Parallel Model Checking for the Alternation Free μ -Calculus

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Abstract. In this paper, we describe the design and implementation of a parallel model checking algorithm for the alternation free fragment of the μ -calculus. It exploits a characterisation of the model checking problem for this fragment in terms of two-person games. Our algorithm determines the corresponding winner in parallel. It is designed to run on a network of workstations. An implementation within the verification tool TRUTH shows promising results.

1 Introduction

Model checking is becoming more and more popular for the verification of complex hardware and software systems. These systems are usually given by a formal description which can be transformed into a transition system. A desired property of the system, on the other hand, is usually specified as a formula of a temporal logic. A model checking algorithm answers the question whether the (transition) system satisfies this property. Numerous case studies have shown that this approach improves the early detection of errors [3].

Despite the developments in the last years, the so-called *state space explosion* still limits its application. While *partial order reduction* [20] or *symbolic model checking* [17] reduce the state space by orders of magnitude, typical verification tasks still last days on a single workstation or are even (practically) undecidable due to memory restrictions (see for example [7]).

On the other hand, cheap yet powerful parallel computers can be constructed of Networks Of Workstations (NOWs). From the outside, a NOW appears as one single parallel computer with high computing power and, even more important, huge amounts of memory. This enables parallel programs to utilise the accumulated resources of a NOW to solve large problems. Hence, it is a fundamental goal to find parallel model checking algorithms which then may be combined with well-known techniques to avoid the state space explosion gaining even more speedup and further reducing memory requirements.

A famous logic for expressing specifications is Kozen's μ -calculus [11], a temporal logic offering boolean combination of formulae and, especially, labelled *next*-state, minimal, and maximal fixpoint quantifiers. For practical applications, however, it suffices to restrict the μ -calculus in order to gain tractable model checking procedures. The alternation free fragment, denoted by L_μ^1 , prohibits the nesting of minimal and maximal fixpoint operators. It allows the formulation of many *safety* as well as *liveness* properties. While this fragment is already

important on its own, it subsumes the logic CTL [6] which is employed in many practical verification tools. It can be shown that the model checking problem for this fragment is linear in the length of the formula as well as the size of the underlying transition system, and, starting with [4], several sequential model checking procedures are given in the literature (see [13] for a comparison). The algorithms can be classified into *global* and *local* algorithms. Global algorithms require the underlying transition system to be completely constructed while local algorithms compute the necessary part of a transition system *on-the-fly*. In plain words, global algorithms typically compute the fixpoints in an inductive manner while the local algorithms decide the problem by a depth-first-search. [13] compares the algorithms in detail.

Before starting to think about a concrete algorithm, we should consider its limitations, i.e. its complexity. In complexity theory, it is a well-accepted view that problems within the class NC admit a promising parallel computing algorithm. NC is based on the Boolean circuit model for computation and describes the problems computable in polylogarithmic time with polynomially many processors. It can be shown that NC is contained in P. Problems outside of NC are consequently considered to be inherently sequential. However, it is not known whether $NC=P$. If not, then especially P-complete problems cannot be in NC. Hence, we call P-complete problems *inherently sequential* [8].

We show that model checking L_μ^1 is inherently sequential, limiting our enthusiasm for finding a (theoretically) good parallel model checking algorithm. Even worse, depth-first-search is also P-complete, hence, promising parallel local algorithms are unlikely to exist [21]. Despite these theoretical limitations, we present a parallel global model checking algorithm. We implemented it within our verification tool TRUTH [14] and found out that it behaves very well for many practical problems.

Our algorithm is based on a characterisation of the model checking problem for this fragment in terms of two-person games due to Stirling [24]. Strictly speaking, we present a parallel algorithm for colouring so-called game graphs corresponding to the underlying model checking problem. This colouring answers the model checking problem and allows a derivation of a winning strategy. The latter may be employed by the user of a verification tool for debugging the underlying system interactively [24].

Another characterisation of this model checking problem can be given in terms of so-called 1-simple-weak-alternating-Büchi automata [12]. However, these correspond to game-based model checking [16]. Hence, our algorithm can also be used for checking the emptiness of these automata in parallel.

Until today, not much effort has been taken to consider parallel model checking algorithms. [25,1] present parallelised data structures which employ further computers within a network as a substitute for external storage. The algorithms described in [19,2] divide the underlying problem into several tasks. However, they are designed in the way that only a single computer can be employed to sequentially handle one task at a time. Stern and Dill [23] show how to carry out a parallel reachability analysis. The distribution of the underlying struc-

ture is similar to the one presented here. But their algorithm is not suitable for model checking temporal logic formulae. [9] presents a parallel reachability analysis algorithm for BDDs. They argue that many safety properties can be formulated as a reachability problem. In this way, their algorithm allows checking safety formulae. However, *liveness* properties which can be expressed within L_μ^1 are not supported. Furthermore, [10] argues that explicit state representation as well as BDDs have application domains in which they outperform the other one. Moreover, BDD-based algorithms generally do not provide counter examples, which are important in practice. Our main contribution is the first parallel model checking algorithm for L_μ^1 that supports interactive debugging.

The syntax and semantics of the μ -calculus are introduced in the next section. Furthermore, it is shown that L_μ^1 is inherently sequential. In Section 3, we describe model checking games for the μ -calculus and provide an important characterisation of the game graph which will be the basis for our parallel algorithm. Section 4 discusses our parallel model checking procedure and is followed by experimental results. We conclude by summing up our approach as well as giving directions for future work.

2 The μ -Calculus

In this section, we recall the syntax and semantics of the (modal) μ -calculus and its alternation free fragment. Furthermore, we show that model checking this fragment is inherently sequential.

2.1 Syntax and Semantics

Let Var be a set of variables and $\mathcal{A}$ a set of actions. Formulae of the modal μ -calculus over Var and $\mathcal{A}$ in positive form as introduced by [11] are defined as

$$\varphi ::= \mathbf{false} \mid \mathbf{true} \mid X \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid [K]\varphi \mid \langle K \rangle \varphi \mid \nu X.\varphi \mid \mu X.\varphi$$

where $X \in Var$ and K ranges over subsets of actions $\mathcal{A}$. Like [24], we allow sets of actions instead of single actions appearing in modalities.¹ It is a simple exercise to extend our approach towards the propositional μ -calculus (cf. Section 4).

A formula φ is *normal* if every occurrence of a binder μX or νX in φ binds a distinct variable, and no free variable X in φ is also used in a binder μX or νX . Every formula can easily be converted into an equivalent normal formula. If a formula φ is normal, every bound variable X of φ *identifies* a unique subformula $\mu X.\psi$ or $\nu X.\psi$ of φ where X is a free variable of ψ .

Let $\mathcal{T} = (S, T, \mathcal{A}, s_0)$ be a *labelled transition system* where S is a finite set of states, $\mathcal{A}$ a set of actions, and $T \subseteq S \times \mathcal{A} \times S$ denotes the transitions. As usual, we write $s \xrightarrow{a} t$ instead of $(s, a, t) \in T$. Furthermore, let $s_0 \in S$ be the *initial state* of the transition system. We employ valuations V mapping a variable X to a set of states $V(X) \subseteq S$. Let $V[X/E]$, $E \subseteq S$, be the valuation which is the same as V except for X where $V(X) = E$. Given a labelled transition system

¹ $\langle - \rangle \varphi$ is an abbreviation for $\langle \mathcal{A} \rangle \varphi$.

$\mathcal{T} = (S, T, \mathcal{A}, s_0)$ and a formula φ over Var and $\mathcal{A}$, the satisfaction of φ wrt. $\mathcal{T}$ and a state $s \in S$ is defined for **true**, **false**, disjunction, and conjunction in the usual way, and for the temporal and fixpoint operators as follows:

$$\begin{aligned} \mathcal{T}, s \models_V [K]\varphi &\text{ iff } \forall a \in K: \text{ if } s \xrightarrow{a} t \text{ then } \mathcal{T}, t \models_V \varphi \\ \mathcal{T}, s \models_V \langle K \rangle \varphi &\text{ iff } \exists a \in K: s \xrightarrow{a} t \text{ and } \mathcal{T}, t \models_V \varphi \\ \mathcal{T}, s \models_V \mu X. \varphi &\text{ iff } \forall E \subseteq S: \text{ if } s \notin E \text{ then } \exists t \in S: t \notin E \text{ and } \mathcal{T}, t \models_{V[X/E]} \varphi \\ \mathcal{T}, s \models_V \nu X. \varphi &\text{ iff } \exists E \subseteq S, s \in E: \text{ and } \forall t \in E: \mathcal{T}, t \models_{V[X/E]} \varphi \end{aligned}$$

We shorten $\mathcal{T}, s \models_V \varphi$ by $\mathcal{T}, s \models \varphi$ for a formula φ without any free variables and $\mathcal{T}, s_0 \models_V \varphi$ by $\mathcal{T} \models_V \varphi$ and use identifiers like $\varphi, \psi, \dots$ for formulae, $s, t, \dots$ for states, and $a, b, \dots$ for actions of the transition system under consideration. K denotes a set of actions. Whenever the sort of the fixpoint does not matter, we use σ for either μ or ν . For a formula of the μ -calculus, we introduce the notion of *subformulae* and *free* and *bound* variables as usual.

The *alternation free fragment* of the μ -calculus is that sublogic of the μ -calculus where no subformula ψ of a formula φ contains both a free variable X bound by a μX in φ as well as a free variable Y bound by a νY in φ .

Given a labelled transition system $\mathcal{T}$ and a formula φ , *model checking* is the problem whether $\mathcal{T}$ satisfies φ , i.e. whether $\mathcal{T} \models \varphi$. The *combined complexity* of the model checking problem is its complexity wrt. the product of the size of the transition system and the size of the formula. Its *program complexity* considers the complexity only wrt. the size of the transition system.

In [26], it was shown that the combined complexity for the alternation free μ -calculus is P-complete and, for a version of the alternation free μ -calculus employing two actions, that its program complexity is P-complete. [12] shows the latter result by using a formula with two propositions. We strengthen both results by employing neither propositions nor any action labelling.

Lemma 1. *The program complexity of the alternation free μ -calculus is P-hard.*

Proof. We reduce the P-complete *Game Problem* [8] to checking a formula of the alternation free μ -calculus wrt. a corresponding labelled transition system. A two player game is a tuple $G = (P_1, P_2, M, W_0, s)$. P_1 and P_2 , $P_1 \cap P_2 = \emptyset$, are positions, in which it is the turn of Player 1 or Player 2, resp. $M \subseteq (P_1 \times P_2) \cup (P_2 \times P_1)$ is the set of moves the respective player can make. $W_0 \subseteq P_1 \cup P_2$ denotes the *succeeding positions*, $s \in P_1$ the *starting position*. The players move alternately beginning with Player 1. We call $x \in P_1 \cup P_2$ *winning* iff either x is succeeding ($x \in W_0$), or $x \in P_1$ and there is a winning $y \in P_2$ such that $(x, y) \in M$, or $x \in P_2$ and for all $(x, y) \in M$, y is winning. The Game Problem is the question whether s is winning.

Corresponding to this, we define a transition system $\mathcal{T}_G = (P_1 \cup P_2, T)$ by $T = (M - \{(p, q) \in M \mid p \in W_0\}) \cup \{(p, p) \in P_1 \times P_1 \mid p \notin W_0 \text{ and there is no transition from } p \text{ in } M\}$.

T is defined in the way that every deadlock state, i.e. state with no outgoing edges, is a state of W_0 or a state of P_2 in which Player 2 is not able to move. Hence, deadlocks are winning. They can be characterised in the μ -calculus by $\varphi_{W_0} = [-]\mathbf{false}$ where a formula $[-]\varphi$ indicates that φ is satisfied in all successor

states. Further winning positions for Player 1 are states of P_1 such that there is a successor state (in P_2) whose direct successors (in P_1) are all winning. Hence, the formula $\varphi = \mu X.(\langle - \rangle[-]X \vee \varphi_{W_0})$ is satisfied in exactly those positions of P_1 which are winning where $\langle - \rangle\varphi$ guarantees the existence of a successor state in which φ holds. Note that φ may be satisfied in further positions of P_2 which does not bother us. We conclude that s is winning in the game $G = (P_1, P_2, M, W_0, s)$ iff $\mathcal{T}_G, s \models \varphi$. Observe that the construction of the transition system can be done in LOGSPACE. Note, we do not make use of propositions. Furthermore, we manage without actions at all by slightly adapting the modal fragment of our logic.

Together with a lineartime algorithm [12,13], we have the following theorem:

Theorem 1. *Model checking the alternation free μ -calculus is inherently sequential wrt. the combined complexity as well as the program complexity.*

3 Games for the μ -Calculus

Given a labelled transition system $\mathcal{T} = (S, T, \mathcal{A}, s_0)$ and a formula φ over Var and $\mathcal{A}$, we are able to define the *model checking game*. Its board is the Cartesian product $S \times Sub(\varphi)$ of the set of states and the set of subformulae. The game is played by two players, namely $\forall$ belard (the pessimist), who wants to show that $\mathcal{T}, s_0 \models \varphi$ does *not* hold, whereas $\exists$ loise (the optimist) wants to show the opposite.

The model checking game $G(s, \varphi)$ is given by all its *plays*, i.e. (possibly infinite) sequences $C_0 \rightarrow_{p_0} C_1 \rightarrow_{p_1} C_2 \rightarrow_{p_2} \dots$ of *configurations*, where $C_0 = (s, \varphi)$ and for all i , $C_i \in S \times Sub(\varphi)$ and p_i is either $\exists$ loise or $\forall$ belard. We write $\rightarrow$ instead of $\rightarrow_{p_i}$ if we abstract from the player. The players do not have to move alternately, instead, the next turn is determined by the current subformula of φ . Hence, the second component of a configuration C_i determines the player p_i who is to choose the next move. $\forall$ belard makes universal $\rightarrow_{\forall}$ -moves, $\exists$ loise makes existential $\rightarrow_{\exists}$ -moves. More precisely, whenever

1. $C_i = (s, \mathbf{false})$, then the play is finished.
2. $C_i = (s, \psi_1 \wedge \psi_2)$, then $\forall$ belard chooses $\varphi = \psi_1$ or $\varphi = \psi_2$, and $C_{i+1} = (s, \varphi)$.
3. $C_i = (s, [K]\psi)$, then $\forall$ belard chooses $s \xrightarrow{a} t$ with $a \in K$ and $C_{i+1} = (t, \psi)$.
4. $C_i = (s, \nu X.\psi)$, then $C_{i+1} = (s, \psi)$.
5. $C_i = (s, \mathbf{true})$, then the play is finished.
6. $C_i = (s, \psi_1 \vee \psi_2)$, then $\exists$ loise chooses $\varphi = \psi_1$ or $\varphi = \psi_2$, and $C_{i+1} = (s, \varphi)$.
7. $C_i = (s, \langle K \rangle \psi)$, then $\exists$ loise chooses $s \xrightarrow{a} t$ with $a \in K$ and $C_{i+1} = (t, \psi)$.
8. $C_i = (s, \mu X.\psi)$, then $C_{i+1} = (s, \psi)$.
9. $C_i = (s, X)$ and X identifies φ , then $C_{i+1} = (s, \varphi)$.

As the moves 1,4,5,8 and 9 are deterministic, no player needs to be charged with them. With regard to the winning strategies and the algorithm, we will speak of *$\forall$ belard-moves* in cases 1-4 and 9 if $\sigma = \mu$, and *$\exists$ loise-moves* in all other cases. C_i is called $\forall$ -configuration or $\exists$ -configuration, respectively.

$\forall$ belard wins a play G , iff

- $G = C_0 \rightarrow \cdots \rightarrow C_n$ and $C_n = (s, \mathbf{false})$ for any state s .
- $G = C_0 \rightarrow \cdots \rightarrow C_n$ and $C_n = (s, \langle K \rangle \varphi)$ and $\nexists t : s \xrightarrow{a} t$ for any $a \in K$.
- $G = C_0 \rightarrow \cdots$ has infinite length and the outermost fixpoint which is unwinded infinitely often is a μ -fixpoint.

Dually, Eloise wins a play G , iff a configuration with the formula **true** is reached, $\forall$ belard gets “stuck”, or the outermost fixpoint which is unwinded infinitely often is a ν -fixpoint [24].

Please note that, given a transition system and a formula, there are several plays and these not necessarily have the same winner.

To characterise plays, we introduce the notion of witnesses and judgements. Configurations C in which no move is possible are called *judgements*. A judgement is further called $\exists$ -judgement ($\forall$ -judgement) iff it is an $\exists$ -configuration ($\forall$ -configuration). For finite plays, it is obvious that Eloise ($\forall$ belard) wins the play iff it contains an $\exists$ -judgement ($\forall$ -judgement).

Configurations of the form (s, X) where s is any state of the transition system and X is a variable are called *witnesses* since in the following move, X is unwinded. If X is bound by a μ -quantifier, it is called an $\forall$ -witness, otherwise $\exists$ -witness. Witnesses have a natural partial order given by the nesting within the originating formula. A witness (s, X) is less than (s, Y) iff $\sigma X.\varphi(X, Y)$ is a subformula of $\sigma'Y.\psi(X, Y)$ where $\eta(X, Y)$ means that η may contain the free variables X, Y . For infinite plays, it is easy to see that it has a unique maximal witness and that the winning condition can be formulated as *$\forall$ belard wins iff this witness is an $\forall$ -witness, Eloise wins iff it is an $\exists$ -witness*. Please note that no configuration can be a judgement as well as a witness.

A *strategy* is a set of rules for a Player p telling her or him how to move in the current configuration. A *winning strategy* now guarantees that the play which p plays regarding the rules will be won by p . [24] shows that the model checking problem for the μ -calculus is equivalent to finding a winning strategy for one of the players: Let $\mathcal{T}$ be a transition system with starting state s , and let φ be a μ -calculus formula. $\mathcal{T}, s \models \varphi$ implies that Eloise has a winning strategy starting at (s, φ) , and $\mathcal{T}, s \not\models \varphi$ implies that $\forall$ belard has a winning strategy starting at (s, φ) . Since a formula either holds or is falsified, this result also implies that model checking games are *determined*, i.e., for every game, either $\forall$ belard or Eloise has a winning strategy.

All possible plays for a transition system $\mathcal{T}$ and a formula φ are captured in the *game graph* whose nodes are the elements of the game board (the possible configurations) and whose edges are the players' possible moves. The game graph can be understood as an *and/or*-graph where the *or*-nodes (denoted by $\vee$) are $\exists$ -configurations and *and*-nodes (denoted by $\wedge$) are $\forall$ -configurations. Furthermore, the notion of witnesses and judgements carries over without any modification. A play corresponds to a path in the game graph and vice versa.

In the following, we concentrate on the alternation free μ -calculus. The following characterisation of the game graph for this fragment is useful for formulating a sequential algorithm and essential for our parallel algorithm.

Theorem 2. *Let $\mathcal{T}$ be a labelled transition system and φ a formula of the alternation free μ -calculus. Furthermore, let (Q, E) be their game graph. Then there exists a collection of $Q_1, \dots, Q_m$ such that the following holds:*

1. *the collection of the Q_i is a partition of Q , i.e., $Q = \bigcup_{i \in \{1, \dots, m\}} Q_i$ and for all $i, j \in \{1, \dots, m\}$ with $i \neq j$, it holds $Q_i \cap Q_j = \emptyset$.*
2. *The subgraph induced by Q_i is exactly one of*
 - (a) *a non-trivial maximal strongly connected component (Type I).²*
 - (b) *a singleton containing a judgement (Type II).*
 - (c) *a maximal directed acyclic graph without any judgements (Type III).*
3. *every Q_i of Type I either contains at least one $\exists$ -witness and no $\forall$ -witnesses or contains at least one $\forall$ -witness and no $\exists$ -witnesses.*
4. *there is a partial order $\leq$ on the Q_i 's such that for every $q \in Q_i$ and $q' \in Q_j$ with an edge from q to q' , we have $Q_j \leq Q_i$. Thus, moves from a configuration in Q_i lead to configurations in either the same Q_i or a lower Q_j .*

Proof. The proof is inspired by a characterisation of the game graph in terms of weak alternating automata [16]. First, consider the nodes of maximal non-trivial strongly connected components. These only occur because of unwinding fixpoint formulae. Hence, they contain witnesses. Alternation freeness now guarantees that these are all of the same kind, i.e. either $\exists$ -witnesses or $\forall$ -witnesses. Second, consider the leaves of the game graph, i.e. configurations without outgoing edges. These cannot be members of strongly connected components of Type I. By definition, every such Q_i is a judgement. Third, it is now easy to see that all remaining nodes belong to directed acyclic graphs which do not contain any leaves of the original game graph. Please note that maximality of the strongly connected components guarantees the order defined to be a partial order.

To prove our parallel algorithm to be free from deadlock, we need the following insight which holds since on every cycle, a fixpoint formula is unwinded.

Proposition 1. *Every strongly connected component of a game graph with more than one element contains at least one witness.*

Let us sketch a sequential algorithm deciding which player has a winning strategy [12]. It labels a configuration q by *green* or *red*, depending on whether Eloise or Vbelard has a winning strategy for the game starting in this configuration q . It will be extended to a parallel version in the next section.

Let us consider a game graph. By Theorem 2, there exists a partition of its nodes Q into disjoint Q_i of Type I–III, and every Q_i of Type I either contains $\exists$ -witnesses or $\forall$ -witnesses. Also, there exists a partial order $\leq$ on the Q_i such that for $q \in Q_i$ and $q' \in Q_j$ for which there is a possible move from q to q' , we have $Q_j \leq Q_i$. As seen before, every infinite play gets trapped within a single Q_i , and the winner depends on whether Q_i contains $\exists$ -witnesses or $\forall$ -witnesses. By Prop. 1, every infinite play visits such a witness infinitely often.

The game graph can be coloured by processing all Q_i upwards according to the partial order. To make the algorithm deterministic, enlarge the partial order

² A component is called non-trivial if it contains a least two nodes.

on the Q_i to a total order. Let Q_i be minimal wrt. $\leq$. Then it is of Type I or Type II. Furthermore, from any configuration of Q_i , every move leads to Q_i . If in Q_i there is an $\exists$ -witness or Q_i consists of an $\exists$ -judgement, all its nodes are labelled with *green*, otherwise *red*.

Once a configuration $q \in Q_i$ is labelled with *red* or *green*, its predecessors are labelled if possible. That means, an $\wedge$ -node is labelled with *red* if q is *red*, but labelled *green*, if all successors are *green*. An $\vee$ -node is treated dually. Furthermore, if a node could be labelled, its outgoing edges are erased. Such labelling is propagated further.

Let Q_j be the next set of configurations wrt. the total order. Then, all configurations in $Q_i \leq Q_j$ are already coloured by either *red* or *green*. If Q_j is of Type III, all of its configurations must be labelled due to the propagation described before. For a set of Type I, some unlabelled configurations might remain. These are labelled according to the type of witnesses in Q_j , i.e. with *red* if Q_j has $\forall$ -witnesses, otherwise with *green*.

Fig. 1. State graph (a) and its partition (b)

Let us consider Figure 1. Part (b) is a partition of the game graph shown in (a). It contains components of types I, II, and III. Q_1 is a directed acyclic graph, Q_2 a non-trivial maximal strongly connected component, and Q_3 is a singleton containing a $\exists$ -judgement (denoted by $+$). The components can be ordered like $Q_3 < Q_2 < Q_1$. Since the minimal Q_3 contains an $\exists$ -judgement, it will be labelled with *green*. The $\vee$ -node from Q_2 to Q_3 and subsequently all nodes will therefore be labelled with *green* as well. Note how the $\forall$ -witness (denoted by $-$) has no influence here, since all nodes are coloured already.

4 Parallel Model Checking

Given a transition system and an L_μ^1 -formula, our approach is both to construct the game graph as well as to determine the colour of its nodes in parallel.

4.1 Distributing the Game Graph

It is obvious that the construction of the game graph can be carried out in parallel by a typical breadth-first strategy. Given a node q , determine its successors $q_1, \dots, q_n$. Now, the successors can be processed in the same manner in parallel.

However, to obtain a terminating procedure, only exactly the q_i *not* processed before must be expanded. All states generated have to be stored within the NOW, and load sharing must be guaranteed. On a shared memory architecture, this does not involve big conceptual problems. For distributed memory machines, however, this is a little bit more difficult.

A first idea might be to distribute the first $q_1, \dots, q_n$ to the first n processors, and these process the q_i as described before and distribute the successors to the next processors. However, deciding whether a q_i was processed before becomes an expensive operation. Every processor could have processed q_i and should therefore be consulted. In the worst case, for every node, such a broadcast is required. This yields no reasonable algorithm.

A different, often-employed way to store graphs on a distributed memory machine is to divide the graph's adjacency matrix $M \in \{0, 1\}^{|V| \times |V|}$ into equal sized *blocks* and to store each block on a single processor [22]. This has several advantages. First, the blocks of the matrix can be generated in parallel. Second, given nodes $p, q \in V$, it is easy to check whether there is an edge from p to q , i.e. whether $M_{p,q} = 1$. Since there is a unique location for the block of the matrix containing the value for the pair (p, q) , a single communication is needed. Third, every processor gets the same amount of data.

For our problem, this approach cannot be used. The number of nodes of our graph is unknown a priori but computed while constructing the graph.³ Hence, the partition of the game graph into blocks cannot be determined in advance.

We propose the following way to construct and store the graph which is inspired by the work pool presentation of [15] and is similarly applied in [23]. Let f be a function mapping the states of the game graph to a processor of our network. Usually, one takes a function in the spirit of a hash function assigning to every state an integer and subsequently its value modulo the number of processors. Then, f determines the location of every state within the network uniquely and without global knowledge. In a breadth-first manner, starting with the initial state q_0 of the game graph, the state space can be constructed in parallel with the help of f in the following way. Given a state q (and possibly some of its direct predecessors), send it to its processor $f(q)$. If q is already in the local store of $f(q)$, then q is reached a second time, hence the procedure stops. If predecessors of q were sent together with q , the list of predecessors is augmented accordingly. If q is not in the local memory of $f(q)$, it is stored there together with the given predecessors as well as all its successors $q_1, \dots, q_k$, the states within the formula $\delta(q, a)$ which are computed. These are sent in the same manner to their (wrt. f) processors, together with the information that q is a direct predecessor. The corresponding processes update their local memory similarly.

4.2 Labelling the Game Graph

Given the game graph, a first possibility for labelling the nodes with *red* and *green* would be to apply a depth-first-search as done by sequential algorithms.

³ In the context of model checking, the transition system is not given explicitly but expanded at run-time from a formal system description.

However, since this problem is P-complete [21], there is no hope to get a suitable parallel algorithm by adapting the ideas of depth-first-search, and there is no reason to do this.

Another possibility for labelling the nodes with *red* and *green* is applying typical algorithms for generating strongly connected components of our game graph in parallel which will be labelled in a second step [22]. However, labelling a connected component requires the knowledge whether the component can be “successfully abandoned” (cf. Section 3). It is neither clear how to obtain this information from the graph computed by these algorithms nor to modify these algorithms in the way that this information can be gathered easily.

We therefore propose the following method. The parallel colouring process is carried out *speculative*. For example, given an $\exists$ -witness q , it determines an accepting component. This node is coloured *green* unless there is an $\wedge$ -node with an edge to a lower component which is coloured *red*. The parallel algorithm, however, labels the witness q with *green*. Furthermore, a notification is sent to its direct predecessors $q_1, \dots, q_k$. This notification tells each q_i that q changed its colour. Hence, they recompute their own colour according to the following obvious rule: If q_i is an $\vee$ -node, then it is labelled with *green* if one of its successors is *green*. If all successors are *red*, it is labelled with *red*. Otherwise, some successors have no label yet and no colour is assigned to the current node. For an $\wedge$ -node, the dual is carried out. If the colour of q_i has changed, it sends a notification to its predecessors where the same procedure starts again. Otherwise, the procedure is done. It is clear that the predecessors can be processed in parallel. The whole algorithm stops if all notifications are processed.

Theorem 3. *The algorithm described before labels a node (s, ψ) of the game graph with green if $\mathcal{T}, s \models \psi$. Otherwise, the node is labelled with red.*

Proof. We give a sketch of the proof. The *termination* of the algorithm can easily be seen by recalling that the game graph can be divided into components as aforementioned which are partially ordered. Labelling notifications are sent either within the current component or propagated to a higher one (wrt. the partial order). Since the colour operation is monotone (in the obvious sense), only a finite number of labelling notifications is generated.

When the algorithm terminates, the game graph is entirely labelled (*completeness*), because the components are either leaves (in which case they are labelled as described before), they contain a witness (cf. Proposition 1), in which case the speculative part of the algorithm jumps in and starts the labelling of this component, or completely depend on a lower component.

The sequential algorithm labels the nodes with a correct colour by processing the mentioned components according to the partial order. The labelling remains *correct* if the labelling of higher (wrt. the partial order) components is done speculative and corrected as soon as the correct colour of the lower component is determined. Note that the colour of *leaves* and *leaf components* (i.e. components which are minimal for the partial order) are correctly labelled from the beginning.

It should be mentioned that for an implementation, the two steps of constructing the game graph and labelling the nodes are carried out concurrently

(cf. Section 4.3). The combined algorithm stops if no further labelling or state expansion steps have to be processed. For this task, we employ the DFG token termination algorithm, as presented in [5]. Due to space constraints, we will not go into the details of this algorithm.

4.3 The Algorithm

To describe our approach in more detail, we show an algorithm in *pseudo code* which combines the task of constructing and labelling the game graph in parallel. The termination check is omitted to simplify the presentation. This algorithm runs on every machine within the NOW. Note that initially one machine has to send the root of the game graph to its processor to start the procedure.

Let us consider Figure 2. Each processor's part of the graph is stored there in a relational structure consisting of one tuple for every node already processed of the form (**node**, **colour**, **preds**, **succs**) where **node** is the node together with its **colour**, its predecessors **preds**, and its successors together with their colours (**succs**, line 3). We introduce the colour *white* to denote unlabelled nodes. In line 7, a message is received. If the message requests to expand a given node, it is checked whether this node has already been processed before (line 10). If so, it might have a colour (not equal to *white*) which will be propagated to the new predecessors (lines 12–13). In any case, new predecessors are stored (line 11). If the delivered node has not been processed before (line 14), its successors are computed (with an initially *white* colour) and propagated to the corresponding processors (lines 15–18). Furthermore, it must be checked whether a labelling process must be initiated, i.e. whether the current node fulfils the requirements for being either an $\exists$ - or an $\forall$ -witness. If so, all predecessors are informed about the current node's colour (lines 19–25).

The second type of messages which are received are colouring messages (line 26) informing that a node's child has changed its colour. The current settings for *node* are extracted (line 27), and the new colour of the corresponding child is stored (line 28). If now the evaluation of the node's colour according to Section 4.2 yields a different colour than the old one, then all predecessors are informed in the previous way (lines 29–33).

It is easy to see that the space required by our algorithm is linear in the size of the game graph. The worst case run-time, however, is a factorial of its size. This case might turn up when the components of the game graph are of Type II, linearly ordered and alternating, i.e. Q_i contains an $\exists$ -witness and Q_{i+1} contains an $\forall$ -witness. Now, in every component a (speculative) recolouring up to the maximal (root) component may be initiated.

Since we already observed that the model checking problem for the considered fragment of the μ -calculus is unlikely to be in NC, we were warned that a parallel algorithm might not be optimal in every case. Despite these theoretical limitations, in practice the behaviour turns out to be feasible (cf. Section 5). An explanation for this fact is that the aforementioned kinds of game graphs require formulae with deeply nested fixpoints, which rarely occur as typical specifications.

```

1  Process P
2
3  // graph : Node  $\longrightarrow$  (Colour  $\times$  [Node]  $\times$  [(Node, Colour)])
4
5  begin
6    until hasTerminated do
7      msg  $\leftarrow$  readMessage;
8      case msg of
9        Expand node pred:
10         if (node, colour, oldpreds,  $\_$ ) in graph then
11           addPreds(node, preds)
12           if colour  $\neq$  White and pred  $\notin$  oldpreds then
13             sendMessageTo (Colour pred node colour) f(pred)
14         else
15           succs  $\leftarrow$  computeSuccs(node)
16           succss  $\leftarrow$  [ ( s, White) | s in succs ]
17           for s  $\in$  succs do
18             sendMessageTo (Expand s [node]) f(s)
19           colour  $\leftarrow$  case
20             node is  $\exists$ -witness or  $\exists$ -judgement: Green
21             node is  $\forall$ -witness or  $\forall$ -judgement: Red
22             else: White
23           addGraph (node, colour, preds, succss)
24           if colour  $\neq$  White
25             sendMessageTo (Colour pred node colour) f(pred)
26         Colour node child colour:
27           (node, oldcolour, preds, succs) in graph
28           updateSucc(succs, child, colour)
29           newcolour  $\leftarrow$  computeColour(node, succs)
30           if newcolour  $\neq$  oldcolour then
31             updateGraph(node, newcolour)
32           for p  $\in$  preds do
33             sendMessageTo (Colour p node newcolour) f(p)
34     end
35
36  Function computeColour (node, succs)
37  begin
38    case
39      node is  $\vee$ -node:
40        case
41          all (= Red) succs: Red
42          any (= Green) succs: Green
43          else: White
44      node is  $\wedge$ -node:
45        case
46          all (= Green) succs: Green
47          any (= Red) succs: Red
48          else: White
49  end

```

Fig. 2. A parallel construction and labelling algorithm

Let us consider the graph in Figure 3 as a simple example. It suggests that the distribution function f will map the given nodes onto two processors as shown. Starting with node 1, its successors (2, 5) are computed and sent to processors p_1 and p_2 , resp. Now, 2 and 5 can be expanded in parallel with the effect that nodes 3 and 6 are sent to p_2 . The successors of 3 are 5 and 4 which are delivered to p_2 . Since 6 is an $\forall$ -witness (indicated by $-$), it is labelled with *red*, initiates a relabelling of 5, and issues an expand-2-message to p_1 , which notices that 2 is already expanded and registers 6 as one of 2's predecessors. p_2 carries on with expanding 4, noticing that it is an $\exists$ -judgement (indicated by $+$), and sends this information to 3. Next, the *red*-labelling of 5 is propagated to 3 and 1. However, 3 can now determine *green* as its colour and sends it to its predecessor. 2 propagates *green* to 1 and 6. Finally, the whole graph is labelled *green*.

Fig. 3. A partition

5 Experimental Results

We have tested our approach within our verification platform TRUTH [14]. We implemented the distribution routine on its own as well as the combined labelling routine described in the previous section. As implementation language we have chosen the purely functional programming language Haskell⁴, which enabled us to embed this algorithm in the infrastructure of our verification tool TRUTH and also to prototype a *concise* reference implementation. The actual Haskell source code of the algorithm has less than 300 lines of code. The communication layer of our implementation is based upon MPICH⁵, an implementation of the MPI (Message Passing Interface) standard.

Now we will show some results we achieved when verifying properties of certain system specifications. Figures 4 and 5 show the measured results of state distribution and the speedup when running our implementation on a NOW consisting of up to 52 processors and a total of 13GB main memory. They are connected with a conventional 100MBit Fast-Ethernet network.

The distribution routine shows that our approach is very well suited for constructing large game graphs. We were able to construct graphs with several hundred thousands of states within minutes. The game graph of the largest example we have constructed so far, a quad-parallel instance of the Alternating Bit Protocol [18], consists of more than 1.6 million states, and we get a homogeneous distribution of the state space on the workstations (Figure 4). The distribution depends on our hash function f , and the results are quite good compared to its simplicity. In fact, all our measurements showed similar results, provided the size of the graph is reasonably larger than the number of used workstations.

Our approach also scales very well with regard to the overall runtime (Figure 5). Unfortunately, because of the size of the game graphs we inspected, we

⁴ <http://haskell.org/>

⁵ <http://www-unix.mcs.anl.gov/mapi/mpich/>

Fig. 4. state distribution

Fig. 5. runtime results

did not get results when running the algorithm on less than five workstations due to memory restrictions. Therefore the shown speedups are calculated relative to 5 processors instead of one. We found that we gain a linear speedup for reasonably large game graphs (in fact, for graphs with more than 500.000 states we even got *superlinear* speedups, which we will discuss later). The results are especially satisfying, if one considers that—for reasons of simplicity—we did not try to employ well-known optimisation means, for example reducing the communication overhead by packing several states into one message.

Due to our choice of Haskell as implementation language and its inherent inefficiency, we did not focus on optimising the internal data structures either. We use purely functional data structures like balanced trees and lists rather than destructively updateable arrays or hash tables. This is also the reason for the superlinear speedups we remarked before. We found that the overhead for insertions and lookups on our internal data structures dramatically increases with the number of stored states. We verified this by running all processes on a *single processor* in parallel and replacing the network message passing with inter-process communication. The expected result would have been to find similar runtimes as one process would achieve in this situation, or even slightly worse due to operating system context switches between the processes running in parallel. But we found that there is a significant speedup because the internal data structures are less crowded so that lookups and insertions are considerably cheaper.

Comparing our approach to the implemented sequential game-based depth-first-search model checking algorithm for L_μ^1 , we have to learn that it is not possible for small examples to beat it. There are two reasons for that. In many cases, a formula can be proven or falsified by considering only a part of the game graph. Even parallel power does not outperform in these cases. Second, the communication between processors is dramatically slower within a NOW compared to accessing memory. Hence, as long as a problem fits into main memory, it is difficult to beat a sequential algorithm by a parallel one running on a NOW.

The situation changes completely when most of a huge game graph has to be checked for proving a formula. This situation arises for example in the frequent case that a *NoDeadlock* formula is considered. To check whether a system description contains any deadlock requires the whole game graph to be analysed.

For large systems (several hundred thousand states), the parallel version beats the sequential one. More important, we were able to verify certain systems with the help of our parallel algorithm while the sequential failed due to memory restrictions.

6 Conclusion

In this paper, we presented a *parallel* game-based model checking algorithm for an important fragment of the μ -calculus. The demand for parallel algorithms becomes visible by considering the memory and run-time consumptions of sequential algorithms. Since the employed fragment of the μ -calculus subsumes the well-known logic CTL, it is of high practical interest. We have implemented our approach within the verification platform TRUTH. Systems with a million of states could be constructed within half an hour on a NOW consisting of up to 52 processors. We found out that the algorithm scales very well wrt. run-time and memory consumption when enlarging the NOW. Furthermore, the distribution of states on the processors is homogeneous.

Compared to the also implemented on-the-fly sequential model checking algorithm for L_μ^1 , we learned that for simple examples a parallel *global* algorithm cannot outperform a local one. This is especially true for formulae like $\mathbf{true} \vee \varphi$ being checked, where $\varphi = \nu X. \Phi$ yields a considerably big part of the resulting game graph. A local algorithm would be able to almost instantaneously present a solution, since the formula is dominated by an $\exists$ -judgement ($\mathbf{true}$).

To improve our algorithm for such cases, we head towards an “almost local” variant, which not only uses two colours but *colour weights*, with which e.g. the propagation of *safe* colours (resulting from minimal components Q_i) can be tracked better. So we eventually short-circuit the colouring process.

However, considering real world specifications yielding millions of states, even the here presented parallel algorithm gains the upperhand. Answers are computed more quickly, and, more important, there are numerous cases in which the sequential algorithm fails because of memory restrictions and the parallel version is able to prove a formula. From the practical point of view, it is a central feature of a verification tool to give an answer in as many cases as possible.

While our approach is already of practical interest since it allows to check larger systems, it should also be considered as a further attempt to develop parallel model checking algorithms. More research should be carried out in this direction. Especially, on-the-fly model checking and partial order reduction [20] should be analysed with respect to parallelisation. Furthermore, different (especially non-P-complete) specification logics might provide better parallel model checking algorithms.

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