Transforming Pseudo-Triangulations

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Abstract. We study the problem of transforming pseudo-triangulations in the plane. We show that a pseudo-triangulation with n vertices can be transformed into another one using $O(n \log n)$ flips only. This improves the previous bound $O(n^2)$ [3]. We present an algorithm for computing a transformation between two pseudo-triangulations in $O((f + n) \log n)$ time where f is the number of flips.

1 Introduction

Pseudo-triangles and pseudo-triangulations have received much attention recently because of applications in visibility [8, 9], ray shooting [8], collision detection [7], and rigid motion [4, 10].

A pseudo-triangle is a simple polygon with exactly three convex vertices. For a set S of n points in the plane, a *pseudo-triangulation* T is defined as partition of the convex hull of S into interior disjoint pseudo-triangulations whose vertices are points of S (each point of S is a vertex of T and vice versa). A *minimum pseudo-triangulation* is a pseudo-triangulation with the least number of edges among all pseudo-triangulations of S .

One of the operations on triangulations and pseudo-triangulations is the *flip*, see Fig. 1. Flips in pseudo-triangulations are used for enumeration all pseudo-triangulations defined by a set of vertices [1, 3]. These algorithms imply that a pseudo-triangulation with n vertices can be transformed to any other using at most $O(n^2)$ flips.

In this paper we show that any two pseudo-triangulations with the same set of n vertices can be transformed into each other by $O(n \log n)$ flips only. We present a simple algorithm for finding a sequence of flips based on our proof. The algorithm computes flips in $O((f + n) \log n)$ time where f is the number of flips.

Interestingly, the flip distance between pseudo-triangulations is smaller than one between triangulations. Hurtado *et al.* [6] show that there are $2n$ points in the plane and two triangulations that require more than $(n - 1)^2$ flips to transform one into the other.

2 Preliminaries

Let $\text{conv}(A)$ denote the convex hull of a set A in the plane. We classify vertices of a polygon in the plane: *convex vertices* whose internal angle is less than π

Fig. 1. a) Flip of triangles. b) Flip of pseudo-triangles.

and *reflex vertices* whose internal angle is greater than π . A *pseudo-triangle* is a simple polygon with three convex vertices. A sequence of pseudo-triangle edges between two convex vertices is called a *side* of the pseudo-triangle.

We assume that S is a set of n points in the plane in general position (no three collinear points). Let T be a pseudo-triangulation and e be an edge of T whose flip introduces an edge e' . The edge e' is called a *dual edge of E* , see Fig. 1 b).

An edge e of a pseudo-triangle Δ is called *cross edge for a vertex a* if e is on the side bc and a point of e is visible from a . Note that Δ can have many cross edges for a .

Lemma 1. *Let abc be a pseudo-triangle of a pseudo-triangulation T and let e be a cross edge for a . Let Δ be a pseudo-triangle with vertex a obtained by flipping e . The dual edge of e is not a cross edge of a in Δ .*

Proof. First we note that there can be two pseudo-triangles incident to a after flipping e , see, for example, Fig. 1 a). In this case the dual edge of e (which is the common edge of two new pseudo-triangles) is incident to a and is a cross edge in either pseudo-triangle.

In general, let bcd be the pseudo-triangle that contains the edge e . The union of pseudo-triangles abc and bcd is a *pseudo-quadrangle*, a simple polygon with exactly four convex vertices. The dual edge of e is a part of geodesic path ad in the pseudo-quadrangle $abcd$, see, for example, Fig. 1 b). The lemma follows.

3 Canonical Pseudo-Triangulations

In this Section we introduce *canonical pseudo-triangulations* and show how a pseudo-triangulation can be transformed into a canonical one.

Definition 1 Let S be a set of n points in the plane and let p be a vertex of the convex hull of S . Let $\sigma(p) = (p_1, p_2, \dots, p_n)$ be a sorted sequence of points such that

- (i) $p_1 = p$, and
- (ii) the points $p_2, \dots, p_n$ are sorted in clockwise order around p_1 , and
- (iii) $p_1 p_2$ and $p_1 p_n$ are the edges of the convex hull, see Fig. 2.

A canonical pseudo-triangulation with center p , denoted by $\mathcal{T}(p, S)$, is defined recursively as follows. If $n = 3$ (if $n = 1$ or $n = 2$ $\mathcal{T}(p, S)$ has no pseudo-triangles) then $\mathcal{T}(p, S)$ consists of one triangle $p_1 p_2 p_3$. Suppose that $n \geq 4$. Let $m = \lceil n/2 \rceil$ and $S_1 = \{p_1, p_2, \dots, p_m\}$ and $S_2 = \{p_1, p_{m+1}, p_{m+2}, \dots, p_n\}$. $\mathcal{T}(p, S)$ contains the pseudo-triangles of $\mathcal{T}(p, S_1)$ and $\mathcal{T}(p, S_2)$ and a pseudo-

Fig. 2. Canonical pseudo-triangulation with center p_1 .

triangle that remains if $\text{conv}(S_1)$ and $\text{conv}(S_2)$ is removed from $\text{conv}(S)$, see Fig. 2.

Theorem 2. Let S be a set of n points in the plane in general position and let p be a point of S . Let T be a pseudo-triangulation of S . The canonical pseudo-triangulation with center p can be obtained from T using $O(n \log n)$ flips.

Proof. We apply flips to T and construct the canonical pseudo-triangulation $\mathcal{T}(p, S)$ as follows. If S contains three points then S then they define the unique triangle and T contains it. Suppose that S has at least four points. We consider the sets S_1 and S_2 from Definition 1. Let $P = \text{conv}(S)$ and $P_i = \text{conv}(S_i)$ where $i = 1, 2$. Let Δ be the pseudo-triangle $\Delta = P \setminus P_1 \cup P_2$, see Fig. 3. Let p_{i_1} and p_{i_2} be two convex vertices of Δ where $i_1 < i_2$.

Fig. 3. Edges $e_1, e_2, \dots, e_l$.

If an edge e of the pseudo-triangulation T has both endpoints in S_1 (or S_2) then e does not intersect Δ . Thus, if an edge, say $e = (p_i, p_j), i < j$, intersects Δ then p_i is in S_1 and p_j is in S_2 . Therefore e crosses the pseudo-triangle Δ . Let $E = \{e_1, e_2, \dots, e_l\}$ be a set of the edges of T intersecting Δ where the edges of E are in the order they intersect the side $p_1p_{i_1}$, see Fig. 3. Let s be a segment in Δ that connect p_1 and a point of the segment $p_{i_1}p_{i_2}$.

First we show that, if the set E is empty then the pseudo-triangle Δ is present in T . Suppose that $E = \emptyset$. The segment s lies in a pseudo-triangle of T since none of edges of T intersects s . Let Δ' denote this pseudo-triangle. Clearly, $p_{i_1}p_{i_2}$ is an edge of Δ' . Let C_j where $j = 1, 2$ be the side of Δ between p_1 and p_{i_j} . It remains to show that the pseudo-triangle Δ' contains all the edges of C_1 and C_2 . Suppose that an edge $e = (p_{j_1}p_{j_2})$ of C_1 is not the edge of Δ' . The edge e split Δ' into two polygons, see Fig. 4. Let Q be the polygon that does not contain s . Q has a convex vertex different from p_{j_1} and p_{j_2} (vertex p_k in Fig. 4). The vertices p_1, p_{i_1} and p_{i_2} are convex vertices of Δ' too. Contradiction.

By Lemma 1 the dual edge of e_1 does not intersect the segment s . We flip edges $e_1, e_2, \dots, e_l$ in this order. Let T' be the pseudo-triangulation after l flips. By the previous argument (E is empty) T' contains the pseudo-triangle Δ . Note that p_1 is included in both S_1 and S_2 . We apply flips recursively to the sets S_1 and S_2 . Clearly, this yields the canonical pseudo-triangulation $\mathcal{T}(p, S)$.

Fig. 4. Edge $p_{j_1}p_{j_2}$ must be in Δ' .

It remains to show that the total number of flips is $O(n \log n)$. The number of flips needed to generate the pseudo-triangle Δ is l . $l = O(n)$ since the total number of edges of T is linear in terms of the number of vertices (a better bound $l \leq 2n - 3$ follows from the fact that T is a pseudo-triangulation). Let $F(k)$ be the number of flips required by our procedure applied to a pseudo-triangulation with k vertices. Then

$$F(n) = O(n) + F(\lceil n/2 \rceil) + F(\lfloor n/2 \rfloor + 1), F(1) = F(2) = F(3) = 0.$$

$F(n) = O(n \log n)$ is the solution of the recurrence. The theorem follows.

4 Algorithm

Theorem 3. *Let S be a set of n points in the plane in general position and let T and T'' be two pseudo-triangulations defined on S . A sequence of $f = O(n \log n)$ flips transforming T into T' can be found in $O((f + n) \log n)$ time.*

Proof. Let T and T'' be two pseudo-triangulations defined on the same set of vertices S . We show how to compute a sequence of flips that transforms T into T' . Let p be a point on the convex hull of S . We apply flips according to Theorem 2. The algorithm computes a sequence of flips to transform T into the canonical pseudo-triangulation $\mathcal{T}(p, S)$ and a sequence of flips to transform T'' into the canonical pseudo-triangulation $\mathcal{T}(p, S)$. Combining these two sequences we obtain the transformation of T into T' .

The main computational difficulty is the finding of a dual edge. A simple linear time algorithm leads to $O(fn)$ total running time which can be super-quadratic. We use an approach from [1] for maintaining a pseudo-triangulation. It is based on a data structure for storing a planar subdivision [5]. Applied to pseudo-triangulations the data structure has the following components:

- *faces*. The faces are pseudo-triangles. With each face we store pointers to its sides.

- *sides of pseudo-triangles*. Each side is a sequence of edges that form a convex chain. We store the chain edges in counterclockwise order. We associate a binary tree with every chain. With each side we store a pointer to its pseudo-triangle.
- *edges*. An edge is pair of vertices and we store pointers to these vertices. An edge is incident to either one pseudo-triangle (if its edge of $\text{conv}(S)$) or two pseudo-triangles (otherwise). We keep two pointers for these sides.
- *vertices*. Each vertex is stored one time. With each vertex we store its incident edges.

It is shown [1] that a dual edge can found in $O(\log n)$ time and the data structure can be maintained in $O(\log n)$ time per flip. Therefore the total time for flipping edges and maintaining the data structure is $O(f \log n)$.

It remains to show how to compute the flipping edges. The algorithm for finding flips is recursive we show it for one recursive call. First we compute the segment s that divides vertices of S into two subsets of almost equal size. s has the median slope and the median can be found in linear time [2].

We compute the edges $e_1, e_2, \dots, e_l$ crossing the segment s by checking every edge of T . In order to find the order of the edges we compute incident sides of pseudo-triangles and the pseudo-triangles traversed by s . This allows us to find the order in linear time.

Let $t(n)$ be the total time for computing flipping edges. Then $t(n) = O(n) + 2t(n/2)$ and the theorem follows.

5 Conclusion

We presented an algorithm that transforms a pseudo-triangulation to another one. The number of flips produced by the algorithm if $f = O(n \log n)$ where n is the number of vertices. The running time of the algorithm is $O((f + n) \log n)$. An interesting open question is to close the gap between the trivial lower bound $f = \Omega(n)$ and the upper bound $f = O(n \log n)$ for the smallest number of flips f .

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