Non-manifold Decomposition in Arbitrary Dimensions

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Abstract. In this paper we consider the problem of decomposing a non-manifold n -dimensional object described by an abstract simplicial complex into an assembly of ‘more-regular’ components. Manifolds, which would be natural candidates for components, cannot be used to this aim in high dimensions because they are not decidable sets. Therefore, we define d -quasi-manifolds, a decidable superset of the class of combinatorial d -manifolds that coincides with d -manifolds in dimension less or equal than two. We first introduce the notion of d -quasi-manifold complexes, then we sketch an algorithm to decompose an arbitrary complex into an assembly of quasi-manifold components abutting at non-manifold joints. This result provides a rigorous starting point for our future work, which includes designing efficient data structures for non-manifold modeling, as well as defining a notion of measure of shape complexity of such models.

1 Introduction

A d -manifold M is any subset of the Euclidean space such that the neighborhood of any point of M is locally equivalent to a d -dimensional open ball. Geometric objects that do not fulfill such property at all points are said to be *non-manifold*. Non-manifold objects are very common in practical applications, because of their high expressive power [5,10,20,21,22,25,26,27]. However, most objects considered in the applications have a relatively small number of non-manifold joints (geometric singularities), while most parts of such objects fulfill the manifold condition. Examples are objects containing parts of different dimensionality. Valid representation of non-manifolds is a central issue in the geometric modeling of solids. Non-manifold objects are hard to control and to understand, both from a theoretical and from a practical point of view. Non-manifolds are not well understood and classified mathematically, especially in high dimensions. Non-manifold cell complexes, which can be used to represent such objects, are hard to encode and manipulate with efficient data structures. Models and data structures for representing such objects in computers have been proposed in the literature in two dimensions [13,16,22,25,26,27]. Moreover, most such models and data structures do not scale well with the degree of “non-manifoldness” (i.e., the number of geometric singularities) of the model, thus resulting often verbose and inefficient.

A possible approach in addressing the representation of non-manifold objects consists of decomposing them into simpler parts. So far, this approach has been pursued in the literature in two dimensions for non-manifold objects by using *pseudo-manifold* components [10,21]. A decomposition into manifold components, which seems a natural choice, is actually possible only in two dimensions, but it is not feasible in the general case, since the class of manifolds is not decidable in higher dimensions.

The purpose of our paper is to define a mathematically founded decomposition of non-manifold objects into components that belong to a well-understood class, that we call *d-quasi-manifolds*, which is a decidable proper superset of *d*-manifolds for $d \geq 3$, while it coincides with the class of *d*-manifolds for $d \leq 2$. Our final goal is to use such a decomposition as a basis for designing efficient data structures to represent non-manifold objects in any dimension, as well as defining a measure of structural complexity of singularities. Our approach allows also a better understanding of the combinatorial structure of non-manifolds and represents a contribution in the direction of topology-based geometric modeling [17].

The remaining of this paper is organized as follows. In Section 2, we review related work, from geometric modeling and from combinatorial topology. In Section 3, we introduce basic definitions from combinatorial topology. In Section 4, we develop the notion of quasi-manifold, recall the classic definition of combinatorial manifolds, and compare these two class of objects. In Section 5, we define a decomposition procedure that will split a non-manifold complex into an assembly of quasi-manifold objects. Finally, in Section 6, we draw some conclusions, and discuss future work.

2 Related Work

Work related to the results presented in this paper can be found in the literature on geometric modeling and on combinatorial topology. In geometric modeling, the main interest about non-manifold objects is to develop modeling schemes and related data structures for representing and manipulating such objects. Since CAD applications has motivated most such research, most contributions are concerned with the description of the boundary of solid objects. A common assumption is to consider only those objects that can be described by a general geometric cell complex formed by cells homeomorphic to closed balls. Some proposals also exist that consider objects with open boundaries [16,22]. In the context of this paper we will stick to the above assumption.

A possible approach to the representation of such objects is to encode general cell complexes directly. Existing data structures developed under this approach (e.g., the *incidence graphs* [7,28], the *Radial Edge* [27] and the *Tri-cyclic Cusps* [13]) may be quite verbose and do not scale well with the degree of “non-manifoldness” of the modeled object. In other words, such data structures have been designed by assuming to have a relatively high number of non-manifold situations, and, thus, they are less computationally efficient and compact, when

applied to manifolds, then data structures developed for boundary representation of manifold solids [2,12,18].

The *Facet-Edge* [6] data structure, even if developed for subdivided manifolds, can model some special cases of non-manifold objects, but it is not clear how to extend it to deal with more general situations. However, the representation domain of the Facet-Edge suggested the notion of *regularly adjacent complexes* that we introduce in Section 3.

The *n-G-maps* proposed in [16] are defined in arbitrary dimensions, and they can describe a subclass of pseudo-manifolds (hence, of non-manifolds). The *quasi-manifolds*, that we will introduce in Section 5, can be modeled with n-G-maps, while the reverse is not true, since n-G-maps can model open cells as well. A representation based on *chains* of n-G-Maps [8] can be used to represent arbitrary cell complexes with a mix of open and closed cells.

An alternative approach to modeling consists of decomposing a non-manifold object into simpler and more manageable parts. There are a few proposals that follow this approach, which are limited to the two-dimensional case. In [10, 21], decompositions of two-dimensional r-sets (i.e., uniformly two-dimensional objects) into pseudomanifolds are presented. In [11], the idea of cutting a two-dimensional non-manifold complex into manifold pieces, and to use a two-level representation to stitch them together is exploited in order to develop a geometric compression algorithm.

In order to pursue an approach based on decomposition, a crucial point is to understand the possible structure of the components. Combinatorial topology establishes limits on what can be done, and motivates the fact that d -manifolds cannot be used as building blocks, in general. In order to use objects of a given class as building blocks, we must be able to *decide* whether an object belongs to this class or not. Following the classical main stream from point set to combinatorial topology (see, e.g., [15] for a survey), we restrict our attention to *combinatorial manifolds* (see Section 5). The set of combinatorial d -manifolds is decidable if and only if the $(d - 1)$ -spheres can be recognized. However, while a decision procedure for the recognition of the 3-sphere is known [24], this problem is open for $d = 4$, and d -spheres are not recognizable for $d \geq 5$ (see [14,19] and [23] Chapter 9). Therefore, manifolds are not decidable for $d \geq 6$. For this reason, we will use building blocks from a superclass of combinatorial manifolds, which we will define in Section 4. Such class of objects proves to be decidable in all dimensions, and to coincide with the class of manifolds for $d \leq 2$.

3 Background Notions

In this section, we introduce *abstract simplicial complexes* and their basic related notions [9]. We will use geometric complexes just as examples and completely avoid all definitions from point set topology.

Abstract Simplicial Complexes

Let V be a finite set of points that we call *vertices*. An *abstract simplicial complex* on V is a subset Ω of the set of (non empty) parts of V such that $\{v\} \in \Omega$ for

every point $v \in V$, and, if $\gamma \subset V$ is an element of Ω , then every subset of γ is also an element of Ω . Each element of Ω is called an *abstract simplex*.

We associate an integer $\dim(\gamma)$ with each abstract simplex $\gamma \in \Omega$, called *dimension of γ* , defined by $\dim(\gamma) = |\gamma| - 1$, where $|\gamma|$ is the number of vertices in γ . Whenever no ambiguity arises, we will use the term *cell* or *simplex* to denote an *abstract simplex*. We will use also the term *complex* or *abstract complex* to denote an abstract *simplicial complex*. A cell of dimension s is called an s -cell. A complex Ω is called d -dimensional or a d -complex if $\max_{\gamma \in \Omega}(\dim(\gamma)) = d$; each d -cell of a d -complex Ω is called a *maximal cell* of Ω .

Boundary, Star, Link, Subcomplexes, and Closures

The *boundary* $\partial\gamma$ of a cell γ is defined to be the set of all proper parts of γ . Cells ξ in $\partial\gamma$ are called *faces* of γ . Similarly, the *co-boundary*, or *star*, of a cell γ is defined as $\star\gamma = \{\xi \in \Omega \mid \gamma \subset \xi\}$. Cells ξ in $\star\gamma$ are called *co-faces* of γ . Any cell γ such that $\star\gamma = \{\gamma\}$ is called a *top cell* of Ω . Two distinct cells are said to be *incident* if and only if one of them is a face of the other. Otherwise, they are called *disjoint* cells. Both star and boundary are defined by referring to the set of cells in Ω . We can emphasize this by using ∂_Ω , instead of ∂ and using $\star_\Omega$, instead of $\star$.

The above definition can be easily extended to a set of cells. Thus if Θ is a set of cells, we will have that $\star\Theta = \bigcup_{\gamma \in \Theta} \star\gamma$. The *link* of a set of cells Θ (denoted by $lk(\Theta)$) is the set of all faces of co-faces of cells in Θ , that are not incident to cells in Θ .

A subset of cells $\Omega' \subset \Omega$ which has the property to be a complex is called a *sub-complex* Ω' of Ω . A subset Θ of Ω is *open* (resp. *closed*) (in Ω) if, for every cell γ of Θ , all cells of the co-boundary (resp. boundary) of γ in Ω are also cells of Θ . Note that a closed subset of Ω is a sub-complex of Ω . Given a set of cells Θ , the closure $\overline{\Theta}$ of Θ will be the smallest sub-complex whose cells include those in Θ . Note that the star of an arbitrary set of cells might not be a sub-complex while the link of a set of cells is always a sub-complex. With this definitions, we can write that, for a generic set of cells Θ , $lk(\Theta) = \overline{\star\Theta} - \star\Theta$. Similarly we can emphasize the dependence of the link on the complex Ω by writing $lk_\Omega(\Theta) = \overline{\star_\Omega\Theta} - \star_\Omega\Theta$.

Carrier and Embedding

If cells of a complex Ω are subsets of a the same topological space $\mathcal{E}$, we define the *carrier* $\Delta(\Omega)$ as the topological subspace defined by the union $\bigcup_{\gamma \in \Omega} \gamma$. We will say that $\Delta(\Omega)$ is an *embedding* of Ω into the topological space $\mathcal{E}$. The carrier of an abstract simplicial complex is usually called a *polyhedron*.

Adjacency, Paths, and Connected Complexes

Two cells are called s -adjacent if they share a s -face. In particular, two p -cells are said to be *adjacent* if they are $(p-1)$ -adjacent. A sequence of elements $(\gamma_i)_{i=0}^k$ in a subset Ω' of a complex Ω is called a *path* in Ω' from γ_0 to γ_k if for every two consecutive cells $\gamma_{i-1}, \gamma_i, i = 1, \dots, k$ we have $\gamma_{i-1} \in \partial\gamma_i$ or $\gamma_{i-1} \in \star\gamma_i$. A subset Ω' of a complex Ω is called *connected* if it contains at least a path for any pair

of its points. Equivalence classes of this connectivity relation are sub-complexes called *connected components* of Ω .

Combinatorial Equivalence

Given a d -cell γ in Ω and a point $w \notin \gamma$, we define the *cone* from a vertex w to γ as the cell $\{w\} \cup \gamma$. Since $w \notin \gamma$, the cone is a $(d+1)$ -cell.

An abstract simplicial complex Ω' is a *starring subdivision* of another abstract simplicial complex Ω if Ω' is obtained from Ω by deleting a cell γ (which is not a vertex) and by adding a vertex w plus all the cones from w to all the faces, that are different from γ , of cells in $\star\gamma$.

An abstract simplicial complex Ω' is a *subdivision* of another abstract simplicial complex Ω if Ω' is obtained from Ω by a sequence of starring subdivisions.

Two abstract simplicial complexes will be *combinatorial equivalent*, or *stellar equivalent*, if and only if they have two isomorphic subdivision.

4 Quasi-Manifolds

In this section, we introduce the new concept of *quasi-manifolds*, and discuss the relation between quasi-manifolds and combinatorial manifolds. To this aim, we need to introduce some other definitions.

Definition 1. A d -complex in which all top cells are maximal is called a *regular*, or a *uniformly d -dimensional complex*.

Let γ be a $(d-1)$ -cell in a regular d -complex Ω . We say that γ is a *manifold cell* if and only if there exist at most two d -cells incident at γ . Such incidence is called *manifold adjacency*.

Similarly, we say that γ is a *non-manifold cell* in Ω if and only if there exist at least three d -cells incident at γ .

Let Ω_b^{d-1} be the collection of all $(d-1)$ -cells having only one incident d -cell. The sub-complex $\partial\Omega = \overline{\Omega_b^{d-1}}$ is called the *boundary* of Ω , while the collection of the cells in $\Omega - \partial\Omega$ is called the *set of internal cells* of Ω .

Definition 2. A regular complex which has only manifold cells is called a *combinatorial pseudo-manifold* (with possible boundary).

For a given regular complex Ω , we call the equivalence classes induced by the reflexive and transitive closure of the manifold adjacency relation the *pseudo-manifold components* in Ω . Note that in the literature, d -pseudo-manifolds are required to be connected via manifold adjacency [1]. In definition 2 we do not need this fact. We will mention the connectivity separately when it is necessary.

Figure 1(a) shows an example of a two dimensional pseudo-manifold since it is uniformly two-dimensional and for every 1-simplex there are at most two 2-simplexes incident in it. The complex in Figure 1(b) is the closure of a pseudo-manifold component but it is not a pseudo-manifold. This is due to the fact that segment nm is the face of three triangles.

Fig. 1. (a) An example of a 2-pseudo-manifold; (b) the closure of a 2-pseudo-manifold component that is not a pseudo-manifold.

Using the fact that, for any vertex v , the map that associate with each cell in $\star v - \{v\}$ its face in $lk(v)$ is bijective, the following properties can be easily proven:

- Proposition 1** – *A d -complex, with $d > 0$, is a regular complex if and only if the link of each vertex is a regular $(d - 1)$ -complex. All 0-complexes are regular.*
- *A d -complex, with $d > 1$, is a pseudo-manifold if and only if the link of each vertex is a $(d - 1)$ -pseudo-manifold. For $d = 1$, only regular 1-complexes (i.e. graphs) with at most two 1-simplexes incident at every vertex are pseudo-manifolds.*

We define *regularly adjacent complexes* inductively as follows.

Definition 3. *A regular abstract simplicial 1-complex is a regularly adjacent complex. A regular abstract simplicial d -complex is regularly adjacent if and only if the link of each vertex is a connected regularly adjacent $(d - 1)$ -complex.*

Vertices for which the link condition in the definition above is not satisfied are called *singular vertices*. Complex in Figure 1(a) is a pseudo-manifold which is not regularly adjacent. The link of vertex l is not connected since it is represented by the two thick disconnected 1-simplexes $\{o, p\}, \{q, r\}$. For this reason, we can say that vertex l is a singular vertex. Complex in (b) is regularly adjacent but not a pseudo-manifold. The link of each point is a connected graph. Vertices n and m in Figure 1(b), are singular vertices because they violate link condition for pseudo-manifoldness. The complex in figure 2(a) is a 3-pseudo-manifold but not regularly adjacent. This is due to the fact that the link of vertex p is represented by the strip of triangles pinched at q , which is not regularly adjacent since it has the same property of the complex in Figure 1(a). Complex in Figure 2(b) is a regularly adjacent 3-pseudo-manifold. Therefore, pseudo-manifoldness and regular adjacency are two independent requirements for the regularity of a complex.

Definition 4. *We say that a complex is a quasi-manifold if and only if it is both a pseudo-manifold and a regularly adjacent complex.*

Fig. 2. (a) An example of a 3-pseudo-manifold that is not a regularly adjacent; (b) an example of a 3-pseudo-manifold which is a regularly adjacent

The complex in Figure 2(b) is an example of quasi-manifolds. It is easy to reformulate this definition of quasi-manifolds in term of *link conditions*. This yields to the following inductive characterization of quasi-manifolds.

Proposition 2 *A d -complex is a quasi-manifold if and only if, for $d > 1$, the link of each vertex is a connected $(d - 1)$ -quasi-manifold. Only regular 1-pseudo-manifolds are quasi-manifolds.*

A similar characterization can be given for d -quasi-manifolds with boundary by just considering pseudo-manifolds with boundary. We now introduce the definition of combinatorial manifolds and we discuss their relation with quasi-manifolds.

Definition 5. *A combinatorial d -pseudo-manifold Ω (with possible boundary) is called a combinatorial d -manifold if and only if the link of every vertex v is a $(d - 1)$ -complex that is combinatorially equivalent either to the boundary of the standard d -dimensional simplex, if v is an internal vertex, or to the standard $(d - 1)$ -simplex if v is a boundary vertex.*

The following property, that can be easily proven by induction on the manifold dimension d , gives the relation between manifolds and quasi-manifolds.

Proposition 3 *For any $d \geq 0$, every combinatorial d -manifold is a d -quasi-manifold. Conversely, every 2-quasi-manifold is a 2-manifold; however, for $d > 2$, there exist d -quasi-manifolds that are not d -manifolds.*

Figure 2(b) shows an example of a three dimensional quasi-manifold which is not a combinatorial manifold, because the link of vertex p is a equivalent to a 2-cylinder (and, hence, is not equivalent to the boundary of a sphere or to a disc).

5 Decomposition of a Non-manifold Complex

In this section we briefly describe an algorithmic procedure to decompose a d -complex into a natural assembly of *quasi-manifolds* of dimension $h \leq d$. Details

and proofs about correctness and time complexity of the procedure are given in [4]. The decomposition procedure consists of two sub-procedures, that we will call **SplitVertices** and **CheckTies**. Such two procedures are applied iteratively to the complex until no more singularities remain. It can be shown that a single iteration of procedure **SplitVertices** alone is sufficient to decompose a complex for $d \leq 2$. For $d \geq 3$ procedure **CheckTies** is necessary to remove situations such as those depicted in Figure 4, which are not detected by **SplitVertices**. Procedure **SplitVertices** works recursively on the dimension d of the complex to decompose, on the basis of the following inductive process. A 1-complex is (isomorphic to) a graph possibly with isolated vertices. We define the decomposition of this graph as follows.

- For every vertex v of degree $o > 2$ (i.e., with a number of incident 1-simplexes > 2) we introduce o distinct copies of v , one for each edge incident into v . This will split Ω into a set of disconnected points, cycles and chains.
- If a chain of this expanded graph starts and ends on two copies of the same vertex, we convert it into a cycle by identifying such two copies.

The result of this decomposition is a graph whose connected components are either isolated points, or single cycles, or linear chains of adjacent edges. In this decomposition scheme, connected components will be no further decomposed.

For $d \geq 2$, we consider each vertex v of a d -complex Ω , and, we recursively decompose its link $lk(v)$ (which is a $(d-1)$ -complex) into an assembly of k connected $(d-1)$ -complexes. If $k > 1$, we call vertex v a *splitting* vertex. In this case, given the collection $\{l_i(v), 1 \leq i \leq k\}$ of connected components of $lk(v)$ (decomposed recursively), we will modify the input complex Ω by replacing v with k copies of it. More precisely, for every $i \in \{1, \dots, k\}$, v_i replaces v in all cells of the star of v that are incident at the cells of component $l_i(v)$. We iterate this local decomposition process until no more splitting vertices remain. It can be shown that this procedure works in time linear in the number t of top simplexes in the input complex.

The resulting decomposition is dependent on the order in which we examine vertices of the input complex. In Figure 3 we give an example of two local decompositions of the 2-complex in (a). The first decomposition (b) starts by considering vertex v , while the second decomposition (c) starts by considering vertex u . The components resulting from the two choices are different.

Fig. 3. An example of decomposition depending on the order of choice of vertices.

For d -complexes, with $d > 2$, some singularities may remain after running procedure **SplitVertices**. Such singularities are due to vertices having a link

which remains made of a single connected component, but needs to be decomposed at some of its vertices (without disconnecting it) in order to fulfill the link conditions for quasi-manifold complexes. We call such vertices *ties*. Vertices p and q in Figure 4 are examples of ties.

Note that vertices that are potential ties are detected by procedure **SplitVertices**, depending on the structure of their link components. However, such vertices are left unchanged at that stage, because some such situations can be resolved while splitting some of their neighboring vertices afterwards. Procedure **CheckTies** tests only such potential ties. If a tie v is found, then we introduce a new copy of v , as above, for the closure of each top simplex in the star of v (see Figure 4(b), for an example). Note that it is sufficient to split just vertex p to remove the tie also at q . Tie removal causes a sort of “explosion” of the complex in the neighborhood of a removed tie. This explosion may generate other singularities. Therefore, procedure **SplitVertices** is repeated only on those vertices that might be affected by tie removal, and the process is iterated until no more ties are found.

It can be shown that a single application of procedure **CheckTies** has a complexity linear in the number of cells it visits. Moreover, the total number of iterations of the two procedures is bounded from above by the number t of top simplexes in the input complex. Therefore, the total decomposition process for $d \geq 3$ is completed in time $O(t^2)$. However, we believe that this estimate is quite pessimistic. For instance, under the (reasonable) assumption that the number of cells incident in a vertex is bounded from above by a constant, it can be shown that the complexity of the whole decomposition is always $\Theta(dt)$.

Fig. 4. An example of decomposition of a tie: Complex in (a) contains two ties p and q . Tetrahedra in (b) correspond to the decomposition of tie p .

We can obtain the original complex Ω from the set of quasi-manifold components by identifying all vertex copies of each non-manifold vertex of Ω . Therefore, such a set of quasi-manifold components, together with a simplicial map that identifies all vertex copies of every non-manifold vertex, can describe completely the original complex Ω .

Fig. 5. A non-manifold 2-complex Ω and its decomposition with hyper-graph

An example of the results of the decomposition for a 2-non-manifold is depicted in Figure 5. Consider the complex Ω in Figure 5(a). Points b, i and f are the non-manifold points. The decomposed link of vertex b has five connected components: cycles (c, e, f) , (c, d, e) , (h, i, m) and (p, q, r) , arc (i, f) and isolated point a . The link of vertex i has two connected components: cycle (b, h, j, m) and arc (b, f, g) . The link of f has two connected components: cycle (b, c, e) and arc (b, i, g) . Therefore (b, i) and (i, g) are non-manifold edges. Figure 5(b) shows the manifold decomposition of the complex in Figure 5(a).

6 Concluding Remarks

Results in this paper show that any abstract simplicial complex can be decomposed into an assembly of subcomplexes. Each subcomplex belongs to the class of quasi-manifolds complexes. The resulting decomposition is natural (i.e., not arbitrary) in the sense that it is obtained by induction and depends only on the order in which the vertices are selected.

The decomposition described can be used to define a hierarchical two-level representation for non-manifolds. In a upper level we can describe, through a hypergraph, the way in which quasi-manifold components are stitched together to form the original complex. In the lower level, we describe each quasi-manifold component. This idea is already well understood and developed for 2-dimensional complexes in [3]. For d -dimensional complexes for $d > 2$ we expect to find a compact data structure to encode d -quasi-manifolds. Simplex connectivity in a d -quasi-manifold can be completely described by giving the pairs of $(d - 1)$ -adjacent d -simplexes.

Finally, we note that our decomposition operates on a local basis and, therefore, the structure of the decomposition can be understood by local analysis. This notion will be used to define a measure of the amount of shape

complexity at each non-manifold point. This will be used to guide shape simplification process which preserves the iconic content of an object (see [3] for details).

Acknowledgments. This work has been supported by the European Research Training Network “MINGLE - Multiresolution in Geometric Modelling”, contract number HPRN-CT-1999-00117, and by the National Project founded by the Italian Ministry of Instruction, University and Research (MIUR) “SPADA - Representation and processing of spatial data in Geographic Information Systems”.

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