One Class of Splitting Iterative Schemes

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Abstract. This paper deals with the stability analysis of a new class of iterative methods for elliptic problems. These schemes are based on a general splitting method, which decomposes a multidimensional parabolic problem into a system of one dimensional implicit problems. We use a spectral stability analysis and investigate the convergence order of two iterative schemes. Finally, some results of numerical experiments are presented.

1 Introduction

Iterative methods for solving discrete elliptic problems can be viewed as finite difference schemes for non stationary parabolic problems. The most important difference is that for elliptic problems we select the time step parameter according the requirements of the convergence to the stationary solution and can ignore the approximation error. For splitting methods for elliptic problems we refer to [4, 6]. In particular, [5, 6] involve an alternating direction method, [4] presents factorization schemes. The convergence rate can be increased if optimal non-stationary parameters are used for the definition of each iteration, see e.g. [6].

Recently the multicomponent versions of alternating direction method were proposed in [1]. These schemes are also used for solving multidimensional elliptic problems [2]. While in the previous papers the stability of these splitting schemes was investigated by the energy method, we will use the spectral stability analysis. For symmetric problems it gives necessary and sufficient convergence conditions and enables us to find optimal values of iterative parameters. Such analysis was used also in [3]

The content of this paper is organized as follows. In Section 2 we formulate the multicomponent iterative scheme. The convergence of 2D scheme is investigated in Section 3. In section 4 we investigate the stability of 3D iterative scheme, the analysis is done using numerical experiments. Finally, in Section 5 we study the convergence of the p-dimensional multicomponent iterative scheme and prove the energy stability estimates. In Section 6, the Seidel type scheme is formulated and investigated.

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2 Multicomponent iterative scheme

Let the strictly elliptic problem be given by

$$-\sum_{i=1}^{p} \frac{\partial^2 u}{\partial x_i^2} = f(x), \quad x \in Q\,,$$
$$u(x) = 0 \quad x \in \partial Q\,,$$

where $Q = (0,1)^p$. We introduce a uniform grid in Q and approximate the elliptic problem by the following finite difference scheme

$$\sum_{\alpha=1}^{p} A_\alpha y = f, \tag{1}$$

here A_α denotes the approximation of a differential operator using standard central difference formula.

Let $(,)$ and $\|\cdot\|$ denote the inner product and the L_2 norm of discrete functions, respectively. Generally, we introduce the following assumptions:

$$A_\alpha^* = A_\alpha\,,$$
$$0 < m_\alpha \|y\| \le (A_\alpha y, y) \le M_\alpha \|y\|\,.$$

It is well known that for the Laplace operator these assumptions are satisfied with

$$8 \le m_\alpha, \quad M_\alpha \le \frac{4}{h^2}\,, \tag{2}$$

where h is the spatial step size.

In order to solve the system of linear equations (1) we define p unknown functions $y_\alpha,\ \alpha = 1, 2, \ldots, p$. Then the *multicomponent alternating direction* (MAD) scheme is given as (see, [1]):

$$\frac{\overset{s+1}{y}_\alpha - \overset{s}{\tilde{y}}}{\tau} + pA_\alpha\left(\overset{s+1}{y}_\alpha - \overset{s}{y}_\alpha\right) + \sum_{\beta=1}^{p} A_\beta \overset{s}{y}_\beta = f\,, \quad \alpha = 1, 2, \ldots, p\,, \tag{3}$$
$$\overset{s}{\tilde{y}} = \frac{1}{p}\sum_{\alpha=1}^{p} \overset{s}{y}_\alpha\,,$$

where $\overset{s}{y}_\alpha$ is the s-th iteration of y_α. We note that all equations (3) can be solved in parallel.

We will investigate the convergence of the MAD scheme by using the following error norms:

$$\|\overset{s}{e}\| = \|\overset{s}{\tilde{y}} - y\|\,, \quad \|\overset{s}{r}\| = \|\sum_{\alpha=1}^{p} A_\alpha \overset{s}{y}_\alpha - f\|\,.$$

3 Spectral stability analysis of 2D iterative scheme

In this section we consider a two-dimensional iterative scheme (1)

$$\frac{\overset{s+1}{y}_\alpha - \overset{s}{\tilde{y}}}{\tau} + 2A_\alpha\Big(\overset{s+1}{y}_\alpha - \overset{s}{y}_\alpha\Big) + \sum_{\beta=1}^{2} A_\beta \overset{s}{y}_\beta = f\,, \quad \alpha = 1, 2\,. \tag{4}$$

Let denote the error functions

$$\overset{s}{e}_\alpha = \overset{s}{y}_\alpha - y, \quad \alpha = 1, 2, \dots, p\,, \quad \overset{s}{\tilde{e}} = \overset{s}{\tilde{y}} - y\,.$$

Then the error functions satisfy the following MAD scheme:

$$\frac{\overset{s+1}{e}_\alpha - \overset{s}{\tilde{e}}}{\tau} + 2A_\alpha\Big(\overset{s+1}{e}_\alpha - \overset{s}{e}_\alpha\Big) + \sum_{\beta=1}^{2} A_\beta \overset{s}{e}_\beta = 0\,, \quad \alpha = 1, 2\,, \tag{5}$$

$$\overset{s}{\tilde{e}} = \frac{1}{2}\sum_{\alpha=1}^{2} \overset{s}{e}_\alpha\,,$$

To apply the discrete von Neumann stability criteria to problem (5), we let

$$\overset{s}{e}_\alpha = \sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \overset{s}{d}_{\alpha,jk} \sin(j\pi x_1)\sin(k\pi x_2)\,, \quad \alpha = 1, 2\,, \tag{6}$$

where N is the number of grid points in one-dimensional grid. It is well known, that $\sin(j\pi x_\alpha)$ are eigenvectors of the operator A_α, i.e.:

$$A_\alpha \sin(j\pi x_\alpha) = \lambda_j \sin(j\pi x_\alpha),$$
$$8 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{N-1} \le \frac{4}{h^2}. \tag{7}$$

If we replace $\overset{s+1}{e}_\alpha$ and $\overset{s}{e}_\alpha$ in (5) by expressions of the form given by equation (6) and use (7), we get the following matrix equations

$$\overset{s+1}{\mathbf{d}}_{jk} = Q_2\, \overset{s}{\mathbf{d}}_{jk}\,, \tag{8}$$

where $\mathbf{d}_{jk}$ is the column vector of spectral coefficients

$$\mathbf{d}_{jk} = \begin{pmatrix} d_{1,jk} \\ d_{2,jk} \end{pmatrix}\,,$$

and Q_2 is the stability matrix of MAD scheme

$$Q_2 = \begin{pmatrix} \dfrac{0.5 + \tau\lambda_j}{1 + 2\tau\lambda_j} & \dfrac{0.5 - \tau\lambda_k}{1 + 2\tau\lambda_j} \\[2ex] \dfrac{0.5 - \tau\lambda_j}{1 + 2\tau\lambda_k} & \dfrac{0.5 + \tau\lambda_k}{1 + 2\tau\lambda_k} \end{pmatrix}.$$

Now we consider the necessary conditions for the stability of iterative MAD scheme (4). Since Q_2 is not symmetric matrix, the discrete von Neumann stability criteria can not prove that they are also sufficient for stability of MAD scheme.

Theorem 1. *All eigenvalues of stability matrix Q_2 satisfy inequalities*

$$|q_{jk}| < 1, \quad 1 \le j, k \le N-1$$

unconditionally for any values of parameters τ and h.

Proof. Using simple computations we get that eigenvalues q of the amplification matrix Q_2 satisfy the quadratic equation

$$q^2 - q + \frac{\tau(\lambda_j + \lambda_k)}{(1+2\tau\lambda_j)(1+2\tau\lambda_k)} = 0\,.$$

Then the eigenvalues of Q_2 are

$$q_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{\frac{(1-2\tau\lambda_j)(1-2\tau\lambda_k)}{(1+2\tau\lambda_j)(1+2\tau\lambda_k)}}\,\right)$$

and they are obviously both less than 1. The theorem is proved.

The convergence order of MDA iterative scheme depends on the parameter τ. We will use quasi - optimal parameter τ_0 which solves the minimization problem

$$w(\tau_0) = \min_{\tau} \max_{\lambda_1 \le \lambda \le \lambda_{N-1}} \left| \frac{1-2\tau\lambda}{1+2\tau\lambda} \right|\,.$$

Since the function

$$g(a) = \frac{1-a}{1+a}$$

is strictly decreasing, we find τ_0 from the equation

$$\frac{1-2\tau\lambda_1}{1+2\tau\lambda_1} = \frac{2\tau\lambda_{N-1} - 1}{1+2\tau\lambda_{N-1}}\,.$$

After simple computations we get

$$\tau_0 = \frac{1}{2\sqrt{\lambda_1\lambda_{N-1}}} \approx \frac{h}{4\pi}$$

and all eigenvalues of the amplification matrix Q_2 satisfy the inequality

$$|q_{jk}(\tau_0)| \le \frac{1}{1+\sqrt{\lambda_1/\lambda_{N-1}}} \approx 1 - \frac{\pi}{2}h\,. \tag{9}$$

These estimates are similar to convergence estimates obtained for the standard iterative scheme of alternating directions.

4 Spectral stability analysis of 3D iterative scheme

As it was stated in section 3, the von Neumann stability criteria gives only sufficient stability conditions of MAD iterative scheme. In this section we will use the spectral stability analysis for 3D iterative scheme (3). But instead of finding eigenvalues of an amplification matrix Q_3 we use direct numerical experiments and find the optimal value of the parameter τ. Such methodology gives us a possibility to investigate the convergence order of MAD scheme in the usual l_2 norm.

Let consider the model problem

$$\sum_{\alpha=1}^{3} A_\alpha y = (\lambda_j + \lambda_k + \lambda_l) \sin(j\pi x_1) \sin(k\pi x_2) \sin(l\pi x_3)\,, \tag{10}$$

which has the exact solution

$$y = \sin(j\pi x_1) \sin(k\pi x_2) \sin(l\pi x_3)\,.$$

Then the solution of 3D MAD scheme can be represented as

$$\overset{s}{y}_\alpha = \overset{s}{d}_\alpha \sin(j\pi x_1) \sin(k\pi x_2) \sin(l\pi x_3)\,, \quad \alpha = 1, 2, 3\,,$$

where d_α can be computed explicitly

$$\overset{s+1}{d}_\alpha = \frac{1}{1 + 3\tau\lambda_\alpha} \left(\sum_{\beta=1}^{3} \left(\frac{\overset{s}{d}_\beta}{3} - \tau\lambda_\beta (\overset{s}{d}_\beta - 1) \right) + 3\tau\lambda_\alpha \overset{s}{d}_\alpha \right)\,, \quad \alpha = 1, 2, 3\,.$$

Then the error of the sth iteration $\overset{s}{\tilde{d}}$ is estimated by the following formula

$$\overset{s}{e} = \left| \frac{\overset{s}{d}_1 + \overset{s}{d}_2 + \overset{s}{d}_3}{3} - 1 \right|.$$

Let $S(\tau, \lambda_j, \lambda_k, \lambda_l)$ be the number of iterations required to achieve the accuracy $\overset{s}{e} \leq \varepsilon$ for given eigenvalues λ_α. Then we investigate the whole interval $[m, M]$, which characterize the stiffness of the problem and compute

$$S(\tau) = \max_{m \leq \lambda_\alpha \leq M} S(\tau, \lambda_j, \lambda_k, \lambda_l)\,.$$

This problem is solved approximately by computing $S(\tau, \lambda_j, \lambda_k, \lambda_l)$ for $(K+1)^3$ combinations of eigenvalues $\lambda_\alpha = m + i(M - m)/K$.

First, we investigated the dependence of the number of iterations $S(\tau)$ on the parameter τ. It was proved that there exists the optimal value τ_0, such that

$$S(\tau_0) \leq S(\tau)$$

Table 1. The number of iterations as a function of τ

ε	τ	$S(\tau)$
0.0001	0.0015	210
0.0001	0.0020	159
0.0001	0.0022	145
0.0001	0.0023	149
0.0001	0.0025	162

and this value satisfies the following condition

$$\max_{m \le \lambda_\alpha \le M} S(\tau_0, \lambda_j, \lambda_k, \lambda_l) = S(\tau_0, m, m, m)\,. \tag{11}$$

In Table 1 we present numbers of iterations $S(\tau)$ for different values of τ. These experiments were done with $m = 10, M = 4000$.

The optimal value of the parameter τ depends slightly on ε. In Table 2 we present optimal numbers of iterations $S(\tau_0)$ for different values of ε.

Table 2. The optimal value of τ as a function of ε

ε	τ_0	$S(\tau_0)$
0.001	0.00233	103
0.0001	0.00221	144
0.00001	0.00212	187

Finally, we investigated the dependence of the convergence rate of the MAD iterative scheme on m and M. In Table 3 we present optimal values of τ and numbers of iterations $S(\tau_0)$ for different spectral intervals. We used $\varepsilon = 10^{-4}$ in these experiments.

It follows from results presented in Table 3 that

$$\tau_0 = \frac{c}{\sqrt{mM}}\,, \quad S(\tau_0) = O\left(\sqrt{\frac{M}{m}}\right)\,.$$

The above conclusion agrees well with results of section 3.

Table 3. The optimal value of τ as a function of m and M

m	M	τ_0	$S(\tau_0)$
10	4000	0.00221	144
10	16000	0.00110	284
10	64000	0.00055	563
10	16000	0.00110	284
40	16000	0.00055	145
90	16000	0.00037	98

5 Error estimates for *p*-dimensional *MAD* scheme

In this section we consider p-dimensional iterative scheme (3). Let us introduce the following notation:

$$\overset{s}{y}_{\alpha t} = \frac{\overset{s+1}{y}_\alpha - \overset{s}{y}_\alpha}{\tau}, \quad \overset{s}{\tilde{y}}_t = \frac{\overset{s+1}{\tilde{y}} - \overset{s}{\tilde{y}}}{\tau}, \quad \overset{s}{v}{}^{(\alpha,\beta)} = \overset{s}{y}_\alpha - \overset{s}{y}_\beta ,$$

$$\| \overset{s}{v} \|_3^2 = \sum_{\alpha,\beta=1,\, \alpha>\beta}^{p} \| \overset{s}{v}{}^{(\alpha,\beta)} \|^2 , \quad Q_p(\overset{s}{y}) = \| \overset{s}{r} \|^2 + \frac{1}{p^2\,\tau^2} \| \overset{s}{v} \|_3^2 .$$

In the following theorem we estimate the convergence rate of MAD iterative scheme.

Theorem 2. *Iterative scheme* (3) *produces a sequence converging unconditionally to the solution of problem* (2) *and the convergence rate is estimated as*

$$Q_p(\overset{s+1}{y}) \le \frac{1}{q}\, Q_p(\overset{s}{y}) , \quad q = \min\left(1 + m\,p\,\tau,\ 1 + \frac{1}{2M\,p\,\tau}\right), \tag{12}$$

where m and M are the spectral estimates of the operator A:

$$m = \min_{1\le\alpha\le p} m_\alpha, \quad M = \max_{1\le\alpha\le p} M_\alpha .$$

Proof. Multiplying both sides of (3) by $\overset{s}{y}_{\alpha t}$ and adding all equalities we get

$$\sum_{\alpha=1}^{p}\left(\frac{\overset{s+1}{y}_\alpha - \overset{s}{\tilde{y}}}{\tau}, \overset{s}{y}_{\alpha t}\right) + p\,\tau \sum_{\alpha=1}^{p}(A_\alpha \overset{s}{y}_{\alpha t}, \overset{s}{y}_{\alpha t}) + p\left(\sum_{\beta=1}^{p} A_\beta \overset{s}{y}_\beta - f, \overset{s}{\tilde{y}}_t\right) = 0 . \tag{13}$$

The first term of (13) can be rewritten as

$$\begin{aligned} I_1 &= \frac{1}{p\,\tau} \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} \left(\overset{s+1}{y}_\alpha - \overset{s+1}{y}_\beta,\, \overset{s}{y}_{\alpha t} \right) + \frac{1}{p\,\tau} \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} \left(\overset{s+1}{y}_\beta - \overset{s}{y}_\beta,\, \overset{s}{y}_{\alpha t} \right) \\ &= p\, \| \overset{s}{\tilde{y}}_t \|^2 + \frac{1}{p\,\tau} \sum_{\alpha,\beta=1,\, \alpha>\beta}^{p} \left(\overset{s+1}{v}{}^{(\alpha,\beta)}, \overset{s}{v}{}_t^{(\alpha,\beta)} \right) \\ &= p\, \| \overset{s}{\tilde{y}}_t \|^2 + \frac{1}{2p} \| \overset{s}{v}_t \|_3^2 + \frac{1}{2p\,\tau^2} \left(\| \overset{s+1}{v} \|_3^2 - \| \overset{s}{v} \|_3^2 \right) . \end{aligned} \tag{14}$$

By adding equations (3) we get that

$$\overset{s}{\tilde{y}}_t = - \left(\sum_{\alpha=1}^{p} A_\alpha \overset{s+1}{y}_\alpha - f \right) ,$$

hence using the third term of (13) we can prove that

$$p\, \| \overset{s}{\tilde{y}}_t \|^2 + p \left(\sum_{\beta=1}^{p} A_\beta \overset{s}{y}_\beta - f,\, \overset{s}{\tilde{y}}_t \right) \geq \frac{p}{2} \left(\| \overset{s+1}{r} \|^2 - \| \overset{s}{r} \|^2 \right) . \tag{15}$$

The second term of (13) is estimated similarly. Thus the convergence rate estimate (12) follows trivially. The theorem is proved.

As a corollary of Theorem 2 we can find the optimal value of parameter $\tau_0 = 1/p\sqrt{2mM}$, which is obtained from the equation

$$1 + p\, m\, \tau = 1 + \frac{1}{2M\, p\, \tau} .$$

6 Seidel-type iterative scheme

In this section we investigate the convergence rate of Seidel-type iterative scheme:

$$\frac{\overset{s+1}{y}_\alpha - \overset{s}{y}{}_\alpha^*}{\tau} + \sum_{\beta=1}^{\alpha} A_\beta \overset{s+1}{y}_\beta + \sum_{\beta=\alpha+1}^{p} A_\beta \overset{s}{y}_\beta = f , \quad \alpha = 1, 2, \ldots, p , \tag{16}$$

$$\overset{s}{y}{}_1^* = \overset{s}{y}_1, \quad \overset{s}{y}{}_\alpha^* = 0.5 \left(\overset{s}{y}_\alpha + \overset{s}{y}_{\alpha-1} \right) .$$

6.1 Spectral stability analysis of 2D scheme

To apply the discrete von Neumann stability criteria to problem (16), we write the global error as a series:

$$\overset{s}{e}_\alpha = \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \overset{s}{d}_{\alpha,jk} \sin(j\pi x_1) \sin(k\pi x_2) , \quad \alpha = 1, 2 ,$$

Substituting this expansion into (16), we obtain the equation for coefficients

$$\overset{s+1}{\mathbf{d}}_{jk} = Q_2 \overset{s}{\mathbf{d}}_{jk} \ . \tag{17}$$

where $\mathbf{d}_{jk}$ is the column vector of spectral coefficients and Q_2 is the stability matrix of scheme (16)

$$Q_2 = \begin{pmatrix} \dfrac{1}{1+\tau\lambda_j} & \dfrac{-\tau\lambda_k}{1+\tau\lambda_j} \\ \dfrac{0.5(1-\tau\lambda_j)}{(1+\tau\lambda_j)(1+\tau\lambda_k)} & \dfrac{0.5+\tau\lambda_j+\tau^2\lambda_j\lambda_k}{(1+\tau\lambda_j)(1+\tau\lambda_k)} \end{pmatrix} .$$

Now we consider the necessary conditions for the stability of scheme (16). The eigenvalues of the amplification matrix Q_2 satisfy the quadratic equation

$$q^2 - \left(1 + \frac{0.5(1-\tau\lambda_j)}{(1+\tau\lambda_j)(1+\tau\lambda_k)}\right) q + \frac{0.5}{1+\tau\lambda_j} = 0 .$$

Theorem 3. *All eigenvalues of stability matrix Q_2 satisfy inequalities*

$$|q_{jk}| < 1, \quad 1 \le j,k \le N-1$$

unconditionally for any values of parameters τ and h.

Proof. Application of the Hurwitz criterion gives that $|q_{jk}| \le 1$ is satisfied if and only if

$$\frac{0.5}{1+\tau\lambda_j} < 1 , \qquad \left|1 + \frac{0.5(1-\tau\lambda_j)}{(1+\tau\lambda_j)(1+\tau\lambda_k)}\right| < 1 + \frac{0.5}{1+\tau\lambda_j}.$$

Simple computations prove that both inequalities are satisfied unconditionally. The theorem is proved.

6.2 Spectral stability analysis of 3D iterative scheme

Let consider the model problem (10). The solution of 3D scheme (16) can be represented as

$$\overset{s}{y}_\alpha = \overset{s}{d}_\alpha \sin(j\pi x_1)\sin(k\pi x_2)\sin(l\pi x_3) , \quad \alpha = 1,2,3 ,$$

where $d_\alpha,\ \alpha = 1,2,3$, are computed explicitly

$$\overset{s+1}{d}_\alpha = \frac{1}{1+\tau\lambda_\alpha}\left(\frac{\overset{s}{d}_\alpha + \overset{s}{d}_{\alpha-1}}{2} - \sum_{\beta=1}^{\alpha-1}\tau\lambda_\beta \overset{s+1}{d}_\beta - \sum_{\beta=\alpha+1}^{3}\tau\lambda_\beta \overset{s}{d}_\beta + \sum_{\beta=1}^{3}\tau\lambda_\beta\right) .$$

We estimate the error of the sth iteration $\overset{s}{d}_\alpha$ by the following formula

$$\overset{s}{e} = \max_{1\le\alpha\le3}\left|\overset{s}{d}_\alpha\right| .$$

Table 4. The optimal value of τ as a function of m and M

m	M	τ_0	$S(\tau_0)$
10	4000	0.00173	176
10	16000	0.00089	346
10	64000	0.00045	681
10	16000	0.00089	346
40	16000	0.00042	177
90	16000	0.00028	125

We investigated numerically the dependence of the convergence rate of the iterative scheme (16) on m and M. In Table 4 we present optimal values of τ and numbers of iterations $S(\tau_0)$ for different spectral intervals. We used $\varepsilon = 10^{-4}$ in these experiments.

It follows from results presented in Table 4 that

$$\tau_0 = \frac{c}{\sqrt{mM}}, \quad S(\tau_0) = O\left(\sqrt{\frac{M}{m}}\right).$$

The convergence rate of the Seidel type scheme (16) is the same as of scheme (3).

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