Illuminating Polygons with Vertex π -Floodlights

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Abstract. It is shown that any simple polygon with n vertices can be illuminated by at most $\lfloor (3n - 5)/4 \rfloor$ vertex π -floodlights. This improves the earlier bound $n - 2$, whereas the best lower bound remains $3n/5 + c$.

1 Introduction

The first theorem on *Art Galleries* is due to Chvatál [1] who showed that any simple polygon with n vertices can be illuminated by $\lfloor n/3 \rfloor$ light sources and this bound is tight. The famous proof of Fisk [4] places light sources at vertices of the polygon. It has been shown recently [7], that $\lfloor n/3 \rfloor$ is sufficient even if the light sources can illuminate only a range of angle π (i.e. using π -floodlights). But there, π -floodlights may be placed at any point of the polygon, even two π -floodlights are allowed to be placed at the same point.

Urrutia [2] asked the following question. What is the minimal number of vertex π -floodlights that can collectively illuminate any simple polygonal domain (shortly *polygon*) P with n vertices. A vertex π -floodlight is given by a pair (v, H^v) where v is a vertex of P and H^v is a closed half-plane H^v such that v is on the boundary of H^v . There may be at most one π -floodlight at each vertex of P . A π -floodlight at (v, H^v) *illuminates* $a \in P$ if and only if the closed line segment va is in $P \cap H^v$. All points of P should be *illuminated* by at least one π -floodlight.

F. Santos [9] has produced a family of polygons that requires $\lfloor 3n/5 \rfloor + O(1)$ vertex π -floodlights. Urrutia [2] conjectured that this number is always sufficient to illuminate any polygon with n vertices but proved only the sufficiency of $n - 2$. So far no constant $b < 1$ has been known such that $bn + O(1)$ vertex π -floodlights can illuminate any polygon with n vertices.

Theorem 1. $\lfloor 3(n - 3)/4 \rfloor + 1$ vertex π -floodlights can illuminate any simple polygon with n vertices.

The notion of vertex α -floodlight can be defined for any angle $0 < \alpha < 2\pi$ as a cone of aperture at most α with apex at a vertex of polygon P . Under the

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condition that there may be at most one vertex floodlight at each vertex, it is known [2] that for any angle $\alpha < \pi$, there exist convex polygons P_n with $n \geq n_\alpha$ vertices such that n α -floodlights cannot illuminate P_n .

In this paper, the placement of floodlights is based on a decomposition of the polygon into “dense polygons”. Such decomposition was introduced in [8], and is discussed in our Sect. 2. Any dense polygon with n vertices can be illuminated with at most $\lfloor (3n - 5)/4 \rfloor$ vertex π -floodlights. This does not imply immediately that any polygon P could be illuminated by $3n/4 + O(1)$ floodlights, because at most one vertex π -floodlight can be placed at each vertex of P , and thus there may be conflicts at vertices belonging to several dense sub-polygons. Our floodlight placement algorithm and its analysis is contained in Sect. 4.

2 Dense Polygons

Let P be a simple polygon and let T be a set of triangles in the plane. T is the triangulation of P , if $P = \bigcup T$, the triangles of T are pairwise non-overlapping, and the vertices of the triangles are vertices of P . It is known that every simple polygon has a triangulation, every triangulation consists of exactly $n - 2$ triangles, although the triangulation is not necessarily unique. We define the graph $G(T)$ on a triangulation T . The nodes of the graph correspond to the elements of T , two nodes are adjacent if and only if the corresponding triangles have a common side. $G(T)$ is a tree on $n - 2$ nodes, the maximal degree in $G(T)$ is three, since a triangle $t \in T$ may have a common side with at most three other triangles of T .

Definition 1. *A graph G is dense, if G is a tree and each node of G has degree one or three. A simple polygon S is dense if graph $G(T_S)$ is dense for any triangulation T_S of S .*

Proposition 1. *Any dense graph G has an even number of nodes. If a dense graph G has 2ℓ nodes then it has exactly $\ell + 1$ leaves.*

Proof. Suppose that G has k leaves and l nodes of degree 3. The number of edges is $(k + 3l)/2 = (k + l)/2 + l$, hence $k + l$ is even. G is a tree, so $(k + l)/2 + l = k + l - 1$, that is $k = (k + l)/2 + 1$. $\square$

2.1 Dense Decomposition

The *dense decomposition* L of a polygon P is set of pairwise non-overlapping dense polygons such that $P = \bigcup L$ and the vertices of the dense polygons are vertices of P . We can define the tree $G(L)$ on a dense decomposition L just like $G(T)$. The nodes of the $G(L)$ correspond to the dense polygons of L , two nodes are adjacent if and only if the corresponding polygons have a common side.

The union of the triangulations of elements of a dense decomposition is a triangulation T of P . So T contains an even number of triangles. Clearly, this is impossible if P has an odd number of vertices. We can prove, however, the following Lemma.

Lemma 1. *If P is a simple polygon with an even number of vertices, then P has a dense decomposition.*

Proof. By induction on the number of nodes of $G(T)$. Every quadrilateral is dense. If the polygon P is dense, then the proof is complete.

If P is not dense then there is a triangulation T of P such that there exists a node of degree two in $G(T)$. Consider the tree $G(T)$ as a rooted tree $(G(T), r)$ where an arbitrary leaf r of $G(T)$ is chosen as root. Let $v \in G(T)$ be a node of degree two such that no descendant of v has degree two in $G(T)$. Let S denote the subtree containing v and all its descendant in $(G(T), r)$. According to Proposition 1, the subtree S has an even number of nodes, hence $G(T) \setminus S$ has an even number of nodes as well. The polygons corresponding to S and $G(T) - S$ have dense decompositions by induction, and together they give a dense decomposition of P . $\square$

Fig. 1. Dense decomposition of a polygon P and the corresponding graph G

If P has an odd number of nodes, then let t correspond to a leaf of $G(T)$ in a triangulation T of P . Polygon $P - t$ has already an even number of vertices, and t can be illuminated by one π -floodlight at the unique vertex not adjacent to $P - t$. To establish Theorem 1, it is enough to prove the following.

Lemma 2. *Any simple polygon P with an even number n of vertices can be illuminated by $\lfloor 3(n-2)/4 \rfloor$ vertex π -floodlights.*

2.2 Notation of Dense Polygons

Let L be the dense decomposition of a simple polygon. Consider $G(L)$ as a rooted tree $(G(L), r)$ with an arbitrary leaf r chosen as root. In this way we may

interpret the parent-son relation between dense polygons of L . For any polygon $S \in L$, $S \neq r$, let the *base side* of L be the side adjacent to the parent polygon of S . For the root polygon, let the *base side* be any side not adjacent to any son polygon. A *base vertex* of a polygon $Q \in L$ is a vertex along its base side.

In our method, *convex* and *concave* quadrilaterals of the dense decomposition have different roles. We call *star-polygon* a dense polygon on at least 6 vertices. Fix a triangulation T_S in each $S \in L$.

An *outer vertex* of a star-polygon or concave quadrilateral $S \subset P$ is a vertex of S which belongs to exactly one triangle $t \in T_S$. All other vertices of a star polygon are called *inner*. Every vertex of a convex quadrilateral in L is *outer*. The *corner vertex* of a concave quadrilateral $Q \in L$ is the vertex opposite to the reflex vertex of Q (a π -floodlight at v can illuminate Q).

Proposition 2.

- (1) For every vertex v of P , there is at most one dense polygon $S \in L$ such that v is a non-base vertex of S .
- (2) Every second vertex of a dense polygon is outer.
- (3) Every outer vertex of a dense polygon is convex.
- (4) A star polygon or a concave quadrilateral of L with 2ℓ triangles has exactly $\ell + 1$ outer vertices, one of which is a base vertex.

The proof of the above statements is immediate.

Proposition 3. For two vertices v and x of a star-polygon S , vx cannot be a diagonal of S if v is an outer vertex for a triangulation T_S .

Proof. Let u, v , and w be three consecutive vertices of S such that uvw is a triangle in a triangulation T_S of S . Suppose that vx is a diagonal of S . First we state that there is a vertex y such that both uvy and uvw are diagonals. If $y = x$ does not have this property, then let y be the vertex in uvx where the angle $\angle uvx$ is minimal. There is a triangulation T'_S of S such that $uvy \in T'_S$. In $G(T')$, the node corresponding to uvy has degree 2. $\square$

3 Illuminating Star Polygons

In our method, every dense polygon $S \in L$ will be illuminated by π -floodlights placed at vertices of S . Lemma 2 is best possible in this setting, because there are dense hexagons that cannot be illuminated by less than three vertex π -floodlights.

Definition 2. A π -floodlight (v, H^v) in P is called *complementary*, if P has a reflex angle α at v and the angular domain of α contains the complement of the closed half-plane H^v .

Lemma 3. Any dense polygon $S \in L$ with 2ℓ vertices can be illuminated by at most ℓ vertex π -floodlights at vertices of S : one at an arbitrary outer vertex and at most $\ell - 1$ complementary π -floodlights at reflex vertices.

Proof. Fix a triangulation T of S . Suppose that we put a floodlight at an outer vertex v . v belongs to a unique triangle $xyv \in T$. Let $C(v)$ be the set of points $p \in S$ such that the line segment vp is in S . The rays $\overrightarrow{vx}$ and $\overrightarrow{vy}$ hit the boundary of S at points x' and y' resp.

According to Proposition 3, x' and y' are points of a same side ab of S . We may suppose that b is an outer vertex. $S \setminus C(v)$ consists of at most two simple polygons S_x and S_y such that $x \in S_x$ and $y \in S_y$. Every reflex vertex of S is a vertex of exactly one of S_x and S_y . We may suppose w.l.o.g. that S_x is non-empty.

Visit the reflex angles of S_x along the boundary from x to a in orientation yvx . Consecutively dissect S at each reflex vertex w by a ray emanating from w such that angle $\angle w > 180^\circ$ is partitioned into 180° and $\angle w - 180^\circ$ (e.g. $\angle x$ is dissected by the segment xx'). Repeat this dissection in S_y as well if $S_y \neq \emptyset$. Thus $S \setminus C(v)$ is partitioned into k convex polygons where k is the number of reflex angles in S .

Our proof is complete if S has at most $\ell - 1$ reflex angles (e.g., S is quadrilateral). Suppose that S has ℓ reflex angles, hence x is also reflex vertex. We state that the last convex region C (along ax') is already illuminated by a complementary floodlight at another reflex vertices of S_x . (See Fig. 2 for illustrations.)

Fig. 2. Illuminating star polygons with 8 and 16 vertices resp.

For this, denote by a' the point where ray $\overrightarrow{xa'}$ hits the boundary of S_x . Consider the reflex vertices of S_x along its boundary in orientation axa' from x to a' . Denote by z the last reflex vertex whose dissecting ray hits the boundary of S on the arc az . The complementary floodlight at z illuminates C . $\square$

Remark 1. Consider a simple polygon P with a dense decomposition L and tree $(G(L), r)$. There is a placement of vertex π -floodlights in P such that every dense

polygon of L with 2ℓ vertices is illuminated by ℓ floodlight (i.e., also there is at most one floodlight at each vertex of P).

In each dense polygon $S \in L$, place a π -floodlight at a non-base outer vertex, and another at most $\ell - 1$ complementary π -floodlights at reflex vertices according to Lemma 3. If a vertex v is a common vertex of two dense sub-polygons S_1 and S_2 , and we placed two floodlights at v , then one of them is a complementary floodlight. Hence actually, we place at most one π -floodlight at each vertex.

Such a placement of floodlight is called *basic placement*. If L contains no concave quadrilaterals then a basic placement requires $\lfloor 3(n - 2)/4 \rfloor$ floodlights.

4 Proof of Lemma 2

Fix a dense decomposition L of polygon P , and a triangulation of each $S \in L$. We illuminate every $S \in L$ by π -floodlights at vertices of S . Every star-polygon and convex quadrilateral of L is illuminated by *basic* placement described in Remark 1. Every concave quadrilateral of L is illuminated either by two floodlights of a basic placement or by one floodlight. The main concern of our proof is to guarantee that the majority of the concave quadrilaterals require only one floodlight. Then 2ℓ triangles of ℓ concave quadrilaterals require at most $\lfloor 3(2\ell)/4 \rfloor$ floodlights, proving Lemma 2.

A basic placement is not necessarily unique. A star-polygon or a convex quadrilateral $S \in L$ has at least two non-base outer vertices. If $Q \in L$ is illuminated by a floodlight at its non-base outer vertex v , then we color v red. We make our choice using a voting function: A number of concave quadrilaterals vote for each possible non-base outer vertex. We have another choice to make: a floodlight at a non-base outer vertex v can be directed in two different ways. Again a number of concave quadrilaterals vote at each possible non-base outer vertex. The winners require one floodlight, the losers require two.

It is enough to take care of the following two properties. Every concave quadrilateral of L vote at most once. And every concave quadrilateral which does not vote at all requires one floodlight.

4.1 Classifying Convex Quadrilaterals

Let $\mathcal{R} \subset L$ be the set of concave quadrilaterals of L . Denote by $\mathcal{Q}(v) \subset \mathcal{R}$ the set of concave quadrilaterals whose corner vertex is v .

We define recursively two functions g_1 and g_2 on concave quadrilaterals. Suppose that g_1 and g_2 are defined on all descendants of $Q \in \mathcal{R}$. If the reflex vertex of Q is a base vertex then let $g_1(Q) = g_2(Q) = 0$. Assume that $Q = abcd$ where d is non-base reflex vertex, and b is a base corner vertex. Denote by H_d^+ and H_d^- the two halfplanes determined by the line bd such that $a \in H_d^+$ and $c \in H_d^-$. Partition $\mathcal{Q}(d)$ into two sets, $\mathcal{Q}_d^+$ and $\mathcal{Q}_d^-$, such that reflex vertex of all $W \in \mathcal{Q}_d^+$ (resp. $W \in \mathcal{Q}_d^-$) is in H_d^+ (resp. in H_d^-). Let $Q_d(d) \in \mathcal{Q}(d)$ denote the possible quadrilateral dissected by the line bd . Now let

$$g^+(Q) = \mathcal{Q}_d^+ \cup g^-(Q_d) \quad \text{and} \quad g^-(Q) = \mathcal{Q}_d^- \cup g^+(Q_d).$$

Fig. 3. A dense decomposition of a polygon, where $g_1(abcd)$ is shaded.

We define recursively a function f on non-base outer vertices of dense polygons of L . If f is defined for all non-base outer vertices of all descendant of $S \in L$, then consider a non-base outer vertex v such that u, v , and w are consecutive vertices of S . Let H_1^+ and H_1^- (resp. H_2^+ and H_2^-) be the half-planes determined by uv (resp wv) such that $w \in H_1^+$ (resp. $u \in H_2^+$).

The quadrilaterals in $\mathcal{Q}(v)$ are sorted into tree distinct types. Denote by $\mathcal{Q}_v^A$, $\mathcal{Q}_v^C$, and $\mathcal{Q}_v^D$ the set of quadrilaterals whose reflex angle is in $H_1 \cap H_2$, $H_1^+ \setminus H_2^-$, and $H_2^+ \setminus H_1^+$ respectively. Let $Q_1(v) \in \mathcal{Q}(v)$ and $Q_2(v) \in \mathcal{Q}(v)$ be the possible quadrilaterals dissected by the line uv and wv resp. (Possibly $Q_1(v) = Q_2(v)$.) See Fig. 4 and 5 for illustrations.

Fig. 4. Polygons where $\alpha(v)$ is shaded and $vxyz$ is of type A. Also they are $Q_1(v) = Q_2(v)$ and $Q_1(v)$ resp.

Fig. 5. Polygons where $\alpha(v)$ is shaded and $vx yz$ is of type C , and D resp. Also they are $Q_1(v) = Q_2(v)$ and $Q_2(v)$ resp.

Now let

$$f(v) = \mathcal{Q}_A^v \cup \bigcup_{W \in \mathcal{Q}_A^v} f(W),$$

where $f(W) = f(a)$ for the unique non-base outer vertex a of the concave quadrilateral W . Finally, let

$$h^+(v) = \mathcal{Q}_v^C \cup \bigcup_{W \in \mathcal{Q}_v^C} f(W) \cup g^-(Q_1(v)) \text{ and } h^-(v) = \mathcal{Q}_v^D \cup \bigcup_{W \in \mathcal{Q}_v^D} f(W) \cup g^+(Q_2(v)).$$

Proposition 4.

- (1) $f(v)$, $h^+(v)$, and $h^-(v)$ are disjoint at each vertex v of P .
- (2) $f(u) \cup h^+(u) \cup h^-(u)$ and $f(v) \cup h^+(v) \cup h^-(v)$ are disjoint for two non-base vertices of star-polygons u and v .

Proof. Statement (1) is clear from the tree structure of $(G(L), r)$.

For (2), we assign a non-base vertex $r(Q)$ of a star-polygon to each element of $Q \in \mathcal{R}$. For a $Q \in \mathcal{R}$, let $(Q = Q_0, Q_1, \dots, Q_t)$ be the longest ascending path in $(G(L), r)$ such that $Q_i \in \mathcal{R}$ for all element of the sequence and for all pair (Q_i, Q_{i+1}) , the corner vertex of Q_i is a base vertex. (E.g. if the corner vertex of Q is a non-base vertex, then the longest sequence is $(Q = Q_0)$.) Now observe that $Q \in f(v) \cup h^+(v) \cup h^-(v)$ if the corner vertex $r(Q)$ of Q_t is v and v is a non-base vertex of a star-polygon. $\square$

4.2 Placement of Vertex π -Floodlights

The functions f , g and h were defined recursively in ascending order in $G(L)$. The placement of vertex π -floodlights is done in descending order on the tree $(G(L), r)$. We describe a floodlight-placement algorithm. Step I colors red certain

non-base outer vertices of star-polygons and convex quadrilaterals. Step II colors red or blue concave quadrilaterals. We apply a *basic placement* to star polygons with a floodlight at the red vertex and to red quadrilateral. In Step III, each blue concave quadrilateral is illuminated by adding at most one vertex π -floodlight.

Algorithm:

Step I, In each star polygon and convex quadrilateral of L , color a non-base outer vertex v red, where $|f(v)|$ is minimal. Color all elements of $f(v)$ red, and all elements of $f(w)$ blue for all other non-base outer vertices.

Step II, For each vertex v of P with $h^+(v) \neq \emptyset$ and $h^-(v) \neq \emptyset$, we make a decision. If $h^+(v) \geq h^-(v)$ (resp. $h^+(v) < h^-(v)$) then color every element of $h^-(v)$ (resp. $h^+(v)$) red and every element of $h^+(v)$ (resp. $h^-(v)$) blue.

Color the non-base outer vertex of each red concave quadrilateral red as well.

Step III, Consider a vertex v with $Q(v) \neq \emptyset$.

First suppose that $|Q(v)| = 1$ and v is not red. Place a floodlight at v to illuminate $Q \in Q(v)$. From now on, we assume $|Q(v)| > 1$.

Suppose that v is a non-red convex non-base vertex of a dense polygon S , and $h^+(v) \geq h^-(v)$. Place a floodlight at (v, H_2^-) . We show that each quadrilateral of $f(v) \cup h^+(v)$ can be illuminated by at most one floodlight. Every $Q \in Q(v)$ in H_1^+ is illuminated by (v, H_1^+) , consider the possible case where $abcd = Q_2(v)$ is in $f(v) \cup h^+(v)$. Triangle $abd \subset abcd$ is illuminated by (v, H_1^+) . Place a floodlight at (d, H_d^+) to illuminate triangle abd as well. d is an inner vertex of $abcd$, so basic placements may place at most a complementer floodlight at d .

Suppose that v is a red outer non-base vertex of a dense polygon S and $h^+(v) \geq h^-(v)$. That is, the floodlight at v is assigned to S , and it should illuminate the angular domain uvw . Place a floodlight at (v, H_1^+) . We show that each quadrilateral of $h^+(v)$ can be illuminated by at most one additional floodlight. Every $Q \in Q(v)$ in H_1^+ is illuminated by (v, H_1^+) , consider the possible $abcd = Q_2(v) \in Q(v)$. Triangle $bcd \subset abcd$ is illuminated by (v, H_1^+) , place a floodlight at (d, H_d^+) to illuminate triangle abd as well. d is an inner vertex of $abcd$, so basic placements may place at most a complementer floodlight at d .

If v is a reflex vertex of a star-polygon S , then one π -floodlight at v can illuminate every quadrilateral of $Q(v)$. This is the case also if $v = d$ is a non-base reflex vertex of a concave quadrilateral $abcd$, and there is no floodlight at either (d, H_d^-) nor at (d, H_d^+) .

Suppose that $v = d$ is a reflex vertex of a concave quadrilateral $abcd$ and there is a floodlight at, say, (d, H_d^-) . It illuminates elements of Q_d^+ except a possible $Q_d(d) \in Q(d)$, and elements of Q_d^- are colored red. One triangle of $Q_d(D) = a'b'c'd'$ is illuminated by (d, H_d^-) , place a floodlight at $(d', H_{d'}^+)$ to illuminate the other triangle as well. d' is an inner vertex of $a'b'c'd'$, so basic placements may place at most a complementer floodlight at d' .

During the algorithm, we assigned 0, 1, or 2 floodlights to each concave quadrilateral. We assigned 2 floodlights to a concave quadrilateral if and only if it is colored red. The comparisons of $|f(v)|$ and $|h^+(v)|, |h^-(v)|$ guarantee that the majority of concave quadrilaterals are colored blue.

Fig. 6 illustrates the output of our algorithm on a polygon with a dense decomposition where the base side of the root polygon is the upper horizontal segment.

Fig. 6. Placement of floodlights produced by our algorithm on dense decomposition of a polygon.

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