A Simplified Recognition Algorithm of Digital Planes Pieces

Mohammed Mostefa Mesmoudi

Department of Computer Science and Information Science (DISI)

Science University of Genoa

Via Dodecaneso, 35 -16146 Genoa (Italy)

mesmoudi@disi.unige.it

Abstract. Debled proposed an efficient algorithm for the recognition of rectangular digital planes pieces. However, uses, in some cases called strongly exterior cases, some validity criteria which are only sufficient but not necessary. In this paper we give necessary and sufficient conditions (including strongly exterior cases) to recognize pieces of digital planes. We build up a simplified form of Debled's algorithm. Furthermore, our approach is independent from the rectangular form of pieces considered by Debled.

1 Introduction

Let ν be a 18-connected bounded convex subset of Z^3 which injectively projects on a subset Π of the plane Oxy . Let ω be a positive integer, if there exist integers a, b, c, μ in Z such that ν is the set of solutions of the double Diophantine inequality

$$\begin{cases} \mu \leq ax + by + cz < \mu + \omega \\ (x, y) \in \Pi \end{cases} \quad (1)$$

then we say that ν is a piece of a digital plane $P(a, b, c, \mu, \omega)$ with characteristics a, b, c and lower bound μ . The number ω is called the arithmetic thickness of the plane. When $\omega = \sup(|a|, |b|, |c|)$, the plane $P(a, b, c, \mu, \omega)$ is called a naïve plane and denoted by $P(a, b, c, \mu)$ [1]. Let us consider a family $\{\nu_t\}_{t \in [\mu, \mu + \omega - 1]}$ of real parallel planes ν_t defined by $ax + by + cz = t$. The digital plane ν is the intersection of the above family with Z^3 that can be geometrically represented by a set of voxels as shown in Figure 1. Let us suppose that the greatest common divisor of a, b, c is 1. We say that ν is recognized if it contains sufficiently points to compute all the characteristics a, b, c, μ, ω . To this aim, it is sufficient that ν possesses four affinely independent points that generate the bounding real planes ν_μ and $\nu_{\mu + \omega - 1}$. The problem of studying pieces of digital naïve planes can be reduced to the case where $0 \leq a \leq b \leq c$. Indeed, the general case can be obtained from the former case by rotations and symmetries [3]. There is a wide literature on the problem of recognizing digital plane pieces. Kim and Rosenfeld showed in [6] that a digital surface is a piece of a naïve digital plane if and only

if there exists a face of the convex hull of the surface such that the distance between the points of the surface and the plane that supports the face is less than 1. They proposed an algorithm based on this property of complexity $O(p^4)$, where p is the number of points on the surface. In 1991, Kim and Stojmenović [7] improved this algorithm to obtain another algorithm of complexity $O(p^2 \log p)$. In the same year, Stojmenović and Tošić [11] presented two other algorithms, the first with complexity $O(p \log p)$ based on the construction of two convex hulls and the second with complexity $O(p)$ and based on linear programming in 3D. These algorithms have a low complexity but they are not incremental which is a drawback in application. Furthermore, the construction of the convex hull in 3D is a delicate and expensive operation. In [13,12], Veelaert, relying on a generalization of a regularity property of digital straight lines introduced by Hung [5], developed a simple algorithm of complexity $O(p^2)$ which is satisfactory for small sets ($p \leq 100$).

Fig. 1. The shape of a possible digital plane piece

The arithmetic definition of digital planes led by relation (1) was introduced for the first time by Reveilles in 1991, [10]. In 1995, Debled proposed, in her PhD thesis [3], an efficient algorithm for the recognition of rectangular pieces of digital planes. This algorithm has more advantages than the other algorithms quoted above. It uses a simple and intuitive geometric properties of digital planes. It has a quadratic complexity and is incremental. It consists in sweeping a piece by sections that are parallel to a co-ordinate plane. At the beginning, one fixes, for instance, $y = 0$ and let x vary. In this case, Debled's algorithm tries to recognize digital straight lines until x reaches its maximum value in Π . Then, y is incremented by 1 and x varies again. In this case, Debled's algorithm tries to recognize pieces of digital planes. At each step the algorithm tries to compute the characteristics of the new plane. Three cases may occur

- If the added point $M(x, y)$ satisfies relation (1), then the same characteristics of the last recognized piece are kept.
- If the added point M satisfies one of the following relations

$$ax + by + cz = \mu - 1 \quad \text{or} \quad ax + by + cz = \mu + c$$

- where (a, b, c) is the normal vector of the last recognized piece, the point M is said to be *1-exterior* to the last piece. Debled conjectured that the new piece is recognized in a new plane. By means of other conjectures, it is possible to construct the new characteristics. All these conjectures have been checked by numerous examples. Mathematical proofs of these conjectures have been presented in [2,9,8]. The complexity of Debled's algorithm corresponding to this step is at most linear in the number of points on the piece, see [3] p.180.
- The third possibility is that M satisfies one of the following inequalities

$$ax + by + cz < \mu - 1 \quad \text{or} \quad ax + by + cz > \mu + c$$

In this case we say that M is *strongly k -exterior*, where k is equal to $\mu - (ax + by + cz)$ or $(ax + by + cz) - \mu - c + 1$ respectively. The new piece belongs to a digital plane if M is not too distant from the piece. Debled gave three validity criteria to be checked for the new piece to conclude its flatness. These criteria are sufficient but not necessary, see [3] p. 172.

In this paper we present necessary and sufficient conditions to recognize pieces of digital planes. We obtain a new algorithm that simplifies and generalizes Debled algorithm. Its complexity is at most quadratic since we exploit, for each added point, only the part of Debled's algorithm used for 1-exterior case.

Next section introduces some fundamental notions of digital planes that will be used throughout the paper. Section 3 surveys our fundamental results on 1-exterior case. Section 4 addresses the problem of recognition in strongly exterior case and two recognition theorems are presented. Section 5 describes our algorithm that simplifies Debled's one, whereas Section 6 explains it by an example.

2 Background

Let us begin by giving the definitions of some notions related to digital planes that we will use in this paper. Let $M(x, y, z)$ be a point of a digital plane $P(a, b, c, \mu)$, the quantity $r(M) = ax + by + cz$ is called the *remainder* of M with respect to P . The Diophantine inequality(1) allows us to define $P(a, b, c, \mu)$ as the set of points $(x, y, z) \in \mathbb{Z}^3$ such that $\frac{\mu - ax - by}{c} \leq z < \frac{\mu - ax - by}{c} + 1$. Since the plane projects injectively on Oxy , we can represent $P(a, b, c, \mu)$ in Oxy by level lines corresponding to values of z . We can also represent the plane $P(a, b, c, \mu)$ by the remainder of its points. In Figure 2, we combine the remainder and level lines representations to represent $P(9, 13, 21, 0)$ on plane Oxy . The real plane defined by $r(a, b, c)(M) = k$ is called the plane of *index k* . The plane of index μ is called *lower leaning plane* of P and the plane of index $\mu + c - 1$ is called *upper leaning plane* of P . We denote them by (Pi) and (Ps) respectively.

A *piece* of plane is a convex subset of voxels of a naïve digital plane. A piece of a digital plane is said to be *recognized* if it possesses four leaning points that satisfy one of the following two cases:

- Three upper (resp. lower) leaning points and one lower (resp. upper) leaning point. This configuration is referred to as CAS3.1, see figure 2(a).

- Two upper leaning points and two lower leaning points. This configuration is referred to as CAS2.2, see figure 2(b).

Fig. 2. (a) CAS3.1. The remainder and level lines representation of a piece of $P(9, 13, 21, 0)$ that contains three lower leaning points and one upper leaning point. (b) CAS2.2. The remainder and level lines representation of a piece of $P(3, 5, 22, 0)$ that contains two lower leaning points and two upper leaning points

When the added point M is k -exterior, many geometric constructions are built and key voxels are extracted. These geometric constructions depend on *leaning polygons* and their positions with respect to M . These leaning polygons are defined as follows:

- If $r(M) < \mu$, we call (PS) the convex hull of the upper leaning points in the piece and we call it the *upper leaning polygon*. We define, in this case, the *convex polygon of pivots CVP* to be the upper leaning polygon PS . In the same way, we call PI the *lower leaning polygon*. In this case, we define the *convex polygon of antipodes CVA* to be the lower leaning polygon PI . Thus, we have $CVP = PS$ and $CVA = PI$.
- If $r(M) \geq \mu + c$, the upper leaning polygon PS is called the convex polygon of antipodes CVA and the lower leaning polygon PI is called the convex polygon of pivots CVP . We have $CVA = PS$ and $CVP = PI$.

In figure 3(a) we give an example of the convex polygons of pivots and antipodes for a recognized piece in the plane $P(5, 6, 7, -1)$.

The *polygonal line of pivot vectors* L is constructed depending on the added k -exterior point and its associated convex polygon of pivots. All constructions are realized on the projections in the plane Oxy . Four cases are possible:

1. The CVP is reduced to one point. In this case, the polygonal line of pivot vectors L is reduced to this point.
2. The CVP is formed by points that are not all collinear. In this case, the polygonal line L of pivot vectors is composed by points of the CVP such that their projections in the plane Oxy are located on the part of the boundary of the convex hull of the CVP projection that disappears when the point M is added, see Figure 3(b).
3. The CVP is composed by collinear points in the plane Oxy :

Fig. 3. The point M is 1-exterior of remainder -2 . In (a) the parallelogram in bold is the convex of pivot points CVP and the dashed triangle is the convex polygon of antipodes CVA . In (b) $U \cup V$ form the polygonal line of pivot vectors L , point N is an antipode for both pivot vectors U and V , It is separating for the vector U . Point N' is an antipode for the vector U only.

- If the projection of M is collinear with the projected points of the CVP , then the polygonal line L of pivot vectors is reduced to the nearest point of the CVP to M .
- If the projection M is not collinear with the projected points of the CVP , then L is equal to all points of the CVP .

An *antipode* A of a pivot vector V associated to a k -exterior point M is a summit of the CVA that has the maximal distance, among all points of the CVA , from the line directed by V and containing M . Furthermore, if the end points of V are separated by the line MA , then A is called a *separating antipode*.

3 The 1-Exterior Case

Let S be a recognized piece of the plane $P(a, b, c, \mu)$ and M an added point to S . If the remainder of M with respect to $P(a, b, c, \mu)$ is between μ and $\mu + c - 1$, then $S' = S \cup \{M\}$ is still recognized in the same plane. If the point M is 1-exterior, Debled's conjectures [3,4] assure the existence of a new plane in which S' is recognized. Furthermore, following the shape of the polygonal line of pivot vectors, these conjectures give methods to construct the new characteristics $A \leq B \leq C$ such that C is the smallest for all possible planes that contain S' . These conjectures have been mathematically proven in recent works [2,8,9].

In fact, we prove in [8] the following key theorem that assures the existence of planes that contain S' . We gave also a method to construct the smallest characteristics.

Theorem([8]). *Let S be a recognized piece of the plane $P(a, b, c, \mu)$ and $M(x_0, y_0, z_0) \in \mathbb{Z}^3$ be a point such that $S' = S \cup \{M\}$ is convex. If M is 1-exterior to S , then there exists $(A, B, C, \mu') \in \mathbb{Z}^4$ with $A \wedge B \wedge C = 1$ such that $S' = S \cup \{M\}$ is a piece of the plane $P(A, B, C, \mu')$.*

The proof of this theorem provides two necessary and sufficient conditions on the

choice of points that allow the construction of the new base. These conditions are:

$$\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0) = c, \quad (2)$$

where (x_2, y_2) , (x_1, y_1) and (α_2, β_2) are respectively the projections on Oxy of a pivot point M_2 , an antipode M_1 and a vector V_2 formed by M_2 and another pivot point.

$$d_m - (\mu + c - 1) + \frac{h_2}{H} \leq \frac{\epsilon_m h_m}{H} \leq d_m - (\mu - 1) \quad \forall m \in S, \quad (3)$$

where ϵ_m is the sign of $\beta_2(x - x_0) - \alpha_2(y - y_0)$, the numbers H, h_2 et h_m are the Euclidean distances between the points M_1, M_2 and m respectively and the real line directed by V_2 and containing M .

4 The Strongly Exterior Case

Let us now suppose that M is strongly exterior to the piece $S \subset P(a, b, c, \mu)$. This implies that there exists an integer $P \geq 2$ such that M satisfies one of the following equalities: $ax_0 + by_0 + cz_0 = \mu - P$ (i.e, M is located under the piece S) or $ax_0 + by_0 + cz_0 = \mu + c - 1 + P$ (i.e, M is located over the piece S). An axial rotation of the plane by π reverses the positions of its points. Points located under the plane get over it and vice versa. The characteristics of the plane remain unchanged. This fact allows us to reduce our study to one case, say M being located under the plane. All relations we will obtain are still valid in the case where M is located over the plane. The following lemma is a direct consequence of convexity.

Lemma 1. *Let S be a piece of a discrete plane $P(a, b, c, \mu)$ and $M(x_0, y_0, z_0)$ a point such that $S' = S \cup \{M\}$ is convex. If M is strongly P -exterior then $P \leq c$.*

Proof. Suppose that the remainder of M is $r(M) = \mu - P$ and let consider the point $M'(x_0, y_0, z_0 + 1)$. The remainder of M' is $r(M') = \mu - P + c$. If $r(M') < \mu$, then M is disconnected from S and S' is not convex. Hence, $r(M') \geq \mu$ which gives $P \leq c$.

4.1 Recognition Theorems

The piece S is bounded by two real leaning planes of indices μ and $\mu + c - 1$. When the added point M is exterior to S , located for instance under S , then, in order to geometrically contain $S' = S \cup \{M\}$ in another digital plane, one should lean the plane of index μ in a suitable direction until M is reached. This latter plane becomes the lower leaning plane of the "eventual" digital plane we are looking for. Then let us consider another plane located over S , tangent to S and parallel to the new lower leaning plane. The distance between these two planes should be less than or equal to 1; otherwise we don't get a digital plane.

The operation of leaning planes can be done by revolving on some parallel straight lines all planes of indices between μ and $\mu + c - 1$. If the line, on which pivots the plane of index $\mu + c - 1$, is too distant from M , then we may lose some upper leaning points (i.e., points of *CVP*) located between this line and M after the rotation, see Figure 4(a). Thus, this line should be chosen as close as possible to the point M . In the same way, the rotation of the lower leaning plane should not exclude the lower leaning points. Therefore, the line on which we pivot this plane should be chosen as far as possible from M so that all points of the *CVA* will be located between this line and M , see Figure 4(b). This explains the importance of polygonal lines of pivots and antipodes defined above.

Fig. 4. Profile view: Axes of rotation are shown by $(\odot)$, the result is delimited by the dashed lines. In the 3D-space points, as M, M''_2 , are not piled up. In (a) the rotation excludes leaning points M_1, M_2 and eventual internal points as M''_1 and M''_2 . In (b) the rotation does not exclude any point of the piece. In (c) the rotation R_1 excludes point M_0 , while rotation R'_1 includes all points of S , its axis contains M'_1 . Leaning point M_4 becomes an internal point.

After this operation M become naturally a lower leaning point of the new digital plane that contains S' . In practice, this operation does not always give a recognized piece of a digital plane. This is due to the fact that when M is too distant from S , the rotation of the above upper (resp. lower) leaning plane may excludes some points (i.e., their remainders are beyond the bounds given by relation 1). To palliate to this problem one can choose the lines, on which the leaning planes pivot, outside the piece. But the result we obtain is that either the new digital plane is not naïve (the distance between the resulting leaning planes is greater than 1) or the piece is not recognized (the number of the new leaning points in the piece is strictly inferior to 4). Thus, to get a satisfactory solution we have to carefully handle the operation of leaning planes. We proceed as follows:

We rotate the upper leaning plane, by a suitable angle, as explained bellow on the nearest line L_p to M . We rotate the lower leaning plane on the most far line L_a from M which is parallel to L_p (in the sense that it is directed by the same vector of L_p) and contains a leaning lower point in S . These lines strongly depend on the polygonal line of pivots L and the polygonal line of antipodes L' . Since each of L and L' contains no more than two independent vectors [3,9] then L_p and L_a are chosen in one of the following configurations:

1. If the polygonal line of pivots L is not reduced to a point and if there exists a separating antipode M_1 , then L_p is the line containing vector V_2 of L which

is separated by M_1 . We take L_a as the line directed by V_2 and containing M_1 .

2. If L is reduced to one point, then, since S is recognized, four leaning points exist that define the normal vector of S and its thickness. These points define at least two independent vectors. Since the leaning points on L are collinear, then other leaning points (antipodes) exist on the CVA that define a transverse direction to L . Among antipodes on this transverse direction we take the line L_a which contains the most far antipode from M . Then we define line L_p to be parallel to L_a and to contain the nearest pivot point to M (which is, in this case, equal to L).
3. If L is not reduced to a point and there is no separating antipode, then we proceed in the same way as in the previous case to define L_a and L_p .

The angle of rotation is chosen such that

- (i) if M is reached without excluding any point of S (i.e., all points of S still satisfy a relation of type (1) after the rotation), then this rotation, say R_1 , gives a new naïve digital plane in which the piece S' is recognized.
- (ii) if the previous rotation excludes a point only on L_p , since all possible points on L_p should have the same remainder, then all these points are excluded. Thus M is too distant. To keep the points of L_p in the new plane we have to stop rotation R_1 before reaching M . In this case, L_p and L_a are still the pivots and antipodes lines of the new plane and there is no way to reach M by a rotation. We note that, in this case, M becomes less strongly exterior but never 1-exterior because in this later case the piece becomes recognizable which is impossible.
- (iii) if the previous rotation R_1 excludes a point M_2 of L_p (or L_a) after having reached a point M_0 of remainder $\mu + c - 2$ (or $\mu + 1$), then we reduce the angle of rotation until M_0 is reached. Let R'_1 be the corresponding rotation. Algorithmically, this can be done by moving line L_a to the next line L'_a that is parallel to L_a , contains antipodes, and its distance to M is bigger than L_a 's one, see Figure 4(c).

The leaning points of L_a and L_p always remain leaning points of the new digital plane, the other leaning points become internal points in general, and M_0 becomes a new leaning point. Thus, we get enough leaning points so that the piece S still be recognized in the new digital plane.

- If point M becomes 1-exterior for the new characteristics, then Debled algorithm allows us to recognize the piece S' in another digital plane.
  - If point M remains strongly exterior, then we have to apply a new rotation. But in this case, point M_2 will be excluded and we are in situation (ii). Therefore, piece $S \cup \{M\}$ can not be recognized in a digital plane.
- (iv) If rotation R_1 excludes a point M_0 of remainder $\mu + c - 2$ (or $\mu + 1$) and points of L_p and L_a remain included by R_1 then we have to stop the rotation before excluding M_0 . This corresponds to an extremal position and piece $S \cup \{M\}$ can not be recognized.

All these cases imply that strongly exterior cases can be recognized only through 1-exterior cases. This proves our first theorem of recognition. However, before doing any rotation, there is an analytic way to check what kind of situation we may have. This analytic way is given by condition 5 in the second theorem of recognition.

Theorem 1. *Let S be a recognized piece of a digital plane $P(a, b, c, \mu)$ and $M \in Z^3$ be a point such that $S' = S \cup \{M\}$ is convex. Suppose that M is strongly P -exterior ($P \geq 2$). Then there exists at most one digital naïve plane in which S is recognized and admits M as a strongly q -exterior for some integer $2 \leq q < P$. When a such number q exists, then S' cannot be recognized in a naïve digital plane.*

Theorem 2. *Let S be a recognized piece of a digital plane $P(a, b, c, \mu)$ and $M(x_0, y_0, z_0) \in Z^3$ be a point such that $S' = S \cup \{M\}$ is convex. Suppose that M is strongly P -exterior ($P \geq 2$). Let $M_2(x_2, y_2, z_2)$ be a point on the polygonal line of pivots and $V_2(\alpha_2, \beta_2, \gamma_2)$ a vector based on M_2 and located either on the polygonal line of pivots if it is not reduced to a point or parallel to the polygonal line of antipodes. Let $V_1(\alpha_1, \beta_1, \gamma_1)$ be a vector linking M to an antipode M_1 sufficiently distant from M in order for it to exceed at least the polygonal line of antipodes.*

Let $A = \beta_1\gamma_2 - \beta_2\gamma_1$, $B = \alpha_2\gamma_1 - \alpha_1\gamma_2$, $C = \alpha_1\beta_2 - \alpha_2\beta_1$ and $k = A \wedge B \wedge C$. Then piece S belongs to a digital plane $P(\frac{A}{k}, \frac{B}{k}, \frac{C}{k}, \mu')$ such that M is a leaning point ($q = 0$) or 1-exterior ($q = 1$) if and only if M_1 is arbitrarily chosen on a line, among at most three parallel lines, directed by V_2 such that the following relations are satisfied:

$$0 \leq A \leq B \leq C, \quad (4)$$

$$P(\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)) - C(P - 1) = kc(1 - q), \quad (5)$$

$$\frac{d_m - \mu - c + P}{P} + \frac{k(1 - q)c}{|V_2|H} \leq \frac{\epsilon_m h_m}{H} \leq \frac{d_m - \mu + P}{P}; \quad \forall m \in S, \quad (6)$$

with $q = 0$ or 1 and H (resp. h_m) being the height of point M_1 (resp. m) to the real line passing by M and directed by V_2 .

The number of lines that may contain M_1 is 1 if $k < P$ and at most 3 if $k = P$.

Remarks.

1. In the algorithm that we propose, only relations (4) and (5) are needed explicitly, relation (6) serves mathematically to prove Debled's conjectures and explains how to choose M_1 between all antipodes.

2. We note that if in theorem 2 we assume that P can take the value 1, relations (4),(5) and (6) become equivalent to those quoted in the theorem of the previous section. Following lemma 2 below, the number $k = A \wedge B \wedge C$ divides P which is assumed to be 1 here. This fact is also true for the 1-exterior case. Thus theorem 2 includes the 1-exterior case.

Proof Idea. Let us suppose that the added point M is located under plane $P(a, b, c, \mu)$. The Point M satisfies the relation $ax_0 + by_0 + cz_0 = \mu - P$. Let M_1 be a lower leaning point of $P(a, b, c, \mu)$ and let $V_1 = \overrightarrow{MM_1} =: (\alpha_1, \beta_1, \gamma_1)$ and $(A, B, C) =: V_1 \wedge V_2$. Vector V_1 satisfies the relation $a\alpha_1 + b\beta_1 + c(\gamma_1 - \frac{P}{c}) = 0$. Suppose that V_2 is linearly independent from $V_1(\alpha_1, \beta_1, \gamma_1 - \frac{P}{c})$. The vectorial product $V_1 \wedge V_2 = (A', B', C')$ is a rational multiple, say λ , of $N(a, b, c)$. The numbers A', B', C' are given by:

$$A' = A + \beta_2 \frac{P}{c} = \lambda a; \quad B' = B - \alpha_2 \frac{P}{c} = \lambda b; \quad C' = C = \lambda c.$$

We can always assume (by considering $-V_2$ in place of V_2) that $0 \leq A'$. Since $0 \leq a \leq b \leq c$, then $0 \leq A' \leq B' \leq C'$ which is equivalent to $0 \leq A + \beta_2 \frac{P}{c} \leq B - \alpha_2 \frac{P}{c} \leq C$. If we increase the length of V_1 , then the numbers A', B', C' also increase, and so does the distance between each other. Note that the quantities $\beta_2 \frac{P}{c}$ and $\alpha_2 \frac{P}{c}$ do not change when we increase the length of V_1 . Let us take V_1 sufficiently long so that

$$0 \leq A \leq B \leq C \quad (4)$$

By dividing A, B, C by their greater common divisor, we obtain the characteristics of the eventual new plane which may contain the piece $S' = S \cup \{M\}$.

Lemma 2. *If the numbers α_2, β_2 and γ_2 are prime together ($\alpha_2 \wedge \beta_2 \wedge \gamma_2 = 1$), then $PGCD(A, B, C) = A \wedge B \wedge C$ divides number P .*

Let us search conditions under which there exists μ' such that the point M is q -exterior to $S \subset P(A, B, C, \mu')$, with $q \in \{0, 1\}$. This means that for all $m(x, y, z) \in S'$ we have:

$$\begin{cases} \mu' \leq Ax + By + Cz < \mu' + C \\ Ax_0 + By_0 + Cz_0 = \mu' - P + q \end{cases} \quad \text{where } q = 0 \text{ or } 1$$

For all $m(x, y, z) \in S \subset P(a, b, c, \mu)$ we have $\mu \leq ax + by + cz \leq \mu + c - 1$. The added point M satisfies the relation $ax_0 + by_0 + cz_0 = \mu - P$. Then $\mu - P \leq ax + by + cz \leq \mu + c - 1$, for all points of S' .

By multiplying all members with λ we obtain:

$$\lambda(\mu - P) - P(\frac{\beta_2}{c}x - \frac{\alpha_2}{c}y) \leq Ax + By + Cz \leq \lambda(\mu - 1) - P(\frac{\beta_2}{c}x - \frac{\alpha_2}{c}y) + C.$$

Let us take $k = PGCD(A, B, C)$. By dividing the previous inequality with k we obtain:

$$\lambda''(\mu - P) - P''(\frac{\beta_2 x - \alpha_2 y}{c}) \leq A''x + B''y + C''z \leq \lambda''(\mu - 1) - P''(\frac{\beta_2 x - \alpha_2 y}{c}) + C'',$$

where $P'' = \frac{P}{k}$, $A'' = \frac{A}{k}$, $B'' = \frac{B}{k}$, $C'' = \frac{C}{k}$ et $\lambda'' = \frac{\lambda}{k} = \frac{C''}{c}$.

Let us take $\mu(x, y) = \lambda''(\mu - P) - P''(\frac{\beta_2 x - \alpha_2 y}{c})$. Then for all points in S' we have: $\mu(x, y) \leq A''x + B''y + C''z \leq \mu(x, y) + C'' + \lambda''(P - 1)$.

Using the facts that M is P -exterior and should become q -exterior and that M_2 (one of the end points of vector V_2) is an upper leaning point of pieces S and S' , we obtain the condition

$$P[\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)] - C(P - 1) = kc(1 - q). \quad (5)$$

This condition expresses the position of points M_2 , M'_2 and M_1 with respect to M . Now let us search in what region we can choose V_1 so that surface S will be a recognized piece of the digital plane of parameters $(A'', B'', C'', \mu(x_0, y_0) + q)$ with M a q -exterior, where $q = 0$ or 1 . The former double inequality can be transformed to get $\frac{d_m - \mu - c + P}{P} + \frac{kc(1-q)}{PC} \leq \frac{\beta_2(x - x_0) - \alpha_2(y - y_0)}{\beta_2\alpha_1 - \alpha_2\beta_1} \leq \frac{d_m - \mu + P}{P}$. This expression can be expressed in terms of heights by:

$$\frac{d_m - \mu - c + P}{P} + \frac{k(1-q)c}{|V_2|H} \leq \frac{\epsilon_m h_m}{H} \leq \frac{d_m - \mu + P}{P} \quad (6)$$

with $d_m \in [\mu - P, \mu + c - 1]$. It is the condition that all points $m(x, y, z) \in S'$ should satisfy so that S' will be a recognized piece in the digital plane of characteristics $(A'', B'', C'', \mu' = \mu(x_0, y_0))$. By studying the sign of the first member of relation (5), we can show that this relation can be satisfied at most by three lines directed by V_2 and containing antipodes.

5 Simplified Recognition Algorithm

The algorithm that we describe in this section directly derives from the discussion and the proofs of the two previous theorems. It uses only the part of Debled's algorithm corresponding to 1-exterior case with small modifications to recognize the general case of rectangular pieces. The complexity of the simplified algorithm decreases to become at most quadratic in the number of points in the piece. Following the necessary and sufficient conditions quoted in theorem 2, the simplified algorithm becomes completely decidable where the validity criteria given by Debled fail in the strongly exterior case. These two advantages are added to the other advantages of Debled's algorithm.

The algorithm begin by sweeping the piece to be recognized following sections parallel to one co-ordinate plane, Oxz for instance, by successively adding voxels. At the beginning we initialize $y = 0$ and we let x vary in its interval of definition. At each added voxel the algorithm tries to recognize a piece of a digital straight line. When all values of x are considered we increment y by 1 and we let x sweep again over all its possible values. At each step the algorithm tries to recognize a piece in a digital plane and compute its characteristics. Three cases are possible:

1. If the added point M satisfies the double inequality (1) for the plane constructed before adding M , then we keep the same characteristics and the updated piece is still recognized in this plane with the same leaning points with a possible addition of M .
2. If the added point is 1-exterior to the plane, then we apply Debled's algorithm to recognize the new piece. This step consists of computing polygonal pivots and antipodes lines to determine the vector V_2 which satisfies relation (2), and also search an adequate antipode M_1 that satisfies (3)).
3. In the third case, if the added point is strongly exterior, we search the polygonal line of pivots and antipodes and then check relation (5). If this relation

is not satisfied for any point M_2 on the polygonal line of pivots, then the piece is not recognizable. If relation (5) is satisfied for some q different from 0 and 1 then the piece is not recognizable. If relation (5) is satisfied for $q = 0$, then we apply 1-exterior part of Debled's algorithm. In this case the piece is recognized with M as a leaning point. If relation (5) is satisfied for $q = 1$, then we again apply Debled's algorithm for the 1-exterior case at most three times to decide the recognition of the piece. Note that in this case, the choice of the antipode M_1 is modified from the choice given by Debled. In the first running of the algorithm, point M_1 is taken as in Debled's algorithm. If M becomes 1-exterior, then we apply Debled's algorithm once more and the piece is recognized. At this point, if after one run there are some points which are excluded, then we choose the antipode M_1 on the following line proposed by theorem 2; if the point M becomes 1-exterior, then we are done. If there are some points which are excluded then we take M_1 on the last line proposed by the theorem and the point becomes 1-exterior. Usually, only two lines are sufficient.

6 Example

In Figure 5(a), we represent a recognized piece S of the plane $P(18, 21, 23, 0)$. The added point $M(3, 5, -7)$ is 2-exterior of remainder $r(M) = 2$. The nearest pivot point to M is $M_2(4, 2, -4)$. The polygonal line of antipodes contains only two points $M_1(0, 0, 0)$ and $M'_1(8, 3, -9)$. The vector V_2 is then $(8, 3, -9)$ and $V_1 = (-3, -5, 7)$. The vectorial product of V_1 and V_2 gives $(A, B, C) = (24, 29, 31)$. With this choice relation (6) is not satisfied. Take for instance the point $m = (9, 0, -7)$. The quantity $\frac{h_m}{H} = \frac{\beta_2(x-x_0)-\alpha_2(y-y_0)}{C} = \frac{58}{31} = 1,86$ is greater than $\frac{d_m-\mu+P}{P} = 3/2 = 1,5$. Let us take the antipode located on the line directed by V_2 and that comes just after V_1 . Let $M''_1(5, -1, -3)$ be a such antipode. The point M''_1 is located outside the piece S . In this case vector V_1 is $\overrightarrow{M''_1M} = (2, -6, 4)$. The vectorial product V_1 with V_2 is $(42, 50, 54) = 2(21, 25, 27)$. Note that $PGCD = k = 2$ divides $P = 2$.

The quantity $P(\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)) - C(P - 1)$ is equal to $2(3(4 - 3) - 8(2 - 5)) - 54(2 - 1) = 54 - 54 = 0$. Moreover, for $q = 1$ we obtain $kc(1 - q) = 0$. Thus, relation (5) is satisfied for $q = 1$. We can check that all points of S satisfy relation (6). For $m(9, 0, -7)$, which did not satisfy (6) for the first basis, we get $\frac{h_m}{H} = \frac{58}{54} = 1,07$ which is small to $\frac{d_m-\mu+P}{P} = 3/2$. The value of $\mu' = \mu(x_0, y_0) + q$ is $-1 + 1 = 0$. Piece S is recognized in the plane $P(21, 25, 27, 0)$, see Figure 5(b).

Thus, we return back to 1-exterior case. The polygonal line of pivots is reduced to the point $M_2(4, 2, -4)$. The polygonal line of antipodes contains two points $M_1(0, 0, 0)$ and $M'_1(9, 0, -7)$. Vector V_2 is then equal to $\overrightarrow{M_1M'_1}(9, 0, -7)$ and $V_1 = \overrightarrow{M_1M} = (-3, -5, 7)$. The vectorial product of V_1 with V_2 gives $(35, 42, 45)$. Piece $S' = S \cup \{M\}$ becomes recognized in the plane $P(35, 42, 45, 0)$ with at least three lower leaning points M, M_1, M'_1 and at least one upper leaning point M_2 , see Figure 5(c).

Fig. 5. In (a), piece S is recognized in the plane $P(18, 21, 23, 0)$. In (b) piece S is recognized in the plane $P(21, 25, 27, 0)$ and M is 1-exterior. In (c), piece S' is recognized in the plane $P(35, 42, 45, 0)$

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