Filtration-Convection Problem: Spectral-Difference Method and Preservation of Cosymmetry

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Abstract. For the problem of filtration of viscous fluid in porous medium it was observed that a number of one-parameter families of convective states with the spectrum, which varies along the family. It was shown by V. Yudovich that these families cannot be an orbit of an operation of any symmetry group and as a result the theory of cosymmetry was derived. The combined spectral and finite-difference approach to the planar problem of filtration-convection in porous media with Darcy law is described. The special approximation of nonlinear terms is derived to preserve cosymmetry. The computation of stationary regime transformations is carried out when filtration Rayleigh number varies.

1 Introduction

In this work we study the conservation of cosymmetry in finite-dimensional models of filtration-convection problem derived via combined spectral and finite-difference method. Cosymmetry concept was introduced by Yudovich [1, 2] and some interesting phenomena were found for both dynamical systems possessing the cosymmetry property. Particularly, it was shown that cosymmetry may be a reason for the existence of the continuous family of regimes of the same type. If a symmetry group produces a continuous family of identical regimes then it implies the identical spectrum for all points on the family. The stability spectrum for the cosymmetric system depends on the location of a point, and the family may be formed by stable and unstable regimes.

Following [1], a cosymmetry for a differential equation $\dot{u} = F(u)$ in a Hilbert space is the operator $L(u)$ which is orthogonal to F at each point of the phase space i.e $(F(u), L(u)) = 0, u \in R^n$ with an inner product $(\cdot, \cdot)$. If the equilibrium u_0 is noncosymmetric, i.e. $F(u_0) = 0$ and $L(u_0) \neq 0$, then u_0 belongs to a one-parameter family of equilibria. This takes place if there are no additional degeneracies.

A number of interesting effects were found in the planar filtration-convection problem of fluid flow through porous media [1–4]. The investigations in [4] were carried out for finite-dimensional approximations of small size, so it is desirable to

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develop appropriate numerical methods for finite-dimensional systems of larger size. It is very important to preserve cosymmetry in finite-dimensional models derived from partial differential equations. It was shown in [5] that improper approximation may lead to the destruction of the family of equilibria. We apply in this work the approach based on spectral expansion on vertical coordinate and finite-difference method in horizontal direction.

2 Darcy convection problem

We will consider the planar filtration-convection problem for incompressible fluid saturated with a porous medium in a rectangular container $\mathcal{D} = [0, a] \times [0, b]$ which is uniformly heated below. The temperature difference δT was held constant between the lower $y = 0$ and upper $y = b$ boundaries of the rectangle and the temperature on the vertical boundaries obeys a linear law, so that a time independent uniform vertical temperature profile is formed. We consider perturbation of the temperature from the basic state of rest with a linear conductive profile.

Because the fluid is incompressible, we introduce a stream function ψ such that horizontal and vertical components of the velocity vector are given as $u = -\psi_y$ and $v = \psi_x$, respectively. The dimensionless equations of the filtration convection problem are:

$$\frac{\partial \theta}{\partial t} = \Delta\theta + \lambda \frac{\partial \psi}{\partial x} + J(\psi, \theta) \equiv F_1 \tag{1}$$

$$0 = \Delta\psi - \frac{\partial \theta}{\partial x} \equiv F_2 \tag{2}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian and $J(\psi, \theta)$ denotes the Jacobian operator over (x, y):

$$J(\psi, \theta) = \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x}.$$

The dependent variables $\psi(x, y, t)$ and $\theta(x, y, t)$ denote perturbations of the stream function and temperature, λ is the Rayleigh number given by $\lambda = \beta g \delta T K l / \kappa\mu$, here β is the thermal expansion coefficient, g is the acceleration due to gravity, μ is the kinematic viscosity, κ is the thermal diffusivity of the fluid, K is the permeability coefficient, l is the length parameter. The boundary conditions are:

$$\theta = 0, \qquad \psi = 0 \qquad \text{on} \quad \partial\mathcal{D}, \tag{3}$$

and the initial condition is only defined for the temperature

$$\theta(x, y, 0) = \theta_0(x, y), \tag{4}$$

where θ_0 denotes the initial temperature distribution. For a given θ_0, the stream function ψ can be obtained from (2), (3) as the solution of the Dirichlet problem via Green's operator $\psi = G\theta_x$.

Cosymmetry for underlying system is given by $(\psi, -\theta)$. Really, multiply (1) by ψ and (2) by $-\theta$, sum and integrate over domain $\mathcal{D}$. Then, using integration by parts and Green's formula we derive

$$\int_{\mathcal{D}} (F_1\psi - F_2\theta)dxdy = \int_{\mathcal{D}} (\Delta\theta\psi - \Delta\psi\theta + \lambda\psi_x\psi + \theta_x\theta + J(\theta,\psi)\psi)dxdy = 0. \quad (5)$$

To establish this we also need the following equality

$$\int_{\mathcal{D}} J(\psi,\theta)\psi dxdy = 0. \quad (6)$$

Moreover, the Jacobian J is antisymmetric with respect to its arguments and the equality takes place

$$\int_{\mathcal{D}} J(\psi,\theta)\theta dxdy = 0. \quad (7)$$

For all values of the Rayleigh number there is a trivial equilibrium. The eigenvalues of the spectral problem for the trivial equilibrium are [2]

$$\lambda_{mn} = 4\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \quad (8)$$

where m, and n are integers. They have multiplicity of two if and only if the diopanthine equation $m^2/a^2 + n^2/b^2 = m_1^2/a^2 + n_1^2/b^2$ has a unique solution with $m_1 = m$ and $n_1 = n$. The lowest eigenvalue corresponds to $m = n = 1$, and when the parameter λ passes λ_{11} a one-parameter family of stationary solutions emerges. This family is a closed curve in the phase space. In [2] it was shown that the spectrum varies along this family and therefore this family can not be an orbit of the action of any symmetry group.

3 Spectral-finite-difference method

The approach based on spectral and finite-difference approximation is applied. We use Galerkin expansion in the direction y and finite-difference method for x. Firstly we take the following

$$\theta(x,y,t) = \sum_{j=1}^{m} \theta_j(x,t) \sin\frac{\pi j y}{b}, \quad \psi(x,y,t) = \sum_{j=1}^{m} \psi_j(x,t) \sin\frac{\pi j y}{b}. \quad (9)$$

After substituting (9) to (1)–(2) and integrating on y we derive:

$$\dot{\theta}_j = \theta_j'' - c_j\theta_j + \lambda\psi_j' - J_j, \quad j = 1 \div m, \quad (10)$$

$$0 = \psi_j'' - c_j\psi_j - \theta_j', \quad j = 1 \div m, \quad (11)$$

here a prime and a dot denote differentiation on x and t respectively, $c_j = j^2\pi^2/b^2$, and for J_j we have

$$J_j = \frac{2\pi}{b}\sum_{i=1}^{m-j}[(i+j)(\theta_{i+j}\psi_i' - \theta_i'\psi_{i+j}) + i(\theta_{i+j}'\psi_i - \theta_i\psi_{i+j}')] \tag{12}$$
$$+\frac{2\pi}{b}\sum_{i=1}^{j-1}(j-i)(\theta_i'\psi_{j-i} - \theta_{j-i}\psi_i'), \quad j = 1 \div m.$$

The boundary conditions (3) may be rewritten as

$$\theta_j(t,0) = \theta_j(t,a), \quad \psi_j(t,0) = \psi_j(t,a), \quad j = 1 \div m.$$

We deduce from initial condition (4) the following

$$\theta_j(x,0) = \int_{\mathcal{D}} \theta_0(x,y)\sin\frac{\pi j y}{b}dy, j = 1 \div m.$$

To discretize (10)–(12) on variable x we apply uniform mesh $\omega = \{x_k | x_k = kh, k = 0 \div n, h = a/(n+1)\}$ and the notions $\theta_{j,k} = \theta_j(x_k,t)$, $\psi_{j,k} = \psi_j(x_k,t)$, $J_{j,k} = J_j(x_k,t)$. The centered finite-difference operators are used and we deduce a system of ordinary differential equations

$$\dot{\theta}_{jk} = \frac{\theta_{j,k+1} - 2\theta_{j,k} + \theta_{j,k-1}}{h^2} - c_j\theta_{jk} + \lambda\frac{\psi_{j,k+1} - \psi_{j,k-1}}{2h} - J_{j,k} \equiv \phi_{1jk}, \tag{13}$$
$$0 = \frac{\psi_{j,k+1} - 2\psi_{j,k} + \psi_{j,k-1}}{h^2} - c_j\psi_{jk} + \frac{\theta_{j,k+1} - \theta_{j,k-1}}{2h} \equiv \phi_{2jk}. \tag{14}$$

The expression for J_{jk}, being discretization of J_j (12) at x_k, will be given below. Finally, the boundary conditions are the following

$$\theta_{j0} = \theta_{jn} = 0, \quad \psi_{j0} = \psi_{jn} = 0. \tag{15}$$

4 Cosymmetry conservation

One can check that a vector

$$L_h = (\psi_{11}, ..., \psi_{n1}, \psi_{12}, ..., \psi_{nm}, -\theta_{11}, ..., -\theta_{n1}, -\theta_{12}, ..., -\theta_{nm})$$

gives a cosymmetry for (13)–(14). So, a cosymmetric equality must be held

$$\sum_{j=1}^{m}\sum_{k=1}^{n}[\phi_{1jk}\psi_{j,k} - \phi_{2jk}\theta_{j,k}] = 0. \tag{16}$$

Substitute (13), (14) into (16) and using summation we deduce that linear parts in (13), (14) nullify and the following relation must be preserved

$$\sum_{k=1}^{n}\sum_{j=1}^{m}(J_j\psi_j)_k = 0. \tag{17}$$

We demand also for $J_{j,k}$ the additional property

$$\sum_{k=1}^{n}\sum_{j=1}^{m}(J_j\theta_j)_k = 0. \tag{18}$$

It should be stressed that usual finite-difference operators do not keep the equalities (17)–(18). To reach correct approximation of nonlinear terms we introduce two operators

$$D_a(\theta,\psi) = \theta'\psi - \theta\psi', \quad D_s(\theta,\psi) = \theta'\psi + \theta\psi'.$$

Then, we rewrite J_j

$$J_j = \frac{2\pi}{b}\left(\sum_{i=1}^{m-j}\chi^1_{j,i} + \sum_{i=1}^{j-1}\chi^2_{j,i}\right), \tag{19}$$

$$\chi^1_{j,i} = \frac{2i+j}{2}\left(D_s(\theta_{i+j},\psi_i) - D_s(\theta_i,\psi_{i+j})\right) - \frac{j}{2}\left(D_a(\theta_{i+j},\psi i) + D_a(\theta_i,\psi_{i+j})\right),$$

$$\chi^2_{j,i} = \frac{j-i}{2}\left(D_s(\theta_i,\psi_{j-i}) + D_a(\theta_{i+j},\psi_i) - D_s(\theta_{j-i},\psi_i) + D_a(\theta_{j-i},\psi_i)\right).$$

Using method of free parameters we derive the special approximation on three-point stencil for D_a D_s

$$d_{a,k}(\theta,\psi) = \frac{\theta_{k+1}-\theta_{k-1}}{2h}\psi_k - \theta_k\frac{\psi_{k+1}-\psi_{k-1}}{2h},$$

$$d_{s,k}(\theta,\psi) = \frac{2\theta_{k+1}\psi_{k+1} + \psi_k(\theta_{k+1}-\theta_{k-1}) + \theta_k(\psi_{k+1}-\psi_{k-1}) - 2\theta_{k-1}\psi_{k-1}}{6h}.$$

5 Numerical Results

We rewrite the (13)–(14) in vector form

$$\frac{d}{dt}\Theta = A\Theta + \lambda B\Psi + L(\Theta,\Psi), \quad 0 = A\Psi - B\Theta, \tag{20}$$

$$\Theta = (\theta_{11},...,\theta_{n1},...,\theta_{1m},...,\theta_{nm}), \quad \Psi = (\psi_{11},...,\psi_{n1},...,\psi_{1m},...,\psi_{nm}).$$

The matrix A consists of m three-diagonal submatrices A_j, nonlinear entries of skew-symmetric matrix $B = \{b_{sr}\}_{s,r=1}^{nm}$ are given by $b_{s,s+1} = -b_{s+1,s} = h/2$, $s = 1 \div nm-1$, and $L(F,G)$ presents the nonlinear terms in (13). The discrete stream function Ψ can be expressed in form of Θ using second equation in (20). It gives the following system of ordinary differential equations:

$$\frac{d\Theta}{dt} = (A + \lambda BA^{-1}B)\Theta + L(\Theta, A^{-1}B\Theta). \tag{21}$$

To carry out computation with (21) we create a code on MATLAB. It allows to analyze convective structures, continue the families of stationary regimes .

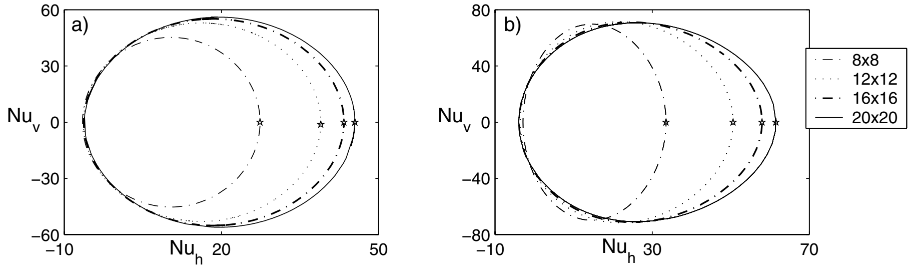

Fig. 1. The families of stationary regimes for different meshes $\lambda = 70, \quad \beta = 1.5$ (left), $\lambda = 55, \quad \beta = 3$ (right)

As stated in [2], if λ is slightly larger than λ_{11}, then all points of the family are stable. Starting from the vicinity of unstable zero equilibrium we integrate the system (21) up to a point Θ_0 close to a stable equilibrium on the family. We have used here the classical fourth order Runge-Kutta method as time integrator. Then a simplified version of the algorithm for family computation may be formulated in the following steps. Correct the point Θ_0 using the modified Newton method. Determine the kernel of the linearization matrix at the point Θ_0 by singular value decomposition. Predict the next point on the family Θ_0 by using fourth order Runge-Kutta method. Repeat these steps until a closed curve is obtained. This method is based on the cosymmetric version of implicit function theorem [6].

We explored the derived technique to calculate the families consisting both of stable and unstable equilibria. We analyze the case of narrow container ($\beta =$

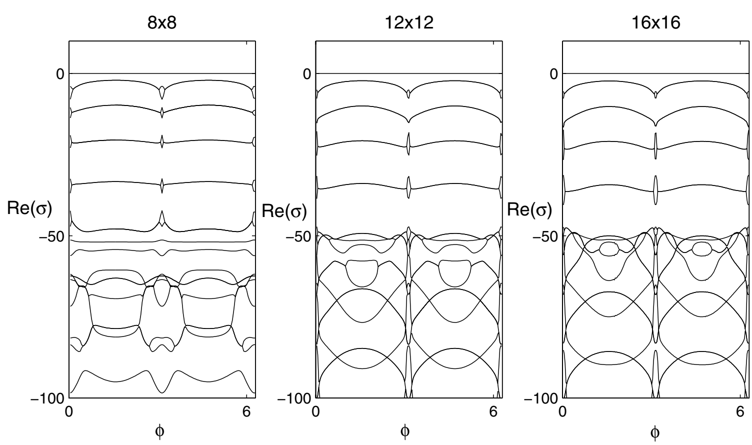

Fig. 2. Spectra for the families computed on different meshes, $\lambda = 55, \quad \beta = 3$

$b/a \geq 1, a = 1$). In Fig. 1 we compare the families computed for different meshes. We use here the coordinates Nu_h and Nu_v [4]

$$Nu_h = \int_0^b \frac{\partial \theta}{\partial x} \,|_{x=a/2}\ dy, \quad Nu_v = \int_0^a \frac{\partial \theta}{\partial y} \,|_{y=0}\ dx$$

The spectra corresponding to these families are given in Fig. 2, where for equilibria we use the parameterization $\phi \in [0, 2\pi]$. In fig.1 we mark by starts

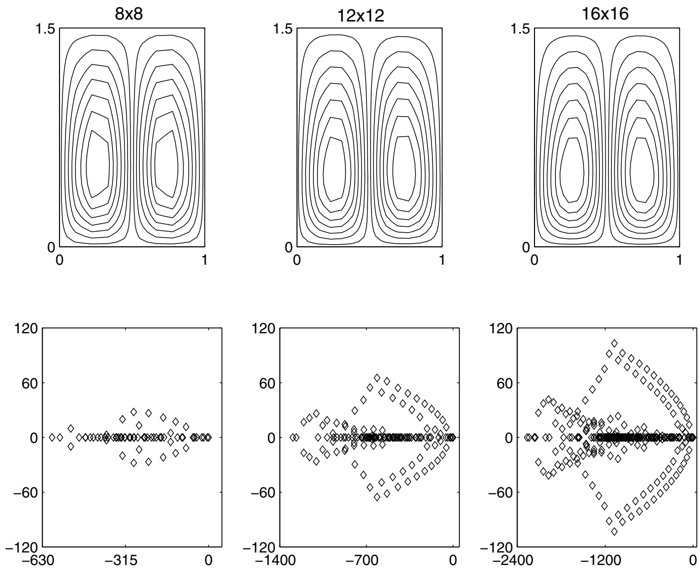

Fig. 3. Stream functions (top) and spectra (bottom) for selected equilibria (marked by stars) in fig. 1, $\lambda = 70, \quad \beta = 1.5$

the stationary regimes for which in fig. 3 stream functions (top) and spectra distributions (bottom) are presented. It is suitable to summarize the results of our computations in fig. 4. To present the dependence of critical values from parameter β we use the ratio λ/λ_{11}, where λ_{11} is the threshold of onset of the family. In fig. 4 the curves 1 and 2 correspond to the monotonic and oscillatory instability respectively, the curve 3 respects to the completely instable primary family ($\lambda = \lambda_o$), and the curve 4 gives the critical values of collision when primary and secondary families collide together ($\lambda = \lambda_c$). The stream functions for the first stationary regimes lost stability are displayed in fig. 5. This picture demonstrates that a number of equilibria lost stability depends on the parameter β. We observe the case of instability at six equilibria for $\beta \approx 2.3$ when the monotonic instability takes place in two points and oscillatory one – in four points.

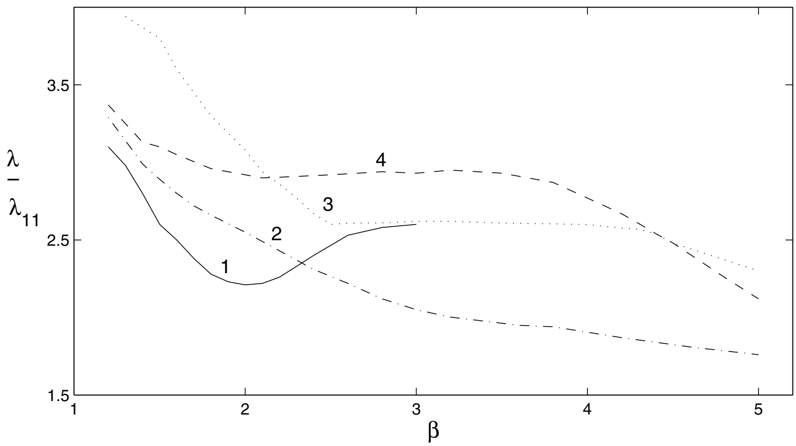

Fig. 4. Critical values for the families of stationary regimes

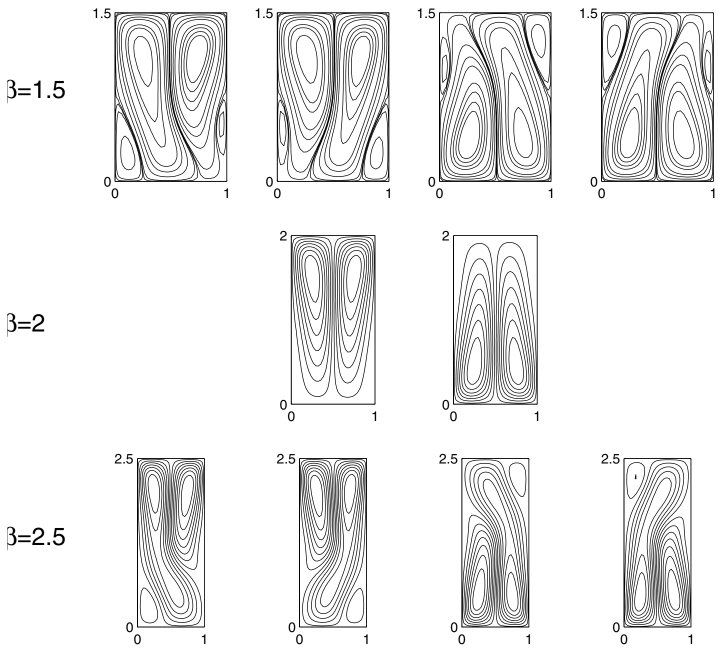

Fig. 5. Stream functions corresponding to first regimes lost stability, $\beta = 1.5, 2, 2.5$

Finally, we demonstrate the scenario of the evolution of the primary family for $\beta = 2$ in fig. 6. We mark by crosses (circles) the regimes lost the stability by monotonic (oscillatory) manner, and draw the secondary family (dotted curve). We use the projection onto two-dimensional unstable manifold of the zero equilibrium (coordinates U_1, U_2).

The primary family of stationary regimes is consists of stable equilibria for $\lambda = 70$, see fig. 6a. When λ increases one can see how the family deforms and at $\lambda_u = 112$ (fig. 6b) two points become unstable by monotonic manner. We present the stream functions for these equilibria in fig. 5. Further two arcs of

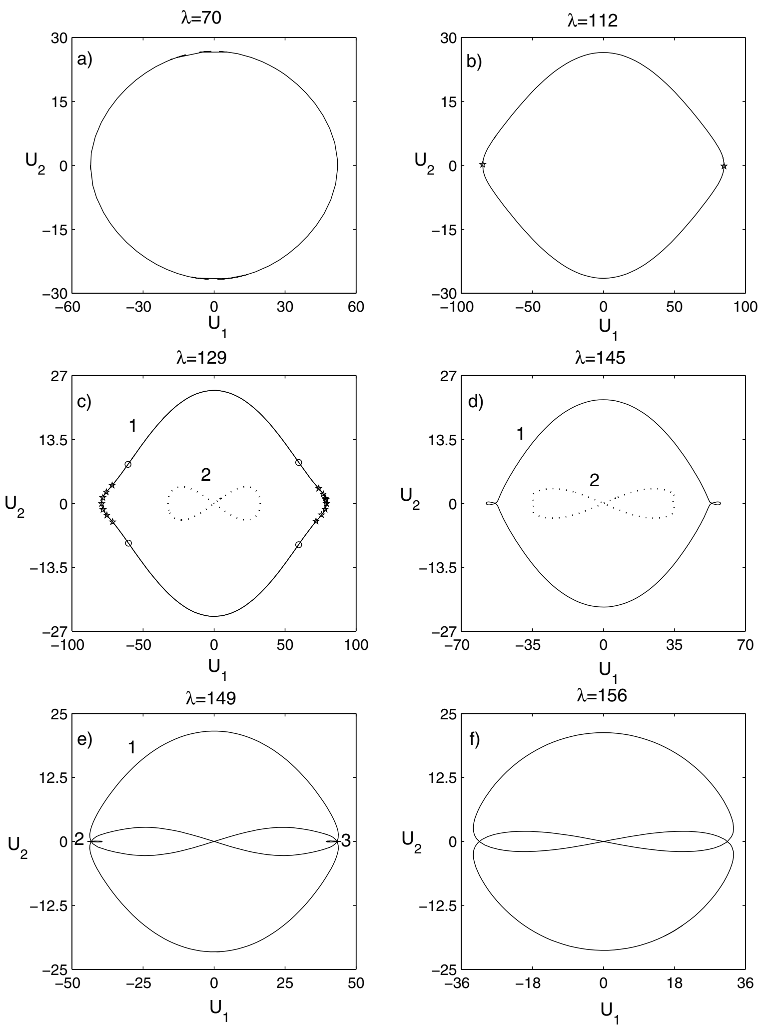

Fig. 6. Primary and secondary families evolution, $\beta = 2$

unstable equilibria are formed, then, for $\lambda = 129$ four regimes lost stability via oscillatory bifurcation (fig. 6c). We mention that for $\lambda > 81.6$ unstable secondary family exists as well (the curve 2 in fig. 6c). In fig. 6d one can see two families for $\lambda = 145$. Collision of primary and secondary families takes place for $\lambda_o = 147.6$. As a result, we see two small families and a combined one, fig. 6. When λ increases the small families collapse and disappear. The combined family for $\lambda = 156$ is completely unstable (fig. 6f).

A quite complete picture of local bifurcations in a cosymmetric dynamical system is presented in [7–9].

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