On Dynamic Generalized Voronoi Diagrams in the Euclidean Metric

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Abstract. The problem of dynamic maintenance of the Voronoi diagram for a set of spheres moving independently in d -dimensional space is addressed in this paper. The maintenance of the generalized Voronoi diagram of spheres, moving along the given trajectories, requires the calculation of topological events, that occur when $d + 2$ spheres become tangent to a common sphere. The criterion for determination of such a topological event for spheres in the Euclidean metric is presented. This criterion is given in the form of polynomial algebraic equations dependent on the coordinates and trajectories of the moving spheres. These equations are normally solved using numerical methods.

1 Introduction

Such areas as motion planning, computer simulation of physical systems, robotics and computer graphics often deal with geometric objects that move with time [12]. In many applied problems from these areas, collection of geometric objects such as points, disks and spheres, is often considered. Objects can be given along with the analytic functions describing their motion, often specified by polynomials of time. The aim in these problems is to answer questions concerning properties of the system of objects, for instance, finding the closest/furthest pair, predicting the next collision or computing the minimum enclosing circle.

The static Voronoi diagram is often employed by for solving the above problems. Extensive libraries of methods for Voronoi diagram construction, maintenance and querying have been developed over time [14,4]. Weighted Voronoi diagrams for a set of circles in the plane have been introduced and their properties have been studied [2,17,9,8,13]. For the dynamic Voronoi diagram of a set of moving points in the plane only a few algorithms are known. These include computation of the moment in time when the convex hull of algebraically moving points reaches a steady state [1], construction and maintenance of the dynamic Voronoi diagram [7], estimation of upper and lower bounds on the number of combinatorial changes to the diagram over time [15] and solving query problems on the set of moving points [3].

Finally, the problem of construction and maintenance of the dynamic Voronoi diagram for a set of moving objects other than points has rarely been addressed in the literature. Some of the works in this field were devoted to the construction of the dynamic Voronoi diagram for a set of moving hyper-rectangles [11], dynamical maintenance of the Voronoi diagram of line segments in the plane

[5] and computation of the time of the topological event in the dynamic Euclidean Voronoi diagram for a set of circles and line segments [10]. In this paper, we extend the result reported in [10] to address the problem of dynamic maintenance of the weighted Euclidean Voronoi diagram for moving spheres in d -dimensional space. An important property of the weighted generalized Voronoi diagram in the Euclidean metric, limiting the number of inscribed spheres, is also established. Based on this result, a criterion to determine the time of the topological event in the Euclidean metric is derived in the form of a system of polynomial algebraic equations.

2 Definitions

Consider a set S of n moving spheres in R^d . Each sphere moves along a given trajectory, described by a function of time. The spheres move in unbounded d -dimensional space. The Voronoi diagram can be used as a data structure to store the topological information about the system of moving objects as well as to answer queries. It is defined as:

Definition 1. The Voronoi diagram for a set of objects S in d -dimensional space is a partitioning of the space into regions, such that each region is the locus of points from S closer to the object $P \in S$ than to any other object $Q \in S$, $Q \neq P$.

The above general definition can be specialized to the set of spheres in the Euclidean metric [14]:

Definition 2. A generalized Euclidean Voronoi diagram (GVD) for a set of sites S in R^d is the set of Voronoi regions $\{\mathbf{x} \in R^d \mid d(\mathbf{x}, P) \leq d(\mathbf{x}, Q), \forall Q \in S - \{P\}\}$, where $d(\mathbf{x}, P)$ is the Euclidean distance function between a point $\mathbf{x}$ and a site $P \in S$.

Following the classification of generalized Voronoi diagrams presented in [14] the Euclidean weighted Voronoi diagram is an instance of the class of additively weighted Voronoi diagrams, where $d(\mathbf{x}, P) = d(\mathbf{x}, \mathbf{p}) - r_p$ (see Fig. 1). The distance $d(\mathbf{x}, \mathbf{p})$ between points $\mathbf{x}(x_1, x_2, \dots, x_d)$ and $\mathbf{p}(p_1, p_2, \dots, p_d)$ in the

Euclidean metric is computed as $d(x, y) = \sqrt{\sum_{i=1}^d (x_i - y_i)^2}$. According to the

definition, the generalized Voronoi region of an additively weighted Voronoi diagram of n sites is obtained as the intersection of $n - 1$ quasi-halfspaces with hyperbolic boundaries. It was shown in [14] that the weighted Euclidean Voronoi diagram for a set of spheres is the set of singly-connected Voronoi regions with the hyperbolic boundaries, star-shaped relative to site P .

The straight-line dual to the Voronoi diagram called a Delaunay tessellation is often used instead of the Voronoi diagram to store topological information for a set of sites:

Definition 3. A generalized Delaunay tessellation corresponding to a generalized Voronoi diagram for a set of spheres S in d -dimensional space is a collection of d -dimensional simplices such that for each generalized Voronoi

Fig. 1. The Euclidean Voronoi diagram and the corresponding Delaunay triangulation

vertex $\mathbf{v} = EVor(P_1) \cap EVor(P_2) \cap \dots \cap EVor(P_{d+1})$ there exists a simplex $(p_1, p_2, \dots, p_{d+1})$ in the generalized Delaunay tessellation.

Fig. 1 represents the Voronoi diagram in the Euclidean metric and the corresponding Delaunay triangulation for four circles in the plane.

In this paper, we first establish the property that the number of inscribed spheres in the Euclidean metric for $d + 1$ spheres in general positions can be either two, one or zero. Following this, the conditions for the topological swap in the generalized Euclidean Voronoi diagram in d -dimensions are derived.

3 Dynamic Generalized Euclidean VD

3.1 Swap Operation in the Euclidean Metric

Consider the problem of maintaining the ordinary dynamic Voronoi diagram in the plane over some period of time. According to Roos [15], a dynamic Voronoi diagram undergoes two types of changes. The first type is a continuous deformation, where the locations of vertices and lengths of Voronoi edges can change, while proximity relationship between Voronoi sites do not change. The second type of change is the topological change, when Voronoi edges appear and disappear. The discrete moments of time when such change can happen are called topological events.

In order to detect such topological events the Delaunay triangulation is often used. Consider first the ordinary Voronoi diagram. When four moving sites in a quadrilateral, comprising two neighboring triangles of the Delaunay triangulation become cocircular, the corresponding edge of the Voronoi diagram gradually shrinks to zero and then a new edge appears. The corresponding diagonal in a quadrilateral in the Delaunay triangulation is flipped (this operation sometimes called a swap operation) and the future topological events for the newly created quadrilaterals are detected. The conditions for the topological event for Voronoi diagram in the Laguerre geometry and under the Euclidean distance function were established in [13,10]. The d -dimensional swap operation is described in

Fig. 2. The Dynamic Euclidean Voronoi diagram transformation

detail in [6,16].

Now, let us consider the topological event in the Euclidean metric. By the dynamic generalized Euclidean Voronoi diagram (referred to as the Euclidean Voronoi diagram in sequel) we mean the generalized Voronoi diagram in the Euclidean metric for a set of sites moving independently along the given trajectories. The topological event in the Euclidean metric can be illustrated on the following example. Consider a case when two circles P_1 and P_3 move towards each other along the straight-line trajectory in the direction shown by arrows (see Fig. 2(a)). Assume for simplicity that other two circles P_2 and P_4 remain in their spatial positions. At some moment of time t four circles become cocircular and the edge between sites P_2 and P_4 is reduced to zero. As the circles P_1 and P_3 continue to move toward each other, the new edge between sites P_1 and P_3 appears and its length increases with time (Fig. 2(b)). The topology of the Euclidean Voronoi diagram changes. The corresponding changes in the Euclidean Delaunay triangulation are shown in Fig. 2 by dashed lines.

The following conclusions can be drawn from the above discussion. First, for a finite set of sites S the topological structure of the Euclidean Voronoi diagram is locally stable, i.e. only continuous deformations take place, under sufficiently small continuous motion of sites. Secondly, the topological changes in the structure of the Euclidean Voronoi diagram are characterized by swaps of adjacent triangles (tetrahedrons) in the Delaunay triangulation (tessellation).

3.2 Dynamic Euclidean Voronoi Diagram Maintenance

The algorithm for maintenance of the Voronoi diagram for n circles, presented in [10], is now extended to handle the d -dimensional case.

1. (Preprocessing) Construct the Delaunay tessellation for the original site distribution. For every existing d -dimensional quadrilateral (quadrilateral formed by $d + 2$ neighboring spheres) in the Delaunay tessellation calculate

the next topological event. Insert all such events into the event priority queue sorted according to the time order.

2. (Iteration) Take the next topological event from the event queue. Update the Delaunay tessellation corresponding to the d -dimensional swap operation
3. Delete all topological events from the event queue, planned for no longer existing d -dimensional quadrilaterals.
4. Compute the new topological events for all new d -dimensional quadrilaterals and insert them into the event queue.

Preprocessing step requires $O(n^{\lceil d/2 \rceil + 1})$ time (see [6,16], for example). The swap operation takes $O(1)$ time, insertion to the queue in sorted order requires $O(\log n)$ time, and deletion from the queue takes $O(1)$ time when for each d -dimensional quadrilateral we store pointers on the events scheduled for this quadrilateral. The maximum size of the queue is $O(n^{\lceil d/2 \rceil})$, since there is at most one event scheduled at any moment of time for each quadrilateral. The space required to store the tessellation is $O(n^{\lceil d/2 \rceil})$. The total number of topological events depends on the trajectories of the moving sites and the elapsed time (upper bounds estimates were obtained for certain types of trajectories in [15]). The above is now summarized.

Lemma 1. The algorithm for maintenance of the Voronoi diagram for a set of sites in plane takes $O(n^{\lceil d/2 \rceil + 1})$ preprocessing time, $O(n^{\lceil d/2 \rceil})$ space and each topological event uses $O(d \log n)$ time.

To determine the time of the topological event, or the moment when $d + 2$ Voronoi sites are co-spherical, it is required to find the minimal root t_0 of the equation $INCIRCLE(P_1, P_2, \dots, P_{d+2}) = 0$. In the above, $P_i = P_i(t)$, $i = 1..d + 2$ are the coordinates of the $d + 2$ moving spheres, where t denotes the time. The form of the *INCIRCLE* function depends on the metric being used. If the value of the *INCIRCLE* function is positive then the empty-sphere condition is satisfied, when it is equal to zero, then the topological event occurs. It was shown that in the planar case the *INCIRCLE* function in power metric can be computed as a 4×4 determinant [9] and that in Euclidean metric it can be represented as a 6th degree polynomial of time [10]. Now, let us consider the generalization to d -dimensions under the Euclidean metric.

3.3 The Number of Inscribed Spheres in the Euclidean Metric

Let $P_i = \{\mathbf{p}_i = (x_{i1}, x_{i2}, \dots, x_{id}), r_i\}$, $i = 1..d+1$ be $d+1$ spheres in d -dimensional space. We will show how to obtain the coordinates and the radius of an inscribed sphere $C = \{\xi = (\xi_1, \xi_2, \dots, \xi_d), \rho\}$ and establish the number of such spheres. Let us first formally define the sphere inscribed among the $d + 1$ spheres in the d -dimensional space.

Definition 4. A sphere $C = \{\xi, \rho\}$ inscribed among $d+1$ spheres $P_1, P_2, \dots, P_{d+1}$ is a sphere with center $\xi = (\xi_1, \xi_2, \dots, \xi_d)$ and radius ρ , such that $\rho = d(\xi, P_1) = d(\xi, P_2) = \dots = d(\xi, P_{d+1})$.

Now, let us reduce the values of the radii of the $d+1$ spheres by the radius of the smallest sphere. Without loss of generality assume that the smallest sphere has index $d+1$. Let us define a coordinate system with the center of coordinates at point $\mathbf{p}_{d+1}$. Then the transformed coordinates of the given spheres are $P_i^* = \{\mathbf{p}_i^* = (x_{i1}^*, x_{i2}^*, \dots, x_{id}^*), r_i^*\}$, $i = 1..d$, where $x_{ij}^* = x_{ij} - x_{d+1,j}$, $i, j = 1..d$, and $r_i^* = r_i - r_{d+1}$. The last sphere is transformed into a point at the origin of coordinates. The unknown inscribed sphere coordinates will change to $\xi_j^* = \xi_j - x_{d+1,j}$, $j = 1..d$, and $\rho^* = \rho + r_{d+1}$.

We will use the fact that the coordinates of the inscribed sphere satisfy the equations

$$d(\xi^*, \mathbf{p}_i^*) = \rho^* + r_i^*, i = 1..d + 1. \quad (1)$$

Expanding the distance function, we get:

$$\begin{cases} (x_{i1}^* - \xi_1^*)^2 + (x_{i2}^* - \xi_2^*)^2 + \dots + (x_{id}^* - \xi_d^*)^2 = (\rho^* + r_i^*)^2, i = 1..d \\ (\xi_1^*)^2 + (\xi_2^*)^2 + \dots + (\xi_d^*)^2 = (\rho^*)^2 \end{cases} \quad (2)$$

The last equation can be subtracted from the remaining equations to cancel the quadratic terms

$$\begin{cases} 2x_{i1}^*\xi_1^* + 2x_{i2}^*\xi_2^* + \dots + 2x_{id}^*\xi_d^* + 2\rho^*r_i^* = w_i^*, i = 1..d \\ (\xi_1^*)^2 + (\xi_2^*)^2 + \dots + (\xi_d^*)^2 = (\rho^*)^2 \end{cases} \quad (3)$$

where $w_i^* = (x_{i1}^*)^2 + (x_{i2}^*)^2 + \dots + (x_{id}^*)^2 - (r_i^*)^2$, $i = 1..d$.

The solution for this system can be obtained by the following steps. The first d equations are linear in $(\xi_1, \xi_2, \dots, \xi_d, \rho)$. This linear system has d equations and $d+1$ variables. Denote the matrix of the system by $\mathbf{A}$, the column of unknowns by $\mathbf{x}$ and the right-hand side column by $\mathbf{b}$.

The following three cases are possible:

Case 1. $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) = d$.

This is the general case. The linear system can be resolved for d of the variables leaving one of the variables as a free parameter. To determine which variable is left free, a variable that can be moved into the right-hand side of the system must be found so that the determinant of the remaining system is non-zero. Thus, a non-zero $[d \times d]$ minor of matrix $\mathbf{A}$ must be found. Assume that ξ_k^* is left as a free variable. The remaining unknowns will all be linear functions of ξ_k^* . They can be substituted into the last equation, which turns into a quadratic equation for ξ_k^* . It can have two, one or no real solutions. Consequently, the following statement is true:

Lemma 2. The number of inscribed spheres in the Euclidean metric for a given $d+1$ spheres in d -dimensional space in general positions (i.e. $\text{rank}(\mathbf{A}) = d$) can be either two, one or zero.

Note that even though the system can have up to two solutions, only those where the radius of the inscribed sphere is positive should be selected. Also

Fig. 3. Infinite number of inscribed spheres.

Fig. 4. Linearly dependent spheres.

note that each of the inscribed spheres corresponds to a distinct Delaunay tetrahedron in the Delaunay tessellation.

Case 2. $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) < d$.

In this case, the linear system has an infinite number of solutions, and, consequently, infinitely many inscribed spheres. An example of such a system is given in Fig. 3.

Case 3. $\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{A}|\mathbf{b}) \leq d$.

In this case, the linear system has no solutions, and consequently, there are no inscribed spheres. An example of such a system is presented in Fig. 4.

Note that cases 2 and 3 both represent degenerate sphere arrangements, because the spheres are linearly dependent when $\text{rank}(\mathbf{A}) < d$.

3.4 Topological Event Computation

Now, let us obtain the formulas to compute the topological event in the Euclidean tessellation of moving sites. Let $P_i = \{(x_i = x_i(t), y_i = y_i(t)), r_i\}$, $i = 1..d + 2$ be a set of spheres with centers $(x_i(t), y_i(t))$ given by analytic functions of time and radii r_i .

Theorem 1. The time of the topological event in a Delaunay d -dimensional quadrilateral of $d + 2$ spheres $P_i = \{(x_i = x_i(t), y_i = y_i(t)), r_i\}$, can be found as the minimum real root t_0 of the equation

$$A_1^2 + A_2^2 + \dots + A_d^2 = A_{d+1}^2, \quad (4)$$

where

$$A = \begin{vmatrix} x_{11} - x_{d+2,1} & x_{12} - x_{d+2,2} & \dots & x_{1d} - x_{d+2,d} & r_1 - r_{d+2} \\ x_{22} - x_{d+2,1} & x_{22} - x_{d+2,2} & \dots & x_{2d} - x_{d+2,d} & r_2 - r_{d+2} \\ \dots & \dots & \dots & \dots & \dots \\ x_{d+1,1} - x_{d+2,1} & x_{d+1,2} - x_{d+2,2} & \dots & x_{d+1,d} - x_{d+2,d} & r_{d+1} - r_{d+2} \end{vmatrix}$$

and A_i is obtained by replacing the i -th column of A by the column $[w_j^*]_{j=1}^{d+1}$,

where $w_i = \sum_{j=1}^d (x_{ij} - x_{d+2,j})^2 - (r_i - r_{d+2})^2, i = 1..d + 1$.

Proof.

The *INCIRCLE* function for $d + 2$ spheres can be obtained by determining the coordinates of the inscribed sphere(s) for the first $d + 1$ spheres, and then computing the distance from the last sphere to the inscribed sphere(s):

$INCIRCLE(P_1, \dots, P_{d+2}) = \rho - d(\xi, P_{d+2})$. In the Euclidean metric, this formula is transformed to:

$$INCIRCLE(P_1, P_2, \dots, P_{d+2}) = \rho - \left(\sqrt{\sum_{i=1}^d (\xi_i - x_{d+2,i})^2 - r_{d+2}^2} \right), \quad (5)$$

where $(\xi_1, \xi_2, \dots, \xi_d, \rho)$ are the coordinates of the sphere inscribed among the spheres $P_1, P_2, \dots, P_{d+1}$.

The condition can be rewritten as the following system of equations:

$$d(\mathbf{p}_i, \xi) = r_i + \rho, i = 1..d + 2. \quad (6)$$

Performing transformations similar to those described above reduces the radii of all spheres by the radius of the smallest sphere (assume that this is the $(d + 2)$ -nd sphere). The origin of the coordinates is moved to the center of the smallest sphere. Denote $x_{ij}^* = x_{ij} - x_{d+2,j}$ and $d_i^* = d_i - d_{d+2}$.

Then the second-degree terms in the first $d + 1$ equations can be cancelled

$$\begin{cases} 2x_{i1}^* \xi_1^* + 2x_{i2}^* \xi_2^* + \dots + 2x_{id}^* \xi_d^* + 2\rho^* r_i^* = w_i^*, i = 1..d + 1 \\ (\xi_1^*)^2 + (\xi_2^*)^2 + \dots + (\xi_d^*)^2 = (\rho^*)^2 \end{cases} \quad (7)$$

In the above system, $w_i^* = (x_{i1}^*)^2 + (x_{i2}^*)^2 + \dots + (x_{id}^*)^2 - (r_i^*)^2, i = 1..d + 1$.

The first $d + 1$ equations represent a linear system

$$2 \begin{bmatrix} x_{11}^* & x_{12}^* & \dots & x_{1d}^* & r_1^* \\ x_{21}^* & x_{22}^* & \dots & x_{2d}^* & r_2^* \\ \dots & \dots & \dots & \dots & \dots \\ x_{d+1,1}^* & x_{d+1,2}^* & \dots & x_{d+1,d}^* & r_{d+1}^* \end{bmatrix} \begin{bmatrix} \xi_1^* \\ \dots \\ \xi_d^* \\ \rho^* \end{bmatrix} = \begin{bmatrix} w_1^* \\ w_2^* \\ \dots \\ w_{d+1}^* \end{bmatrix} \quad (8)$$

Assuming that the determinant of the linear system is non-zero, the system will always have a unique solution. The formulas for the center and the radii of the

sphere inscribed among the $d+2$ spheres with the modified radii can be explicitly written using Cramer's rule:

$$\xi_i^* = \frac{1}{2} \frac{A_i}{A}, i = 1..d; \rho^* = \frac{1}{2} \frac{A_{d+1}}{A}, \tag{9}$$

where, returning to the original coordinates,

$$A = \begin{vmatrix} x_{11} - x_{d+2,1} & x_{12} - x_{d+2,2} & \dots & x_{1d} - x_{d+2,d} & r_1 - r_{d+2} \\ x_{22} - x_{d+2,1} & x_{22} - x_{d+2,2} & \dots & x_{2d} - x_{d+2,d} & r_2 - r_{d+2} \\ \dots & \dots & \dots & \dots & \dots \\ x_{d+1,1} - x_{d+2,1} & x_{d+1,2} - x_{d+2,2} & \dots & x_{d+1,d} - x_{d+2,d} & r_{d+1} - r_{d+2}, \end{vmatrix}$$

and A_i is obtained by replacing the i -th column of A by the column $[w_j^*]_{j=1}^{d+1}$

$$w_i = \sum_{j=1}^d (x_{ij} - x_{d+2,j})^2 - (r_i - r_{d+2})^2, i = 1..d + 1.$$

Then the formulas for the center and radii of the inscribed sphere are substituted into the last quadratic equation from the system (7), arriving at the condition

$$A_1^2 + A_2^2 + \dots + A_d^2 = A_{d+1}^2. \tag{10}$$

The theorem is now proven.

The additional condition, requiring that the radius of the inscribed sphere must be positive, must be imposed as well.

$$\rho = \frac{1}{2} \frac{A_{d+1}}{A} - r_{d+2} > 0. \tag{11}$$

Note that all coordinates are analytical functions of time. When the spheres move by straight-line trajectories, the condition turns into an equation which is an 8th degree polynomial of time. As the spheres move with time, (10) and (11) can be written as

$$f(t) = 0 \tag{12}$$

$$g(t) > 0 \tag{13}$$

Note that the function $f(t)$ is in general non-zero. The first t_0 satisfying $f(t_0) = 0$ and the condition (13) represents the first topological event encountered. The complexity of the solution of (12) and (13) for t_0 clearly depends on the nature of the functions describing the movements of the circles. Even if these functions are linear, solving $f(t) = 0$ reduces to finding the zeros of a high degree polynomial of time. This will require an iterative numerical method, for example Newton's method.

4 Conclusion

Criterion for determination of the time of topological event in a Voronoi diagram for moving spheres in d -dimensions has been presented. The results are given in an algebraic form and can be applied to compute the dynamic generalized Voronoi diagram in the Euclidean metric.

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