Restarted Simpler GMRES Augmented with Harmonic Ritz Vectors

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Abstract. We describe a method for improving the convergence of the Simpler GMRES method for problems with small eigenvalues. We augment the Krylov subspace with harmonic Ritz vectors corresponding to the smallest harmonic Ritz values. The advantage over augmented GMRES is that the problem of finding the minimal residual solution reduces to an upper triangular least-squares problem instead of an upper-Hessenberg least-squares problem. A second advantage is that harmonic Ritz pairs can be cheaply computed. Numerical tests indicate that augmented Simpler GMRES(m) is superior to Simpler GMRES(m) and requires a lesser amount of work than augmented GMRES(m).

1 Introduction

Restarting the GMRES method after each cycle of m iterations slows down the convergence of the iterative method as each restart discards the information on the smallest eigenvalues which compose the solution. There are basically two approaches to reduce the negative effects of a restart: a preconditioning approach [1] and an augmented subspace approach [3]. These two approaches have been shown to improve the convergence of GMRES for problems with small eigenvalues. A cheaper implementation of GMRES can be achieved via the Simpler GMRES algorithm of Walker and Zhou [5]. The advantage is that the problem for finding the minimal residual solution reduces to an upper triangular least-squares problem instead of an upper-Hessenberg least-squares problem for GMRES. We show that for problems with small eigenvalues, it is possible to improve the convergence of restarted Simpler GMRES(m) by augmenting the Krylov subspace with harmonic Ritz vectors corresponding to the smallest harmonic Ritz values which are the roots of the GMRES polynomial.

2 Augmented Simpler GMRES

Consider the nonsymmetric linear system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ is nonsingular and $x, b \in \mathbb{R}^n$. Let x_0 be an initial guess such that $r_0 = b - Ax_0 \neq 0$. Suppose that we have applied the Arnoldi process with $w_1 = Ar_0 / \|Ar_0\|_2$ to produce the orthonormal set of vectors $\{w_1, w_2, \dots, w_{p-1}\}$ for some $p > 1$. Let $\mathcal{K}_p(w)$ denote the Krylov subspace

$$\mathcal{K}_p(w) \equiv \text{span} \{w, Aw, \dots, A^{p-1}w\} . \quad (2)$$

If the GMRES iterate x_{p-1} is not a solution of (1), then $\mathcal{K}_p(w_1) = A(\mathcal{K}_p(r_0))$ has dimension p [5]. Suppose we want to augment the Krylov subspace with k harmonic Ritz vectors $u_1, \dots, u_k$.

Denote $m = p + k$ and let Y_m be the $n \times m$ matrix whose last k vectors are the harmonic Ritz vectors u_i for $i = 1, \dots, k$. Also let W_m to be the $n \times m$ matrix whose first p columns are the vectors $w_1, \dots, w_p$ and whose last k columns are formed by orthogonalizing the vectors $\tilde{u}_i = Au_i$, for $i = 1, \dots, k$ against the previous columns of W_m by the following algorithm:

AugmentedSArnoldi

1. *Start:* Input k approximate eigenvectors U_k , $\tilde{U}_k = AU_k$, m and k . Let $p = m - k$; $w_1 = Ar_0$; $R_{1,1} = \|w_1\|_2$; $w_1 = w_1 / R_{1,1}$;
2. *Iterate:* For $i = 2 : m$
 If $i \leq p$, $w_i = Aw_{i-1}$; else, $w_i = \tilde{u}_{i-p}$;
 For $j = 1, \dots, i - 1$, $R_{j,i} = w_i^T w_j$; $w_i = w_i - R_{j,i} w_j$;
 $R_{i,i} = \|w_i\|_2$; $w_i = w_i / R_{i,i}$;
3. *Output:* If $k = 0$, $Y_m = [r_0 \ W_{m-1}]$; else, $Y_m = [r_0 \ W_{p-1} \ U_k]$.

The result of the above algorithm is the decomposition

$$AY_m = W_m R_m , \quad (3)$$

where R_m is an $m \times m$ upper triangular matrix. If $k = 0$, $Y_m = [r_0 \ W_{m-1}]$ else $Y_m = [r_0 \ W_{p-1} \ U_k]$ where U_k is a matrix whose k columns are the k harmonic Ritz vectors.

Writing the restarted problem in the form $A(\tilde{x} - x_0) = r_0$, we find that the residual vector is given by

$$r_m = r_0 - W_m \tilde{w} , \quad (4)$$

where $\tilde{w} = R_m \hat{y} = (\xi_1, \xi_2, \dots, \xi_m)^T$. From (4) and by the minimal residual criterion [2], we have $W_m^T r_m = 0$ and thus $r_m = r_{m-1} - \xi_m w_m$. The residual norm can be easily updated using

$$\|r_m\|_2 = \sqrt{(\|r_{m-1}\|_2^2 - \xi_m^2)} = \|r_{m-1}\|_2 \sin(\cos^{-1}(\xi_m / \|r_{m-1}\|_2)) . \quad (5)$$

The approximate solution $\tilde{x} - x_0$ is a combination of the columns of Y_m such that $\tilde{x} - x_0 = Y_m \hat{y} = z_m$ and $\hat{y} = R_m^{-1} \tilde{w} = (\eta_1, \eta_2, \dots, \eta_m)^T$. It then follows

that the augmented Simpler GMRES(m) correction is given by $z_m = Y_m \hat{y}$ where

$$z_i = \begin{cases} r_0 \hat{y}_1, & \text{if } i = 1, \\ (r_0, w_1, w_2, \dots, w_{i-1}) \hat{y}_i, & \text{if } 1 < i \leq p, \\ (r_0, w_1, w_2, \dots, w_{p-1}, u_1, \dots, u_{i-p}) \hat{y}_i, & \text{if } p < i \leq m. \end{cases} \quad (6)$$

In the Gram-Schmidt implementation of the augmented Simpler GMRES(m) algorithm, the correction z_i can be rewritten as

$$z_i = \begin{cases} \eta_1 r_0, & \text{if } i = 1, \\ \eta_1 r_{i-1} + \sum_{j=1}^{i-1} (\eta_{j+1} + \eta_1 \xi_j) w_j, & \text{if } 1 < i \leq p, \\ \eta r_{i-1} + \sum_{j=1}^{p-1} (\eta_{j+1} + \eta_1 \xi_j) w_j \\ \quad + \sum_{j=p}^{i-1} (\eta_1 \xi_j w_j + \eta_{j+1} u_{j-p+1}), & \text{if } i > p. \end{cases} \quad (7)$$

Harmonic Ritz vectors are computed by applying the minimal residual criterion. This criterion for finding an approximate eigenvector $u = Y_m \tilde{g}_m$ of A in $AK_m(r_0)$ can be written as [2]

$$(Au - \lambda u) \perp AK_m(r_0) \Leftrightarrow W_m^T (AY_m \tilde{g}_m - \lambda Y_m \tilde{g}_m) = 0. \quad (8)$$

Using (3), we have

$$R_m \tilde{g}_m = \lambda W_m^T Y_m \tilde{g}_m. \quad (9)$$

Denoting $(W_m^T Y_m)^{-1}$ by T_m , (9) becomes

$$T_m R_m \tilde{g}_m = \lambda \tilde{g}_m. \quad (10)$$

Noting that $T_m(R_m T_m)T_m^{-1} = T_m R_m$, the projected eigenvalue problem (9) becomes

$$R_m T_m g_m = \tilde{R}_m g_m = \lambda g_m, \quad (11)$$

and the approximate eigenvector becomes $u = Y_m \tilde{g}_m = Y_m (T_m g_m)$.

If $k = 0$, $Y_m = [r_0 \ W_{m-1}]$, and thus T_m and $\tilde{R}_m$ can be easily obtained as shown in [2]. Now if $k > 0$, then $Y_m = [r_0 \ W_{p-1} \ U_k]$ and

$$T_m^{-1} = \begin{pmatrix} \xi_1 & 1 & & & w_1^T u_1 & \dots & w_1^T u_k \\ \xi_2 & 0 & 1 & & w_2^T u_1 & \dots & w_2^T u_k \\ \vdots & & \ddots & \ddots & \vdots & & \\ \xi_{p-1} & & & 0 & 1 & w_{p-1}^T u_1 & \dots & w_{p-1}^T u_k \\ \xi_p & & & & 0 & w_p^T u_1 & \dots & w_p^T u_k \\ \vdots & & & & & \vdots & & \vdots \\ \xi_m & & & & & w_m^T u_1 & \dots & w_m^T u_k \end{pmatrix}. \quad (12)$$

We then find that the matrix T_m has the following structure:

$$T_m = \begin{pmatrix} 0 & & \tau_{1,1} & \dots & \tau_{1,k+1} \\ 1 & 0 & \tau_{2,1} & \dots & \tau_{2,k+1} \\ & 1 & 0 & \dots & \tau_{3,k+1} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & 0 \\ & & & \tau_{m-1,1} & \dots & \tau_{m-1,k+1} \\ & & & 1 & \tau_{m,1} & \dots & \tau_{m,k+1} \end{pmatrix}. \quad (13)$$

It is thus seen that the matrix $\tilde{R}_m = R_m T_m$ is an upper Hessenberg matrix whose entries can be cheaply computed as follows: $\tilde{R}_{i,j} = \sum_{q=i}^{q=m} R_{i,q} \tau_{q,j-p+1}$, for $i = 1, \dots, m$, $j = p, \dots, m$ and $\tilde{R}_{i,j} = R_{i,j+1}$, for $j = 1, \dots, p-1$, $i = 1, \dots, j+1$.

In the following, we describe the Modified Gram-Schmidt implementation of the augmented Simpler GMRES algorithm:

SGMRESE

1. *Start:* Choose m , maximum size of the subspace, and k , the desired number of approximate eigenvectors. Let ϵ be the convergence stopping criteria. Let $x_0 = 0$ such that $r_0 = b$, $\rho_0 = \|r_0\|_2$, $\rho = 1$, $p = m - k$ and $w_1 = Ar_0 / \|Ar_0\|_2$, $r_0 = r_0 / \rho_0$.
2. *First cycle:* Apply the standard Simpler GMRES to produce Y_m , W_m , R_m , r_m , x_m , $\tilde{w}$. Let $r_0 = r_m$ and $x_0 = x_m$. If $\|r_m\|_2$ is satisfied then stop, else $r_0 = r_0 / \|r_0\|$.
3. *Form approximate eigenvectors:* Compute $T_m = (W_m^T Y_m)^{-1}$ and $\tilde{R}_m = R_m T_m$ in order to solve $\tilde{R}_m g_j = \lambda_j g_j$, for appropriate g_j . Form $u_j = Y_m(T_m g_j)$ and $\tilde{u}_j = W_m(\tilde{R}_m g_j)$ for $j = 1 : k$.
4. *Other cycles:* Apply *AugmentedSArnoldi* with $\tilde{U}_k$, U_k , r_0 , A , k and m as inputs to generate Y_m , W_m , R_m . Find $\tilde{w} = [\xi_1, \xi_2, \dots, \xi_m]^T$ and $r_m = r_0 - W_m \tilde{w}$ during the Simpler Arnoldi orthogonalization, that is, at each i^{th} orthogonalization step we find $\xi_i = w_i^T r_{i-1}$ and $r_i = r_{i-1} - \xi_i w_i$. We also update ρ using $\rho = \rho \sin(\cos^{-1}(\xi_i) / \rho)$ and if $\rho \rho_0 \leq \epsilon$ we go to 5.
5. *Form the approximate solution:* $\hat{y} = R_m^{-1} \tilde{w}$ and $x_m = x_0 + \rho_0 Y_m \hat{y} = x_0 + \rho_0 z_m$ where z_m is given by (7).
6. *Restart:* If $\|r_m\|_2 \leq \epsilon$ is satisfied then stop, else $x_0 = x_m$, $\rho_0 = \|r_m\|_2$ and $r_0 = r_m / \rho_0$. Set $\rho = 1$ and go to 3.

2.1 Implementation Using Householder Reflectors

In order to maintain orthogonality of the vectors to a high accuracy, it is preferable to use Householder orthogonalizations. Let Q_j denote the Householder reflector that transforms only the j^{th} through n^{th} entries of the vectors on which

it acts. Suppose $Q_1, Q_2, \dots, Q_i$ are such that

$$\begin{pmatrix} R_i \\ 0 \end{pmatrix} = \begin{cases} Q_i A r_0, & \text{if } i = 1, \\ Q_i \cdots Q_1 A(r_0, w_1, \dots, w_{i-1}), & \text{if } 1 < i \leq p, \\ Q_i \cdots Q_1 A(r_0, w_1, \dots, w_{p-1}, u_1, \dots, u_{i-p}), & \text{if } p < i \leq m. \end{cases} \quad (14)$$

Since R_i is invertible and

$$\begin{cases} A r_0 = w_1 R_1, & \text{if } i = 1, \\ A(r_0, w_1, \dots, w_{i-1}) = (w_1, \dots, w_i) R_i, & \text{if } 1 < i \leq p, \\ A(r_0, w_1, \dots, w_{p-1}, u_1, \dots, u_{i-p}) = (w_1, \dots, w_i) R_i, & \text{if } p < i \leq m. \end{cases} \quad (15)$$

Then,

$$Q_i \cdots Q_1(w_1, \dots, w_i) = \begin{pmatrix} I_i \\ 0 \end{pmatrix}, \quad I_i \in \mathbb{R}^{i \times i}. \quad (16)$$

and $w_j = Q_1 \cdots Q_i e_j = Q_1 \cdots Q_j e_j$ for $j = 1, \dots, i$ where e_j is the j^{th} column of identity matrix of order n . Letting $\hat{y}_i = (\eta_1, \dots, \eta_i)^T$, and using (6) and (16), we have

$$z_i = \begin{cases} Q_1(\eta_1 Q_1 r_0), & \text{if } i = 1, \\ Q_1 \cdots Q_i [\eta_1 r_i + (\eta_2, \dots, \eta_i, 0, \dots)^T], & \text{if } 1 < i \leq p, \\ Q_1 \cdots Q_i [\eta_1 r_i + (\eta_2, \dots, \eta_p, 0, \dots)^T] + \sum_{j=1}^{i-p} \eta_{p+j} u_j, & \text{if } p < i \leq m. \end{cases} \quad (17)$$

Since $(Q_i \cdots Q_1 r_i)^T e_j = (Q_i \cdots Q_1 r_i)^T (Q_i \cdots Q_1 w_j) = r_i^T w_j = 0$ for $j = 1, \dots, i$, it follows from (4) and (16) that $\xi_i = e_i^T Q_i \cdots Q_1 r_0$, and

$$r_i = Q_1 \cdots Q_i \begin{pmatrix} 0 \\ I_{n-i} \end{pmatrix} r_{i-1}, \quad I_{n-i} \in \mathbb{R}^{(n-i) \times (n-i)}. \quad (18)$$

The following algorithm describes orthogonalization using Householder transformations.

AugmentedSHArnoldi

1. *Start:* Input U_k , k approximate eigenvectors, $\tilde{U}_k = AU_k$, m and k . Let $p = m - k$; $w_1 = Ar_0$;
Find Q_1 such that $Q_1 w_1 = [R_{1,1}, 0, \dots, 0]^T$;
 $w_1 = Q_1 e_1$;
2. *Iterate:* For $i = 2 : m$
If $i \leq p$, $w_i = Q_{i-1} \cdots Q_1 (Aw_{i-1})$;
else, $w_i = Q_{i-1} \cdots Q_1 \tilde{u}_{i-p}$;
Find Q_i such that $Q_i w_i = [R_{1,i}, \dots, R_{i,i}, 0, \dots, 0]^T$;
 $w_i = Q_1 \cdots Q_i e_i$;
3. *Output:* If $k = 0$, $Y_m = [r_0 \ W_{m-1}]$; else, $Y_m = [r_0 \ W_{p-1} \ U]$.

For the augmented Simpler GMRES algorithm using Householder reflections, we give below the steps that are different from the Gram-Schmidt version of the algorithm.

SHGMRESE

2. *First cycle:* Apply standard Simpler Householder GMRES.
4. *Other cycles:* Apply *AugmentedSHArnoldi* with $\tilde{U}_k$, U_k , r_0 , A , k and m as inputs to generate Y_m , W_m , R_m . Find $r_i = Q_i r_i$ and $\xi_i = e_i^T r_i$ at each i^{th} orthogonalization step. We also update ρ using $\rho = \rho \sin(\cos^{-1}(\xi_i)/\rho)$ and if $\rho\rho_0 \leq \epsilon$, we go to 5.
5. *Form the approximate solution:* $\hat{y} = R_m^{-1}\tilde{w}$ and $\tilde{x}_m = x_0 + \rho_0 Y_m \hat{y} = x_0 + \rho_0 z_m$ where z_m is given by (17).

3 Numerical Experiments

We describe the results of numerical experiments using Matlab. We compare restarted Simpler GMRES, SGMRES(m) with the two augmented versions of the algorithm, namely, SGMRESE(p , k), the implementation using Gram-Schmidt orthogonalization and SHGMRESE(p , k), the Householder version. We also give the results for GMRES augmented with eigenvectors, GMRESE(p , k) [3]. For each method, we compare the relative residual norm against the number of matrix-vector products (matvecs) and amount of work required (flops). We use $p = 21$ Krylov vectors and $k = 4$ harmonic Ritz vectors, and therefore $m = 25$. In all experiments, the initial guess is the zero vector.

Experiment 1. We consider the first four test problems considered by Morgan [3]. The matrices are bidiagonal with 0.1 in each superdiagonal position. In Problem 1, the matrix has entries 1, 2, 3, $\dots$, 999, 1000 on the main diagonal. For Problem 2, the diagonal entries are 0.01, 0.02, 0.03, 0.04, 10, 11, $\dots$, 1004, 1005, for Problem 3, we have -2 , -1 , 1, 2, $\dots$, 997, 998 on the main diagonal and for Problem 4, the diagonal entries are 1, 1.01, 1.02, 1.03, 1.04, 2, 3, 4, $\dots$, 995, 996. The right-hand side vector for each problem is a vector of ones.

We show in Fig. 1 the convergence history for Problem 2. This problem is difficult because of the small eigenvalues 0.01, 0.02, 0.03, 0.04. In this case, SGMRES(25) stagnates because the Krylov subspace is not large enough to develop approximations to the small eigenvalues. For roughly the same amount of work, SGMRES reaches a relative residual norm of 2.02E-2 whereas SGMRESE reaches a norm of 8.95E-11. On the other hand, we observe that the three augmented methods are convergent, each reaching a relative residual norm of 2.8E-5 after 207 matvecs. However, the work required by SGMRESE, the Gram-Schmidt implementation, is 2.29E7 which is less than that required by GMRESE, approximately 3.48E7 flops. Also, SHGMRESE requires 6.12E7 flops to reach the same relative residual norm.

Fig. 1. Convergence history for example 2 of [3]

Table 1. Results for problems 1 to 4 of [3]

Problem	Method	matvecs	$\ r_i\ _2 / \ r_0\ _2$	flops
1	SGMRES	523	9.73E-11	3.30E7
	GMRESE	239	9.38E-11	4.28E7
	SGMRESE	239	9.38E-11	2.73E7
	SHGMRESE	239	9.38E-11	7.07E7
2	SGMRES	500	2.02E-2	3.17E7
	GMRESE	280	8.95E-11	4.89E7
	SGMRESE	280	8.95E-11	3.22E7
	SHGMRESE	280	8.95E-11	8.54E7
3	SGMRES	500	8.89E-3	3.17E7
	GMRESE	399	9.64E-11	6.92E7
	SGMRESE	399	9.64E-11	4.55E7
	SHGMRESE	399	9.64E-11	1.18E8
4	SGMRES	511	9.86E-11	3.40E7
	GMRESE	425	9.78E-11	7.66E7
	SGMRESE	425	9.78E-11	4.95E7
	SHGMRESE	425	9.78E-11	1.32E8

Table 1 shows the results for the four test problems. For Problem 1, we note that SGMRES requires 523 matrix-vector products and a work of 3.30E7 to reach a relative residual norm of 9.73E-11 whereas the augmented algorithm SGMRESE needs 239 matrix-vector products and only 2.73E7 flops to reach a norm of 9.38E-11. Also observe that SGMRESE performs better than GMRESE.

The Householder version of augmented SGMRES requires more work because of the higher orthogonalization costs. The matrix in Problem 3 is indefinite and the results show that after 500 matrix-vector products, the relative residual norm reached by SGMRES is 8.89E-3 whereas SGMRESE and GMRESE reach a residual norm of 9.64E-11 after 399 matrix-vector products. However, note that the work required by GMRESE is approximately 50% more than the work required by SGMRESE. As mentioned by Morgan [3], Problem 4 is designed to be difficult for augmented methods. The results seem to confirm this as the augmented version of Simpler GMRES is not better than SGMRES. We note that SGMRESE takes a greater amount of work than SGMRES to reach approximately the same relative residual norm of 9.8E-11.

For Problem 3, we show in Table 2 the residual norm reached and the amount of flops required by SGMRESE and GMRESE after 5, 10, 15 and 20 runs. Note that a run is the iteration between two successive restarts. Both GMRESE and SGMRESE, require the same number of matrix-vector products for the different runs. However we remark that in each case, the amount of work required by SGMRESE is less than the work required by GMRESE to reach the same residual norm.

Table 2. Residual norm and flops after different runs for Problem 3

	After 5 runs		After 10 runs		After 15 runs		After 20 runs	
Method	$\ r_i\ _2$	flops	$\ r_i\ _2$	flops	$\ r_i\ _2$	flops	$\ r_i\ _2$	flops
SGMRESE	0.22	1.14E7	8.3E-5	2.38E7	2.1E-8	3.62E7	5.4E-12	4.86E7
GMRESE	0.22	1.74E7	8.3E-5	3.65E7	2.1E-8	5.55E7	5.4E-12	7.46E7

Similar results for Problem 2 show that, after 12 runs, GMRESE requires approximately 4.32E7 flops to reach a residual norm of 1.69E-7, whereas SGMRESE requires 2.88E7 flops to reach the same norm.

Experiment 2. We consider matrices, SHERMAN4, SHERMAN1 and SAYLR3 from the Harwell-Boeing sparse matrix collection. SHERMAN1 is nonsymmetric of dimension 1000 and has 3750 nonzero entries, SHERMAN4 is nonsymmetric, has dimension 1104 and 3786 nonzero entries and SAYLR3 is a nonsymmetric matrix, has dimension 1000 and 375 nonzero entries. The convergence history for SHERMAN4 in Fig. 3 shows that SGMRESE performs better than the other methods. In Table 3, we show the number of matrix-vector products, the relative residual norm and the amount of work required for the three matrices. For SHERMAN1, we observe that Simpler GMRES requires 5032 matrix-vector products to reach a relative residual norm of 9.99E-11 whereas the augmented version SGMRESE requires only 1304 matrix-vector products to reach approximately the same norm. Comparing the amount of work for the two methods, we

observe that SGMRESE requires a lesser amount. For all the three test problems, we observe that SGMRESE performs better than GMRESE.

Fig. 2. Convergence for the SHERMAN4

Table 3. Results for Harwell-Boeing matrices

Matrix	Method	matvecs	$\ r_i\ _2/\ r_0\ _2$	flops
SHERMAN4	SGMRES	891	9.96E-11	6.48E7
	GMRESE	208	9.11E-11	3.57E7
	SGMRESE	208	9.11E-11	2.39E7
	SHGMRESE	208	9.11e-11	4.63E7
SHERMAN1	SGMRES	5032	9.99E-11	3.36E8
	GMRESE	1304	9.90E-11	2.26E8
	SGMRESE	1304	9.90E-11	1.48E8
	SHGMRESE	1304	9.90e-11	3.18E8
SAYLR3	SGMRES	3951	9.99E-11	2.64E8
	GMRESE	1142	9.98E-11	2.06E8
	SGMRESE	1142	9.98E-11	1.37E8
	SHGMRESE	1142	9.98E-11	3.48E8

We show in Table 4 the residual norm reached and the amount of work required after different runs for SHERMAN4. Observe that the amount of work required by augmented GMRES is roughly 40% more than the work required by augmented Simpler GMRES.

Table 4. Residual norm and flops after different runs for SHERMAN4

Method	After 5 runs		After 8 runs		After 10 runs	
	$\ r_i\ _2$	flops	$\ r_i\ _2$	flops	$\ r_i\ _2$	flops
SGMRESE	4.2E-3	1.21E7	7.2E-7	1.98E7	4.8E-9	2.39E7
GMRESE	4.2E-3	1.69E7	7.2E-7	2.88E7	4.8E-9	3.57E7

4 Conclusion

We have described a method to improve the convergence of Simpler GMRES for problems with small eigenvalues. Numerical tests indicate that using Gram-Schmidt orthogonalizations to implement the algorithm, significant reduction in the amount of work required for convergence can be achieved over augmented GMRES. More numerical experiments using preconditioning techniques for other problems will be reported in another work.

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