier surface determines the rest of control points.

Remark 4. This fact is analogous to what happens in the Gergonne surface: given two border lines, the inverse diagonals of two opposed faces of a cube, the Gergonne surface is fully determined (See Figure I). In our case, given two opposed lines of border control points, the harmonic Bézier surface is fully determined.

Remark 5. The first four equations in (7) imply that the four inner control points are fully determined by the four corner points. So, there are two different kind of masks depending if the point is an inner control point or not:

$$P_{11} = \frac{1}{9} \times \begin{matrix} 4 & \bullet & \bullet & 2 \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ 2 & \bullet & \bullet & 1 \end{matrix}, \qquad P_{10} = \frac{1}{3} \times \begin{matrix} 4 & \bullet & \bullet & 2 \\ -4 & \bullet & \bullet & -2 \\ 2 & \bullet & \bullet & 1 \\ 0 & \bullet & \bullet & 0 \end{matrix}. \qquad (8)$$

The other points have similar masks.

Remark 6. Let us insist in the fact that harmonic chart need not to be minimal. We have obtained the conditions to be harmonic. In order to be minimal, more conditions are needed. Let us split the control points of a Bézier surface into two subsets: the inner points $\{P_{ij}\}_{i,j=1}^{n-1,m-1}$ and the border points. It is not true that given an arbitrary set of border points, there is a unique set of inner control points such that the Bézier surface associated to the whole control net is of minimal area. What we can say is that given just a few border control points, the rest of control points are determined. In the next section we will find which bicubical Bézier surfaces are minimal.

4 Bicubical minimal Bézier patches

We have seen before that the unique biquadratic minimal Bézier patch is the plane. In the cubical case we know that at least there is another minimal Bézier surface, the Enneper's surface. What we want to determine is if this is the only possibility. In order to do that we need to change the methods to attack the problem. The second great moment in the theory of minimal surfaces was the introduction of the methods of complex variable. Let us recall here the main results.
Let $\vec{x}(u, v)$ be an isothermal minimal chart and let us define

$$(\phi_1, \phi_2, \phi_3) = \frac{\partial \vec{x}(u,v)}{\partial u} - \mathbf{i}\frac{\partial \vec{x}(u,v)}{\partial v}.$$

The functions (ϕ_1, ϕ_2, ϕ_3) verify

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0. \qquad (9)$$

Lemma 1. *([10] Lemma 8.1) Let D be a domain in the complex z-plane, $g(z)$ an arbitrary meromorphic function in D and $f(z)$ an analytic function in D having the property that at each point where $g(z)$ has a pole of order m, $f(z)$ has a zero of order at least $2m$. Then the functions*

$$\phi_1 = \frac{1}{2}f(1-g^2), \quad \phi_2 = -\frac{\mathbf{i}}{2}f(1+g^2), \quad \phi_3 = fg \qquad (10)$$

will be analitic in D and satisfy (9). Conversely, every triple of analytic functions in D satisfying (9) may be represented in the form (10), except for $\phi_1 = i\phi_2, \phi_3 = 0$.

Note that both functions can be computed by

$$f = \phi_1 + \mathbf{i}\phi_2, \qquad g = \frac{\phi_3}{\phi_1 + \mathbf{i}\phi_2}. \tag{11}$$

Note also that Equations (10) are not exactly those of Lemma 8.1 in [10]. There is an slight difference in the sign of ϕ_2. Anyway the statement is equivalent.

Lemma 2. *([10] Lemma 8.2) Every simple-connected minimal surface in* $\mathbb{R}^3$ *can be represented in the form*

$$\vec{x}(u,v) = (Re\int_0^z \phi_1(z)dz, Re\int_0^z \phi_2(z)dz, Re\int_0^z \phi_3(z)dz) + P_0, \tag{12}$$

where the ϕ_k *are defined by (10), the functions* f *and* g *having the properties stated in Lemma 1, the domain* D *being either the unit disk or the entire plane, and the integral being taken along an arbitrary path from the origin to the point* $z = u + \mathbf{i}v$.

So, a minimal surface is determined by the pair of complex functions f and g. For example, the most obvious choice: $f(z) = 1, g(z) = z$, leads to the Enneper's surface.
We are going to consider now the following problem: to determine all bicubical polynomial minimal surfaces.
The number of possible choices of the two functions f and g in such a way that the chart given by (12) is a polynomial of degree 3 is not reduced just to $f(z) =$ constant and g a degree 1 polynomial in z. Another possibility is $f(z) = (p(z))^2$ and, $g(z) = \frac{q(z)}{p(z)}$, where $p(z), q(z)$ are polynomials of degree 1. Therefore, the problem we are interested in is not so obvious.

Theorem 2. *Any bicubical polynomial minimal chart,* $\vec{x} : U \to \mathbb{R}^3$*, is, up to an affine transformation of* $\mathbb{R}^3$*, an affine reparametrization of the general Enneper's surface of degree* 3*, i.e, there are* $H_3 \in Aff(\mathbb{R}^3)$ *and* $H_2 \in Aff(\mathbb{R}^2)$ *such that*

$$\vec{x}(u,v) = H_3(\vec{x}^{Enneper}(H_2(u,v))),$$

for any $(u,v) \in U$.

Proof. We can suppose that the chart is isothermal. On the contrary, a well known result of the theory of minimal surfaces states that the chart is a reparametrization of an isothermal chart.
All along the proof we will consider polynomial patches not in the Bernstein polynomial basis, but in the usual polynomial basis.
Let us consider a bicubical, polynomial, isothermal and harmonic chart

$$\vec{x}(u,v) = (\sum_{i,j=0}^{3} a_{ij}u^iv^j, \sum_{i,j=0}^{3} b_{ij}u^iv^j, \sum_{i,j=0}^{3} c_{ij}u^iv^j),$$

where $a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$.
Let us denote by $\vec{v_{ij}}$ the vector (a_{ij}, b_{ij}, c_{ij}).

Note first that thanks to a translation, we can suppose that $\overrightarrow{v_{00}} = 0$.
As the chart is orthogonal ($F = 0$), then, by inspection on the higher degree terms, it is possible to deduce the following relations

$$\overrightarrow{v_{33}} = \overrightarrow{v_{32}} = \overrightarrow{v_{23}} = \overrightarrow{v_{31}} = \overrightarrow{v_{13}} = \overrightarrow{v_{22}} = 0.$$

For example, the coefficient of u^5v^5 in F is the norm of $3\overrightarrow{v_{33}}$, therefore if $F = 0$ then $\overrightarrow{v_{33}} = 0$. To deduce the other relations proceed analogously with the coefficients of $u^5v^3, u^3v^5, u^5v^1, u^1v^5$ and u^3v^3 in that order.
Now, $\Delta\overrightarrow{\mathbf{x}} = 0$ iff

$$\overrightarrow{v_{20}} = -\overrightarrow{v_{02}}, \qquad \overrightarrow{v_{21}} = -3\overrightarrow{v_{03}}, \qquad \overrightarrow{v_{12}} = -3\overrightarrow{v_{30}}.$$

As the chart is isothermal, from the coefficients of v^4 in $E = G$ and $F = 0$, respectively, we obtain

$$||\overrightarrow{v_{30}}|| = ||\overrightarrow{v_{03}}||, \qquad < \overrightarrow{v_{30}}, \overrightarrow{v_{03}} >= 0.$$

At this point the deduction splits into two cases:

Case A. $\overrightarrow{v_{30}} = 0$

In this case, and after some computations, the chart is a piece of the plain $z = 0$. But the plane can be parametrized using a polynomial chart of degree 1, so, it cannot be considered as a proper solution of the problem.

Case B. $\overrightarrow{v_{30}} \neq 0$

In this case, thanks to a rotation and an uniform scaling, we can suppose that

$$\overrightarrow{v_{30}} = (1, 0, 0), \qquad \overrightarrow{v_{03}} = (0, 1, 0).$$

Therefore, from the coefficient of v in $F = 0$ and $E = G$, we obtain

$$a_{11} = -2b_{02}, \qquad b_{11} = 2a_{02}.$$

It can be proved that the isothermal conditions can be now reduced to just four equations involving the coordinates of the vectors $\overrightarrow{v_{ij}}$. Nevertheless, it is easier at this point of the proof to introduce the use of complex numbers.
Let us consider

$$(\phi_1, \phi_2, \phi_3) = \frac{\partial\overrightarrow{\mathbf{x}}}{\partial u} - \mathbf{i}\frac{\partial\overrightarrow{\mathbf{x}}}{\partial v},$$

where

$$\begin{aligned}
\phi_1 &= a_{10} - 2a_{02}u + 3u^2 - 2b_{02}v - 3v^2 - \mathbf{i}(a_{01} - 2b_{02}u + 2(a_{02} - 3u)v),\\
\phi_2 &= b_{10} - 2b_{02}u + (2a_{02} - 6u)v - \mathbf{i}(b_{01} + 2a_{02}u - 3u^2 + 2b_{02}v + 3v^2),\\
\phi_3 &= c_{10} - 2c_{02}u + c_{11}v - \mathbf{i}(c_{01} + c_{11}u + 2c_{02}v).
\end{aligned} \tag{13}$$

The chart $\overrightarrow{\mathbf{x}}$ is isothermal iff Equation (9) is verified.
Now, we can compute the pair of complex functions, $f^{\overrightarrow{\mathbf{x}}}, g^{\overrightarrow{\mathbf{x}}}$, according to (11).
In our case, we obtain that $f^{\overrightarrow{\mathbf{x}}}$ is a constant function and that $g^{\overrightarrow{\mathbf{x}}}$ is a polynomial in z of degree 1. Indeed,

$$\begin{aligned}
f^{\overrightarrow{\mathbf{x}}}(z) &= a_{10} + b_{01} + \mathbf{i}(-a_{01} + b_{10}),\\
g^{\overrightarrow{\mathbf{x}}}(z) &= \tfrac{c_{10} - \mathbf{i}c_{01} + -(2c_{02} + \mathbf{i}c_{11})z}{a_{10} + b_{01} + \mathbf{i}(-a_{01} + b_{10})} = mz + n
\end{aligned}$$

Let us denote the coefficient m by $\rho e^{\mathrm{i}t} \in \mathbb{C}$, where $\rho > 0$ and $t \in [0, 2\pi[$. Now, let us consider the chart

$$\vec{y}(u,v) = \vec{x}(\frac{1}{\rho}(\cos(t)u + \sin(t)v), \frac{1}{\rho}(-\sin(t)u + \cos(t)v)).$$

Note that $\vec{y}$ is also an isothermal chart. It is easy to check that the pair of complex functions, $f^{\vec{y}}, g^{\vec{y}}$ associated to $\vec{y}$ are now

$$f^{\vec{y}}(z) = a \in \mathbb{C}, \qquad g^{\vec{y}}(z) = z.$$

Let us also denote a in the form $\mu e^{\mathrm{i}s} \in \mathbb{C}$, where $\mu > 0$ and $s \in [0, 2\pi[$, and let T be the linear transformation of $\mathbb{R}^3$ defined as the composition of the uniform scaling with factor $\frac{1}{\mu}$ and the spatial rotation with respect to the z-axis and angle $-s$. A well known property of the Enneper surface says that the minimal surface defined by $f(z) = a, g(z) = z$ is the image by T of the Enneper surface. □

5 Acknowledgements

We would thank J. Nuo Ballesteros for a careful reading of the manuscript and to the referees for their valuable comments and suggestions.

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